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A VARIATION OF THE KOEBE MAPPING IN A DENSE SUBSET OF S

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1. Introduction. Let $H(U)$ be the linear space of holomorphic functions defined on the unit disk U endowed with the topology of normal (locally uniform) convergence. For a subset $E \subset H(U)$ we denote by $\bar{E}$ the closure of E with respect to the above topology. The topological dual space of $H(U)$ is denoted by $H'(U)$.

Let D, $0 \in D$, be a simply connected domain in $\mathbf{C}$. The unique univalent conformal mapping ϕ from U onto D, normalized by $\phi(0) = 0$ and $\phi'(0) > 0$ will be called "the Riemann Mapping onto D". Let S be the set of all normalized univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$$

in $H(U)$. In connection with the famous Theorem of De Branges: $|a_n(f)| \leqq n$ for all $f \in S$ and $n \in \mathbf{N}$, and equality holds only for the Koebe function

$$k_1(z) = z/(1 - z)^2$$

and its rotations

$$k_\eta(z) = z/(1 - \eta z)^2; \ |\eta| = 1,$$

Bombieri [2] proved that

$$\varliminf_{a_2 \to 2; f \in S} [n - \operatorname{Re} a_n(f)]/[2 - \operatorname{Re} a_2(f)] > 0 \text{ for even } n, \text{ and}$$

$$\varliminf_{a_3 \to 3; f \in S} [n - \operatorname{Re} a_n(f)]/[3 - \operatorname{Re} a_3(f)] > 0 \text{ for odd } n$$

and conjectured that

$$(1.1) \quad \varliminf_{a_2 \to 2; f \in S} [n - \operatorname{Re} a_n(f)]/[m - \operatorname{Re} a_m(f)] = \inf_{\theta \in [0, 2\pi]} s_n(\theta)/s_m(\theta)$$

where

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$$s_n(\theta) = \sin(n\theta) - n \sin(\theta).$$

He uses there a combination of Schiffer's variation and the Loewner differential equation. Other studies on such variations can be found in [3].

In this article we shall study an elementary variation of k_1. Let

$$L = \{f(z) = z(1 - h(z))/(1 - z)^2 = k_1(z)(1 - h(z)); \quad h \in H(\bar{U})\} \text{ and}$$

$$L_0 = \{f \in L, h(1) \neq 1\}.$$

In Section 2 we show that

$$\overline{L_0 \cap S} = S \quad \text{and} \quad \overline{L_0 \cap S_{\mathbf{R}}} = S_{\mathbf{R}},$$

where

$$S_{\mathbf{R}} = \{f \in S; a_n(f) \in \mathbf{R} \text{ for all } n \in \mathbf{N}\}$$

is a subset of

$$T_{\mathbf{R}} = \left\{t(z) = z + \sum_{n=2}^{\infty} a_n(t)z^n \in H(U); \right.$$
$$\left. a_n(t) \in \mathbf{R} \text{ for all } n \in \mathbf{N} \text{ and } \operatorname{Re}\{(1 - z^2)t(z)/z\} > 0 \text{ in } U\right\}.$$

It is therefore reasonable to study variations of k_1 in $L \cap S$ ($L \cap S_{\mathbf{R}}$ respectively). Our main result is given in Theorem 3.1; the first part of which connects some variations of k_1 with functions in $T_{\mathbf{R}}$. More surprising is the second part, where explicit variations of k_1 in $S \cap L$ are constructed.

In Section 4 we give two applications of our variations. In Theorem 4.1 we give an elementary proof of

$$(1.2) \quad \inf_{\theta \in [0,2\pi]} s_n(\theta)/s_m(\theta) \leqq [n - a_n(f)]/[m - a_m(f)]$$
$$\leqq \sup_{\theta \in [0,2\pi]} s_n(\theta)/s_m(\theta)$$

for all $f \in T_{\mathbf{R}}$, $f \neq k_1$ and k_{-1}, and we show that these bounds are best possible in $S_{\mathbf{R}}$. In the second application we show that Bombieri's Conjecture, stated above, holds for all variations of k_1 stated in Theorem 3.1.

Finally, we shall use frequently the notations

$$\Delta(R) = \{z; |z| < R\}, \quad \Delta(R = 1) = U,$$

$$A(R_1, R_2) = \{z; R_1 < |z| < R_2\} \text{ and } \Delta^2(R) = \Delta(R) \times \Delta(R).$$

Furthermore, if $f_\epsilon \in H(U)$, we write

$$F_\epsilon(z, \zeta) = [f_\epsilon(z) - f_\epsilon(\zeta)]/(z - \zeta) \in H(U^2)$$

where

$$F_\epsilon(z, z) = f'_\epsilon(z).$$

2. A dense subclass in S. Let L be defined by

$$L = \left\{ f(z) = \frac{z(1 - h(z))}{(1 - z)^2};\ h \in H(\bar{U}) \right\} \quad \text{and}$$

$$L_0 = \{f \in L;\ h(1) \neq 1\}.$$

If $f \in L \cap S$ and $h \not\equiv 0$, then $h(0) = 0$ and $\operatorname{Re} h'(0) > 0$. Indeed,

$$\operatorname{Re} a_2(f) = 2 - \operatorname{Re} h'(0) < 2.$$

In this section we show

THEOREM 2.1. *We have*: $\overline{L_0 \cap S} = S$. *In other words, the set of functions in* S *having a pole of order two at* $z = 1$ *is dense in* S.

