SOME INTEGRABILITY THEOREMS

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1. P. Heywood [3] proved the following theorems:

THEOREM A. *If* $0 \leqq \gamma < 2$, *if* $x^{\gamma-1}g(x) \in L(0, \pi)$, *and if*

$$b_n = \frac{2}{\pi}\int_0^\pi g(x) \sin nx \, dx \tag{1.1}$$

for $n = 1, 2, 3, \ldots$, *then the series* $\sum_1^\infty n^{-\gamma} b_n$ *is convergent.*

THEOREM B. *If* $0 \leqq \gamma < 1$, *if* $x^{\gamma-1}f(x) \in L(0, \pi)$, *and if*

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx \, dx \tag{1.2}$$

for $n = 1, 2, 3, \ldots$, *then the series* $\sum_1^\infty n^{-\gamma} a_n$ *is convergent.*

THEOREM C. *Suppose that* $g(x) \in L(0, \pi)$, *that* b_n *is defined by* (1.1) *for each* n, *that* $0 < \gamma \leqq 1$, *and that the series* $\sum_1^\infty n^{\gamma-1} | b_n |$ *converges. Then the integral*

$$\int_{\to 0}^\pi x^{-\gamma} g(x) \, dx$$

exists as a Cauchy-Lebesgue integral.

THEOREM D. *Suppose that* $f(x) \in L(0, \pi)$, *that* a_n *is defined by* (1.2) *for* $n = 1, 2, \ldots$, *that* $0 < \gamma < 1$, *and that the series* $\sum_1^\infty n^{\gamma-1} | a_n |$ *converges. Then the integral*

$$\int_{\to 0}^\pi x^{-\gamma} f(x) \, dx$$

exists as a Cauchy-Lebesgue integral.

When the index γ satisfies $1 < \gamma < 2$, Siobhan O'Shea [6] has proved the following theorem:

THEOREM E. *Suppose that* $1 < \gamma < 2$. *Then the series*

$$\sum_{n=1}^\infty b_n \sin nx \quad (b_n \geqq 0) \tag{1.3}$$

converges everywhere to a function $g(x)$ *satisfying* $x^{-\gamma}g(x) \in L(0, \pi)$, *if and only if* $\sum_1^\infty n^{\gamma-1} b_n < \infty$.

The present note is concerned with generalizations of these theorems. We shall make use of a class of asymptotic functions which have previously been defined in [2]. By $\phi(x) \sim [a, b]$, $0 \leqq a \leqq b < \infty$ or $-\infty < a \leqq b \leqq 0$, we denote a non-negative function $\phi(x)$, not identically zero, such that $x^{-a}\phi(x)$ is non-decreasing and $x^{-b}\phi(x)$ is non-increasing, as x increases in $(0, \infty)$. By $\phi(x) \sim \langle a, b\rangle$, we denote $\phi(x)$ such that there exists some positive constant ε for which $\phi(x) \sim [a+\varepsilon, b-\varepsilon]$. We define $\phi(x) \sim [a, b\rangle$ and $\phi(x) \sim \langle a, b]$ in a similar way. We shall establish the following theorems:

THEOREM 1. *If $x^{-1}\phi(x^{-1})g(x) \in L(0, \pi)$, where $\phi(x) \sim [-1, 0\rangle$ or $\phi(x) \sim \langle -2, -1\rangle$, and if b_n is defined by* (1.1) *for $n = 1, 2, 3, \ldots$, then $\sum_1^\infty \phi(n)b_n$ is convergent.*

THEOREM 2. *If $\phi(x) \sim \langle -1, 0\rangle$ and $x^{-1}\phi(x^{-1})f(x) \in L(0, \pi)$, and if a_n is defined by* (1.2) *for every n, then $\sum_1^\infty \phi(n)a_n$ is convergent.*

THEOREM 3. *Suppose that $g(x) \in L(0, \pi)$, that b_n is defined by* (1.1) *for $n = 1, 2, 3, \ldots$, and that $\sum_1^\infty n^{-1}\phi(n^{-1}) \mid b_n \mid < \infty$, where $\phi(x) \sim \langle -2, 0\rangle$; then the integral*

$$\int_{\to 0}^{\pi} \phi(x)g(x)\,dx$$

exists as a Cauchy-Lebesgue integral.

