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THE INTEGERS AS DIFFERENCES OF A SEQUENCE

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ABSTRACT. It is shown that there exists a sequence of integers $a_1 < a_2 < \cdots$ such that each positive integer is a difference of elements of the sequence in exactly one way, and such that a_k does not exceed a constant times k^3. In fact we construct such a sequence with each a_k in $[C(k-1)^3, Ck^3)$, where C is an absolute constant.

Paul Erdös has asked for a sequence of integers $a_1 < a_2 < \cdots$ such that each positive integer is the difference of two a's in exactly one way, and such that a_k does not exceed a constant times k^3. Prof. Erdos informs us he has constructed such a sequence using the "greedy algorithm"; we construct one with considerable regularity.

THEOREM. *There exists an absolute constant C and a set of integers A such that*

(a) *for each $i \geq 0$, exactly one element of A is in the interval $[Ci^3, C(i+1)^3)$,*

(b) *each $n > 0$ can be written as the difference of elements of A in exactly one way.*

Proof. For convenience the constant C, which will be specified later, will be taken to be an integer. Given a set S, we denote by $D(S)$ the set of all $s_2 - s_1 > 0$, s_1 and s_2 in S. We will consider certain finite sets F having the properties

(c) if $i \geq 0$ at most one element of F is in $[Ci^3, C(i+1)^3)$,

(d) each $n > 0$ can be written as the difference of elements of F in at most one way.

We will describe two constructions C1 and C2. Each construction will define a set F' properly containing F and still satisfying (c) and (d). Construction C1 will put n in $D(F')$, where n is the least positive integer not in $D(F)$. We will only apply C1 when the intervals $[Ci^3, C(i+1)^3)$ intersecting F are consecutive, starting with $[0, C)$. Construction C2 will put into F' an integer in the first interval $[Ci^3, C(i+1)^3)$ not intersecting F. We will start our construction with $F = \{0\}$. The theorem will follow if we show we can always apply C1 or C2 as described above.

Construction C1

Suppose F consists of $0 = a_1 < a_2 < \cdots < a_k$. Let n be the least positive integer not in $D(F)$. Choose an integer r such that $Cr^3 \geq 2a_k + 3$ and

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$C(r-1)^3 > a_k$. Set $a' = Cr^3 - 1$, $a'' = a' + n$, and $F' = F \cup \{a', a''\}$. Then it is easily checked that F' satisfies (c) and (d), and that $n = a'' - a'$ is in $D(F')$.

Construction C2

Since C1 will only be used when F intersects consecutive intervals, we can assume $F = F_1 \cup F_2$, where $F_1 = \{a_1, \ldots, a_k\}$, $C(i-1)^3 \le a_i < Ci^3$ for $i = 1, 2, \ldots, k$, F does not intersect $[Ck^3, C(k+1)^3)$, and F_2 contains at most two elements. We will show we can pick a_{k+1} in $[Ck^3, C(k+1)^3)$ to add to F while retaining condition (d). Actually we will pick a_{k+1} from the smaller set $[Ck^3, C(k^3+k^2))$.

To start we will assume F_2 is empty. Then we must choose a_{k+1} in $[Ck^3, C(k^3+k^2))$ so that $a_{k+1} - a_t = a_i - a_j$ does not hold for t, i, and $j \le k$, $j < i$. That is, a_{k+1} must avoid all the integers $a_t + a_i - a_j$. It will suffice for us to show there are fewer than Ck^2 such numbers to avoid. By symmetry we can assume $i \le t \le k$ and $j < t$.