Proof. The proof is given in six steps. Let $f \in S$.

Step 1. Since $f_r(z) = f(rz)/r$, $0 < r < 1$, converges normally to $f(z)$ as r tends to one, we may assume that $f(U)$ is bounded by an analytic Jordan curve.

Step 2. Let γ be an open arc of $\partial f(U)$ and Γ be a half straight line in $\mathbf{C}\backslash f(U)$ joining $(\partial f(U)\backslash\gamma)$ with infinity. Denote by Ω the domain

$$\Omega = \mathbf{C}\backslash[(\partial f(U)\backslash\gamma) \cup \Gamma]$$

and let f_γ be the Riemann Mapping onto Ω. By the Carathéodory Kernel Theorem f_γ converges normally to f as the length of γ, $|\gamma|$, tends to zero. Therefore we may assume that $f = f_\gamma/f'_\gamma(0)$ for an appropriate γ. Observe that f has now a pole of order two at some point $\eta_0 \in \partial U$.

Step 3. If $\eta_0 = 1$, put $w_r(z) = f(z)$ and go to Step 6; if $\eta_0 = -1$, replace $f(z)$ by $e^{-i\theta}f(e^{i\theta}z)$ where θ is a small positive number. Therefore, we may assume that $\eta_0 \neq \mp 1$.

Step 4. Let $\eta_0 = e^{i\phi_0}$. With no loss of generality we may assume that $0 < \phi_0 < \pi$; otherwise consider $\overline{f(\bar{z})}$. Fix $r \in (0, 1)$ and consider the domains

$$D_{r,t} = U\backslash\{[r, 1] \cup E_{r,t}\}, \quad 0 < t < \pi$$

where

$$E_{r,t} = \{z = re^{i\theta};\ |\theta| \leqq t\}.$$

Let g_t be the Riemann Mapping onto $D_{r,t}$.

Denote by $E'_{r,t}$ the set of primend of $E_{r,t}$ attained by the radial limits

$$\lim_{\rho\uparrow r} \rho e^{i\theta}, \quad |\theta| \leqq t.$$

Since

$$\omega(E'_{r,t}, 0, D_{r,t}) > t/\pi$$

we have

$$g_t(re^{it}) = e^{i\beta(t)}, \quad t < \beta(t) < \pi,$$

and therefore $g_t^{-1}(1)$ consists of two points $\eta_1 = e^{-i\phi_1(t)}$ and $\bar{\eta}_1$ where $t < \phi_1(t) < \pi$. Hence we have

$$0 < \phi_0 < \phi_1(t) < \pi \text{ for all } t \in [\phi_0, \pi].$$

Put $\alpha = (\phi_0 + \pi)/2$. Then $\zeta = e^{-i\psi}$, $\psi > 0$, can be chosen so that

$$0 < \phi_0 < \phi_1(\alpha) - \psi < \pi.$$

In what follows we adopt the notation $\zeta \cdot E = \{\zeta w; w \in E\}$ whenever E is a set in **C**. Consider the Riemann mapping

$$q(z) = \bar{\zeta} g_\alpha(\zeta z)$$

from U onto $\bar{\zeta} \cdot D_{r,\alpha}$. Then

$$q^{-1}(1) = e^{-i\phi_2} \quad \text{where } 0 < \phi_0 < \phi_1(\alpha) - \psi < \phi_2 < \pi.$$

Note that q is analytic at the preimage of one, i.e., at $e^{-i\phi_2}$.

In the next step r is kept fixed and a continuous chain of domains is considered varying between $\bar{\zeta} \cdot D_{r,\alpha}$ and U in order to exhibit a Riemann Mapping h_{r,t_0} from U into U such that $h_{r,t_0}(\bar{\eta}_0) = 1$ and h_{r,t_0} is analytic at $\bar{\eta}_0$.