THEOREM 4. *If $f(x) \in L(0, \pi)$, and if a_n is defined by* (1.2) *for $n = 1, 2, 3, \ldots$, such that $\sum_1^\infty n^{-1}\phi(n^{-1}) \mid a_n \mid < \infty$, where $\phi(x) \sim \langle -1, 0\rangle$, then the integral*

$$\int_{\to 0}^{\pi} \phi(x)f(x)\,dx$$

exists as a Cauchy-Lebesgue integral.

THEOREM 5. *Let $\phi(x) \sim \langle -2, -1\rangle$, and let $b_n \geqq 0$ for every n. Then the trigonometric series $\sum_1^\infty b_n \sin nx$ converges everywhere to $g(x)$ such that $\phi(x)g(x) \in L(0, \pi)$, if and only if $\sum_1^\infty n^{-1}\phi(n^{-1})b_n < \infty$.*

2. It is natural to inquire whether the result in Theorem A can be extended to integrability of the function $x^{-\gamma}\{g(x)\}^p$ for $p > 1$ (cf. *Math. Z.* **66** (1956), 9–12). The answer is in the negative even when $p = 2$. This may be justified by the example: $g(x) = (\pi - x)^{-\frac{1}{2}}$. Here we have $\pi b_n/2 \simeq K(-1)^{n+1}n^{-\frac{1}{2}}$, so that if $\gamma = \frac{1}{3}$, $p = 2$, then $x^{-\gamma}\{g(x)\}^2 \in L(0, \pi)$, but $\sum_1^\infty n^\gamma b_n^p = \sum_1^\infty n^{\frac{1}{3}} b_n^2 = \infty$.

As a particular case in Theorem 1, we may set $\phi(x) = x^{-\gamma}L(1/x)$, where $0 < \gamma < 2$ and $L(t)$ is a slowly increasing function in the sense of Karamata ([4], [5]; cf. also [7, p. 186]).

Similar conditions may be applied to Theorems 2 to 5. By $a(x)\simeq b(x)$ and $a(x)\asymp b(x)$, as $x\to c$, we mean $a(x)/b(x)\to 1$ and $K_1 < a(x)/b(x) < K_2$, respectively, as $x\to c$. Here and later the letter K denotes a positive constant, not necessarily the same at each occurrence.

LEMMA 1. *Let*

$$g(x)=\sum_1^\infty \lambda_n \sin nx, \tag{2.1}$$

where λ_n decreases steadily to zero. If $\phi(x)\sim[-1,\,0\rangle$ and if $\lambda_n\simeq\phi(n)$ as $n\to\infty$, then $g(x)\asymp x^{-1}\phi(x^{-1})$, as $x\to+0$.

Proof. Since $\phi(x)\sim[-1,0\rangle$, by Lemma 1 in [2], $\phi(x)$ is absolutely continuous in (δ,∞), where $\phi(x)$ decreases monotonically. From $\phi(x)\sim[-1,0\rangle$, we obtain $x\phi(x)\sim[0,1\rangle$, where $x\phi(x)$ is non-decreasing in $(0,\infty)$. It follows that

$$g(x)=\sum_1^\infty \lambda_n \sin nx=\sum_{1\leqq n\leqq[1/x]}+\sum_{n>[1/x]}=S_1+S_2, \tag{2.2}$$

where

$$|S_1|\leqq Kx\left|\sum_1^{[1/x]} n\lambda_n\right|\leqq Kx\int_1^{1/x} t\phi(t)\,dt\leqq Kx\left(\frac{1}{x}\right)\left\{\frac{1}{x}\phi\left(\frac{1}{x}\right)\right\}=\frac{K}{x}\phi\left(\frac{1}{x}\right). \tag{2.3}$$

