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ON SPACES AND ω-MAPPINGS

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1. **Introduction.** This note is closely related, as far as methods are concerned, to [3]. In [3] Ponomarev established "In order for a regular space X to be Lindelöf, it is necessary and sufficient that for each open covering ω of the space X there exists an ω-mapping $f: X \to Y$ onto some separable metric space Y." It is the purpose of this note to show that if the word "countable (or finite)" is inserted in the proper place we can obtain an analogous characterization for normal countably paracompact (or normal) spaces.

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2. **Preliminaries.** We require the following definition:

DEFINITION 2.1. Let ω be an open covering of the space X; a continuous mapping f from the space X onto some space Y is called an ω-mapping if for each point $y \in Y$ there exists a neighborhood O_y such that $f^{-1}O_y$ is contained in an element of the covering ω.

In our terminology a normal space need not be a Hausdorff space. Unless otherwise specified, we use the terminology of Kelley [1].

3. **ω-mappings.**

THEOREM 3.1 (Mansfield [2]). *A topological space X is normal and countably paracompact iff each countable open covering of X admits a countable, open star refinement.*

LEMMA 3.1. *If for every countable open covering ω of the space X there exists an ω-mapping $f: X \to Y$, onto a separable metric space Y, then X is normal and countably paracompact.*

Proof. Let ω be a countable open covering of the space X. According to the hypothesis there exists an ω-mapping f of the space X onto some separable metric space Y. For each $y \in Y$ we take O_y to be an open neighborhood of y such that $f^{-1}O_y$ is contained in some $U \in \omega$. Clearly $\mathbf{O} = \{O_y \mid y \in Y\}$ is an open cover of Y. Since Y is separable metric, it is Lindelöf and consequently, $\mathbf{O}$ has a countable subcover $\mathbf{O}' = \{O_{y_i}\}_{i=1}^{\infty}$. Furthermore, Y is a metric space so normal and countably paracompact and therefore by Theorem 3.1, $\mathbf{O}'$ has a countable open star refinement ω_1. Then $f^{-1}\omega_1 = \{f^{-1}v \mid v \in \omega_1\}$ is a countable open star refinement of ω. Hence, by Theorem 3.1, X is a normal and countably paracompact space.

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LEMMA 3.2. *Let X be a normal and countably paracompact space. For each countable open covering ω of the space there exists an ω-mapping $f: X \to Y$ onto some separable metric space Y.*

Proof. Let X be normal and countably paracompact; then by Theorem 3.1 every countable open covering ω has a countable open star refinement. Thus, for X normal and countably paracompact and ω any countable covering of X we can construct a sequence of coverings $\omega = \omega_0, \omega_1, \ldots, \omega_k, \ldots$ where ω_k, $k=1, 2, \ldots$ is a countable open star refinement of ω_{k-1}.

We shall denote by (X, τ) a topological space obtained from X by taking $\{\mathrm{St}\,(x, \omega_i) \mid i=1, 2, \ldots\}$ as a base for the neighborhood system at $x \in X$. For any subset A of X we set

$$\mathrm{int}\,(A; \tau) = \{x \mid \mathrm{St}\,(x, \omega_i) \subset A \text{ for some } i\}.$$

Then $\mathrm{int}\,(A; \tau)$ is open in (X, τ). Let $\mathbf{X}$ be a quotient space obtained from (X, τ) by defining two points x and y to be equivalent if $y \in \mathrm{St}\,(x, \omega_i)$ for $i=1, 2, \ldots$. Let ϕ be a quotient map of (X, τ) onto $\mathbf{X}$. Then we have

$$\mathrm{int}\,(A; \tau) = \phi^{-1}(\phi(\mathrm{int}\,(A, \tau))).$$

This shows that ϕ is a continuous open mapping of (X, τ) onto $\mathbf{X}$. Let ψ be an identity mapping of X onto (X, τ). Then, ψ is obviously continuous. Let us define $f = \phi \circ \psi$, and $Y = \mathbf{X}$. Then, it is clear that Y is a separable metric space and f is a continuous mapping. We need to show that f is an ω-mapping. Suppose $f(x_0) = \mathbf{x}_0$. We shall show that $f^{-1}(\phi\,(\mathrm{St}\,(x_0, \omega_3)))$ is contained in some member of ω_0. Let

$$x \in f^{-1}(\phi\,(\mathrm{St}\,(x_0, \omega_3))), \quad \text{i.e. } x \in (\phi \circ \psi)^{-1}(\phi(\mathrm{St}(x_0, \omega_3)))$$

which implies $x \in \psi^{-1}(\phi^{-1}(\phi(\mathrm{St}(x_0, \omega_3))))$. Hence, $x = \psi(x) \in \phi^{-1}(\phi(\mathrm{St}(x_0, \omega_3)))$ and this implies for some $z \in \mathrm{St}(x_0, \omega_3)$ we have $x \in \mathrm{St}(z, \omega_3)$. Since ω_3 is a star refinement of ω_2 for some $U^3 \in \omega_3$ which contains z and x_0 we have $U^2 \in \omega_2$ such that $\mathrm{St}(U^3, \omega_3) \subset U^2$. Now U^2 contains x and x_0. So for each $x \in f^{-1}(\phi(\mathrm{St}(x_0, \omega_3)))$ there is U^2 containing x and x_0. Since ω_2 is a star refinement of ω_1 it is easy to see that $f^{-1}(\phi(\mathrm{St}(x_0, \omega_3)))$ is contained in some member of ω_1. Finally, since ω_1 is a refinement of ω_0, $f^{-1}(\phi(\mathrm{St}(x_0, \omega_3)))$ is contained in some member of ω_0. Therefore f is an ω-mapping. Hence the lemma is proved.

THEOREM 3.2. *In order that a space X be normal and countably paracompact it is necessary and sufficient that for each countable open covering ω of the space X there exists an ω-mapping $f: X \to Y$ onto some separable metric space Y.*

Proof. Follows from Lemma 3.1 and Lemma 3.2.

It is well known that a topological space X is normal iff each finite open cover of X has a finite open star refinement. Using this fact and the techniques displayed in our proof of Theorem 3.2, it is easy to establish the following result.

THEOREM 3.3. *In order that a space X be normal it is necessary and sufficient that for each finite open covering ω of the space X there exists an ω-mapping $f: X \to Y$ onto some separable metric space.*

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