First we fix t and estimate the number N_t of integers $a_t + a_i - a_j$ in $[Ck^3, C(k^3+k^2))$. For such an integer $C(i-1)^3 \le a_i < C(k^3+k^2) + a_j - a_t$ and $Ck^3 + a_j - a_t \le a_i < Ci^3$, so

$$(k^3 + C^{-1}(a_j - a_t))^{1/3} < i < (k^3 + k^2 + C^{-1}(a_j - a_t))^{1/3} + 1.$$

Thus for j fixed the number $n_j(t)$ of such integers does not exceed

$$(k^3 + k^2 + C^{-1}(a_j - a_t))^{1/3} - (k^3 + C^{-1}(a_j - a_t))^{1/3} + 2.$$

Now $(x+k^2)^{1/3} - x^{1/3}$ is a decreasing function of x for $x > 0$, so

$$n_j(t) < (k^3 + k^2 - t^3)^{1/3} - (k^3 - t^3)^{1/3} + 2,$$

since $a_j - a_t > -Ct^3$. We see

$$N_t = \sum_{1 \le j < t} n_j(t) < (t-1)((k^3+k^2-t^3)^{1/3} - (k^3-t^3)^{1/3} + 2).$$

Thus the total number of integers $a_t + a_i - a_j$ in $[Ck^3, C(k^3+k^2))$ is less than

$$\begin{aligned}
&\sum_{1 \le t \le k} (t-1)((k^3+k^2-t^3)^{1/3} - (k^3-t^3)^{1/3} + 2) \\
&\qquad < k^2 + (k-1)k^{2/3} + \sum_{1 \le t < k} (t-1)((k^3+k^2-t^3)^{1/3} - (k^3-t^3)^{1/3}) \\
&\qquad < 2k^2 + \sum_{1 \le t < k} (t-1)k^2(k^3-t^3)^{-2/3} \\
&\qquad = k^2\left(2 + \sum_{1 \le t < k} (t-1)(k^3-t^3)^{-2/3}\right),
\end{aligned}$$

where we have applied the mean value theorem to the function $x^{1/3}$ on $[k^3 - t^3, k^3 + k^2 - t^3]$. Since there are Ck^2 integers in $[Ck^3, C(k^3+k^2))$ it suffices

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to show

$$S(k)=\sum_{1\le t<k} t(k^3-t^3)^{-2/3}$$

is absolutely bounded.

Let $k=8q+r$, $0\le r<8$. Then

$$\begin{aligned} S(k) &= \sum_{1\le i<q}\ \sum_{8(i-1)<t\le 8i} t(k^3-t^3)^{-2/3}+\sum_{8(q-1)<t<k} t(k^3-t^3)^{-2/3} \\ &\le \sum_{1\le i<q}\ \sum_{8(i-1)<t\le 8i} t((8q)^3-(8i)^3)^{-2/3} \\ &\quad + \sum_{8(q-1)<t\le 8(q+1)} t(k^3-(k-1)^3)^{-2/3} \\ &= \sum_{1\le i<q} (q-\tfrac{7}{16})(q^3-i^3)^{-2/3}+8(16q+1)(k^3-(k-1)^3)^{-2/3} \\ &< S(q)+8q^{-1/3} \quad \text{for} \quad k\ge 8. \end{aligned}$$

Now suppose $k\ge 2^{12}$ and define the integer $p\ge 0$ by $8^p\le k/2^{12}<8^{p+1}$. Let $k=8q_1+r_1$, $0\le r_1<8$. Then $q_1\ge 2^{12}8^{p-1}$ and

$$S(k)<S(q_1)+8q_1^{-1/3}\le S(q_1)+2^{-p}.$$

If $p_1\ge 1$, then $q_1\ge 2^{12}$. Let $q_1=8q_2+r_2$, $0\le r_2<8$. Then $S(k)<S(q_2)+2^{-p}+2^{-(p+1)}$. Continuing in this way we see $S(k)\le M+2$, where M is the maximum of $S(j)$ for $j\le 2^{12}$. We see that if F_2 is empty then taking C any integer $\ge M+4$ will assure that a_{k+1} can be chosen.

Now even if F_2 is not empty it can account for at most $2(k+1)^2$ more numbers of the form $a+a^*-a^{**}$ to avoid, with a in F_2 and a^* and a^{**} in F. A simple calculation shows that taking $C\ge M+14$ makes construction C2 work in any case.

Reference

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