By Abel's transformation, it is easy to verify that

$$|S_2|=\left|\sum_{n>1/x}\lambda_n \sin nx\right|\leqq\frac{K}{x}\phi\left(\frac{1}{x}\right). \tag{2.4}$$

It remains to show that $g(x)>Kx^{-1}\phi(x^{-1})$, as $x\to+0$. To see this, write

$$\left.\begin{aligned} g(x)&=\sum_1^\infty \Delta\lambda_n\frac{\sin^2\frac{1}{2}(n+\frac{1}{2})x}{\sin\frac{1}{2}x}-\frac{\lambda_1}{2}\tan\frac{x}{4}\\ &=\sum_1^\infty \Delta\lambda_n\frac{\sin^2\frac{1}{2}(n+\frac{1}{2})x}{\sin\frac{1}{2}x}+o\left\{\frac{1}{x}\phi\left(\frac{1}{x}\right)\right\}=S_3+S_4,\end{aligned}\right\} \tag{2.5}$$

say. If $\phi(x)\sim\langle -m,0\rangle$, then $x^\varepsilon\phi(x)$ decreases and $x^m\phi(x)$ increases for some $\varepsilon>0$ in $(0,\infty)$. This implies that $n^\varepsilon\phi(n)>(2n)^\varepsilon\phi(2n)$ and $n^m\phi(n)<(2n)^m\phi(2n)$. It follows that $\phi(n)-\phi(2n)>(2^\varepsilon-1)\phi(2n)>2^{-m}(2^\varepsilon-1)\phi(n)=K\phi(n)$. Write $\lambda_n=\{1+o(1)\}\phi(n)$, as $n\to\infty$. Then

$$S_3>\frac{K}{x}\sum_{\pi/2x\leqq n\leqq 3\pi/2x}\Delta\lambda_n\geqq\frac{K}{x}\{\lambda_{[\pi/2x]+1}-\lambda_{[3\pi/2x]}\}>\frac{K}{x}\phi\left(\frac{1}{x}\right), \tag{2.6}$$

as $x\to+0$. Hence $g(x)>(K/x)\phi(1/x)$, as $x\to+0$.

LEMMA 2. *If* $\phi(x) \sim \langle -1, 0\rangle$, *then, for small positive* x,

$$\left|\sum_1^\infty \phi(n)\cos nx\right| \leqq \frac{K}{x}\phi\left(\frac{1}{x}\right), \tag{2.7}$$

and also, for any positive integer N,

$$\left|\sum_1^N \phi(n)\cos nx\right| \leqq \frac{K}{x}\phi\left(\frac{1}{x}\right), \tag{2.8}$$

where K *in* (2.8) *is independent of* N.

The proofs of (2.7) and (2.8) are similar. For brevity, we only prove (2.7) here. Since $\phi(x) \sim \langle -1, 0\rangle$, there exists $\varepsilon > 0$, such that $x^{1-\varepsilon}\phi(x)$ increases and $x^{\varepsilon}\phi(x)$ decreases in $(0, \infty)$. By differentiating these functions we obtain

$$\varepsilon\phi(x)/x \leqq -\phi'(x) \leqq (1-\varepsilon)\phi(x)/x,$$

where $\phi'(x)$ exists almost everywhere. It follows as in the proof of Lemma 1 that

$$f(x) = \sum_1^\infty \phi(n)\cos nx = \sum_{1\leqq n\leqq[1/x]} + \sum_{n>[1/x]} = S_1 + S_2,$$

say. Here we have

$$| S_1 | \leqq K\int_1^{1/x}\phi(t)\,dt \leqq \frac{K}{x}\phi\left(\frac{1}{x}\right) - K\int_1^{1/x} t\phi'(t)\,dt$$

$$\leqq \frac{K}{x}\phi\left(\frac{1}{x}\right) + K(1-\varepsilon)\int_1^{1/x}\phi(t)\,dt \leqq \frac{K}{x}\phi\left(\frac{1}{x}\right),$$

where the last inequality is obtained by shifting the term $K(1-\varepsilon)\int \ldots$ to combine with $K\int_1^{1/x}\phi(t)\,dt$. Also, as in the proof of Lemma 1, we have $| S_2 | < Kx^{-1}\phi(x^{-1})$. The result follows.

LEMMA 3. *If* $\lambda_n \geqq 0$, *and if the series* $\sum_1^\infty \lambda_n \sin nx$ *converges everywhere to a function* $g(x)$ *such that* $x^{-1}g(x) \in L(0, \pi)$, *then* $\sum_1^\infty \lambda_n < \infty$.

LEMMA 4. *If* $\lambda_n \geqq 0$ *and if the series* $\sum_1^\infty \lambda_n \sin nx$ *converges everywhere to the function* $f(x)$, *such that* $\phi(x)f(x) \in L(0, \pi)$, *where* $\phi(x) \sim \langle -1, 0\rangle$, *then*

$$\sum_1^\infty \frac{1}{n}\phi\left(\frac{1}{n}\right)\lambda_n < \infty. \tag{2.9}$$

Lemma 3 is due to R. P. Boas [1]. For the proof of Lemma 4, it is sufficient to prove that the nth partial sum of (2.9) is bounded. Write $\psi(x) = x^{-1}\phi(x^{-1}) \sim \langle -1, 0\rangle$. The condition $\phi(x)f(x) \in L(0, \pi)$ with $\phi(x) \sim \langle -1, 0 \rangle$ implies that $\phi(x) \geqq K_\phi > 0$ in $(0, \pi)$ and that $f(x) \in L(0, \pi)$. Using Lemma 2, we see that

$$\sum_1^n \frac{1}{k}\phi\left(\frac{1}{k}\right)\lambda_k = \sum_1^n \psi(k)\lambda_k = \frac{2}{\pi}\sum_1^n \psi(k)\int_0^\pi f(x)\cos kx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi f(x)\sum_1^n \psi(k)\cos kx\, dx \leqq \frac{2}{\pi}\int_0^\pi |f(x)|\left|\sum_1^n \psi(k)\cos kx\right| dx$$

$$\leqq K\int_0^\pi \frac{1}{x}\psi\left(\frac{1}{x}\right)|f(x)|\, dx = K\int_0^\pi \phi(x)|f(x)|\, dx < K.$$

Lemma 5. *If $\phi(x) \sim \langle -2, -1 \rangle$, and if*

$$g(x) = \sum_1^\infty \phi(n)\sin nx, \tag{2.10}$$

then $g(x) \asymp x^{-1}\phi(x^{-1})$.

Here it should be remarked that $x^{-1}\phi(x^{-1})$ tends to zero as $x \to +0$. So it is not obvious that $\phi(n)$ in (2.10) can be replaced by $\lambda_n \simeq \phi(n)$, as in (2.1) of Lemma 1.

Since $\phi(x) \sim \langle -2, -1 \rangle$, we have $n\phi(n) \to 0$, as $n \to \infty$. By [7, Chap. 5, (1.3)], we see that $g(x) \to 0$, as $x \to +0$. On the other hand,

$$g'(x) = \sum_1^\infty n\phi(n)\cos nx = \sum_1^\infty \psi(n)\cos nx, \tag{2.11}$$

where $\psi(x) \sim \langle -1, 0 \rangle$, and the series (2.11) converges uniformly in (δ, π) for any $\delta > 0$. It follows from Lemma 2 that

$$g(x) = \lim_{\delta \to +0}\int_\delta^x g'(t)\, dt = O\left\{\int_0^x \frac{1}{t}\psi\left(\frac{1}{t}\right) dt\right\} = O\left\{\int_0^x \chi(t)\, dt\right\}, \tag{2.12}$$

as $x \to +0$, where $\chi(t) = t^{-1}\psi(t^{-1}) \sim \langle -1, 0 \rangle$. Then, as in the proof of Lemma 2, we see that the right-hand member of (2.12) is $O\{x\chi(x)\} = O\{x^{-1}\phi(x^{-1})\}$.

Furthermore, it follows as in the proof of Lemma 1 that $g(x) > Kx^{-1}\phi(x^{-1})$. Thus the proof of Lemma 5 is completed.

3. We come now to the proof of Theorem 1. The argument is similar to the proof of Theorem 1 in [3]. For any positive integer N, write

$$\frac{\pi}{2}\sum_1^{N-1}\phi(n)b_n = \int_0^\delta g(x)\sum_1^{N-1}\phi(n)\sin nx\, dx + \int_\delta^\pi g(x)\sum_1^\infty \phi(n)\sin nx\, dx$$

$$-\int_\delta^\eta g(x)\sum_N^\infty \phi(n)\sin nx\, dx - \int_\eta^\pi g(x)\sum_N^\infty \phi(n)\sin nx\, dx \tag{3.1}$$

$$= I_1 + I_2 + I_3 + I_4,$$

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say. Take $\delta = 1/N$, $\eta = 1/\sqrt{N}$. We shall see that I_1, I_3, I_4 tend to zero and I_2 tends to a finite limit. In view of Lemmas 1 and 5, it follows from the hypothesis $x^{-1}g(x)\phi(x^{-1}) \in L(0, \pi)$ that the expression

$$\lim_{N\to\infty} I_2 = \lim_{\delta\to+0} \int_\delta^\pi g(x) \sum_1^\infty \phi(n) \sin nx \, dx$$

is finite. Similarly, in view of (2.5) and Lemma 1,

$$| I_1 | \leqq \int_0^\delta | g(x) | \left| \sum_1^{N-1} \phi(n) \sin nx \right| dx \leqq K \int_0^\delta \frac{1}{x} \phi\left(\frac{1}{x}\right) | g(x) | \, dx = o(1), \tag{3.2}$$

as $N \to \infty$. By similar arguments, it is easy to show that $I_3 = o(1)$ and $I_4 = o(1)$, as $N \to \infty$. This completes the proof of Theorem 1. Similar arguments apply in the proof of Theorem 2. The result follows in a similar way, except that Lemma 1 is replaced by Lemma 2. Here we cannot replace $\phi(x) \sim \langle -1, 0 \rangle$ by $\phi(x) \sim [-1, 0\rangle$. This may easily be seen from the special case $\phi(x) = 1/x$, where $\sum n^{-1} \cos nx \sim -\log x$, as $x \to +0$ [3, p. 174]. This also means that Theorem 2 does not hold for the case $\phi(x) \sim \langle -2, -1 \rangle$.

For the proof of Theorem 3, we write

$$\chi(x) = g(x) - \psi(x),$$

where

$$\psi(x) = \sum_1^N b_n \sin nx, \quad N = [\delta^{-1}],$$

and

$$\int_\delta^\pi \phi(x) g(x) \, dx = \int_\delta^\pi \phi(x)\psi(x) \, dx + \int_\delta^\pi \phi(x)\chi(x) \, dx. \tag{3.3}$$

Let

$$X(x) = \int_0^x \chi(t) \, dt.$$

From Lemma 1 in [2], we see that $\phi(x)$ is absolutely continuous in $[\delta, \pi]$ for any $\delta > 0$. Since $g(x) \in L(0, \pi)$ implies $\chi(x) \in L(0, \pi)$, integration by parts gives

$$\int_\delta^\pi \phi(x)\chi(x) \, dx = \phi(\pi)X(\pi) - \phi(\delta)X(\delta) - \int_\delta^\pi \phi'(x)X(x) \, dx.$$

By similar arguments as in [3], it can readily be shown that the first and second terms on the right tend to zero as $\delta \to 0$. It remains to show that the last term tends to zero as $N \to \infty$. In fact, since $\phi(x) \sim \langle -2, 0 \rangle$ implies that $\phi(x)$ is absolutely continuous, it follows that

$$\left| \int_\delta^\pi \phi'(x)X(x) \, dx \right| \leqq \int_\delta^\pi \{-\phi'(x)\} \, | X(x) | \, dx \leqq \sum_{N+1}^\infty n^{-1} | b_n | \int_\delta^\pi \{-\phi'(x)\} \, dx$$

$$= \sum_{N+1}^\infty n^{-1} | b_n | \{\phi(\delta) - \phi(\pi)\} \leqq \phi(\delta) \sum_{N+1}^\infty n^{-1} | b_n | = o(1),$$

as $N \to \infty$, where $-\phi'(x)$ is positive almost everywhere. Hence

$$\int_\delta^\pi \phi(x)X(x)\,dx \to 0,$$

as $N \to \infty$. Then it is sufficient to consider

$$\int_\delta^\pi \phi(x)\psi(x)\,dx = \int_0^\pi \phi(x)\psi(x)\,dx - \int_0^\delta \phi(x)\psi(x)\,dx$$

$$= \int_0^\pi \phi(x)\psi(x)\,dx - \int_0^\delta \phi(x)\sum_1^M b_n \sin nx\,dx - \int_0^\delta \phi(x)\sum_{M+1}^N b_n \sin nx\,dx \tag{3.4}$$

$$= J_1 + J_2 + J_3,$$

say, where $M = [\delta^{-\frac{1}{2}}]$. Write $\theta_n(t) = \phi(t/n)/\phi(1/n)$ for $n = 1, 2, 3, \ldots$. It is easy to see that

$$t^{-\epsilon} \leqq \theta_n(t) \leqq t^{\epsilon-2} \quad (0<t<1), \qquad t^{\epsilon-2} \leqq \theta_n(t) \leqq t^{-\epsilon} \quad (t>1).$$

Since $\theta_n(t)$ decreases steadily to zero as $t \to \infty$,

$$\left|\int_0^\pi \phi(t)\sin nt\,dt\right| = \frac{1}{n}\phi\left(\frac{1}{n}\right)\left|\int_0^{n\pi} \theta_n(t)\sin t\,dt\right|$$

$$\leqq \frac{1}{n}\phi\left(\frac{1}{n}\right)\left|\left(\int_0^1 + \int_1^\pi\right)\theta_n(t)\sin t\,dt\right|$$

$$\leqq \frac{1}{n}\phi\left(\frac{1}{n}\right)\left\{\int_0^1 t^{\epsilon-2+1}\,dt + \int_1^\pi t^{-\epsilon}\,dt\right\} \leqq \frac{K}{n}\phi\left(\frac{1}{n}\right). \tag{3.5}$$

Hence

$$J_1 = \sum_1^N b_n \int_0^\pi \phi(x)\sin nx\,dx \to \sum_1^\infty b_n \int_0^\pi \phi(x)\sin nx\,dx,$$

as $N \to \infty$, where the last series converges absolutely. It remains to estimate J_2 and J_3. We have

$$|J_2| \leqq \sum_1^{[\sqrt{N}]} |b_n| \frac{1}{n}\phi\left(\frac{1}{n}\right)\int_0^{[1/\sqrt{N}]} \theta_n(t)\sin t\,dt$$

$$\leqq K N^{-\epsilon/2} \sum_1^\infty n^{-1}\phi(n^{-1})\,|b_n| = o(1), \tag{3.6}$$

and

$$|J_3| \leqq \sum_{[\sqrt{N}]}^N n^{-1}\phi(n^{-1})\,|b_n| \int_0^{n/N} \theta_n(t)\sin t\,dt$$

$$\leqq \sum_{[\sqrt{N}]}^N n^{-1}\phi(n^{-1})\,|b_n| \int_0^1 t^{\epsilon-1}\,dt$$

$$\leqq K \sum_{[\sqrt{N}]}^\infty n^{-1}\phi(n^{-1})\,|b_n| = o(1), \tag{3.7}$$

as $N \to \infty$. This completes the proof of Theorem 3.

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Finally, it should be remarked that the proof of Theorem 4 is practically the same as that of Theorem 3. Using Lemma 3 and Lemma 4, the proof of Theorem 5 follows in a similar way as in [6] and is omitted here.

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