Step 5. For given $r \in (0, 1)$ consider the family of increasing domains

$$D_\tau = \begin{cases} \bar{\zeta} \cdot D_{r,\alpha-\tau} & \text{for } 0 \leqq \tau \leqq \alpha \\ \bar{\zeta} \cdot \{U \backslash [c(\tau), 1]\} & \text{for } \alpha \leqq \tau \leqq \pi \end{cases}$$

where

$$c(\tau) = ((1 - r)\tau + \pi r - \alpha)/(\pi - \alpha).$$

Denote by $h_{r,t}$ the Riemann Mapping onto D_τ and put

$$h_{r,\tau}^{-1}(1) = e^{-i\phi_2(\tau)}, \quad 0 < \phi_2(\tau) < \pi.$$

Then $h_{r,\tau}$ defines a Loewner chain from $h_{r,0} = q$ to $h_{r,\pi}$ which is the identity mapping. For

$$F = \{z = e^{i\theta}, 2\psi \leqq \theta \leqq 2\pi\}$$

we have by the symmetry of D_τ with respect to the axis $\{w = \lambda\bar{\zeta}; \lambda \in \mathbf{R}\}$

$$\omega(F, 0, D_\tau) = 1 - (\phi_2(\tau) + \psi)/\pi$$

is a continuous increasing function in τ and hence $\phi_2(\tau)$ decreases continuously from ϕ_2 to zero. Therefore a t_0 exists where $-\phi_2(t_0) = -\phi_0$, i.e.,

$$h_{r,t_0(\overline{\eta}_0)} = 1.$$

Step 6. Put

$$H_r(z) = f(\eta_0 h_{r,t_0}(\overline{\eta}_0 z))$$

and

$$w_r(z) = H_r(z)/H_r'(0).$$

Then $w_r \in S$ and converges normally to f as r tends to one. Observe that w_r has a pole of order two at $z = 1$. We then assume $f = w_r$.

Step 7. For $0 < \rho < 1$, let

$$l_\rho(z) = [w_r(\rho z + (1 - \rho)) - w_r(1 - \rho)]/\rho w_r'(1 - \rho) \in S.$$

Then l_ρ converges normally to w_r as ρ tends to one and has the desired properties.

THEOREM 2.2. *We also have* $\overline{L_0 \cap S_{\mathbf{R}}} = S_{\mathbf{R}}$.

Indeed, Step 1 of the above proof says that we may assume $f \in S_{\mathbf{R}} \cup H(\bar{U})$. In step 2 we take γ to be an arc $\{f(e^{it}), |t| < \delta_0\}$ and Γ the halfline $(-\infty, f(-1)]$. Then we may go directly to Step 7.

3. A variation in $L \cap S$ of the Koebe function. We consider a variation in $L \cap S$ of the Koebe function k_1 of the form

$$f_\epsilon(z) = \frac{z + \epsilon w(z) + g(z, \epsilon)}{(1 - z)^2}; \quad 0 < \epsilon < \epsilon_0,$$

where

$$w(z) \in H(\bar{U}), \quad w(z) + \overline{w(\bar{z})} \not\equiv 0, \quad g(\cdot, \epsilon) \in H(\bar{\Delta}(R))$$

for some $R > 1$ and $g(z, \epsilon)/\epsilon$ converges uniformly to zero in $\bar{\Delta}(R)$ as ϵ tends to zero. By the normalization of the class S we have

$$w(0) = w'(0) = g(0, \epsilon) = \frac{\partial g(0, \epsilon)}{\partial z} = 0.$$

Moreover, since $\operatorname{Re} a_2(f_\epsilon) < 2$, we have

$$\operatorname{Re} w''(0) \leqq 0.$$

Indeed, put

$$g(z, \epsilon) = \sum_{n=2}^{\infty} g_n(\epsilon) z^n,$$

then for every natural $n \geqq 2$, $g_n(\epsilon)/\epsilon$ converges to zero as ϵ tends to zero. This, in conjunction with

$$\text{Re } a_2(f_\epsilon) = \text{Re}\{2 + \epsilon w''(0)/2 + g_2(\epsilon)\} < 2 \text{ for } 0 < \epsilon < \epsilon_1,$$

implies that Re $w''(0) \leqq 0$. Next we show that Re $w''(0) = 0$ cannot occur so that $f_\epsilon(z)$ reduces to the form

$$(3.1)\quad f_\epsilon(z) = \frac{z - \epsilon z h(z) + g(z, \epsilon)}{(1 - z)^2}; \quad \epsilon \in (0, \epsilon_0)$$

where $h \in H(\bar{U})$, $h(0) = 0$, Re $h'(0) > 0$ and g as before. To see this, just note that for every natural $n \geqq 3$ we have

$$\frac{n - \text{Re } a_n(f_\epsilon)}{2 - \text{Re } a_2(f_\epsilon)} = \frac{\text{Re}\left\{\sum_{j=2}^{n} (\epsilon a_j(w) + g_j(\epsilon))(n - j + 1)\right\}}{\text{Re}\{\epsilon a_2(w) + g_2(\epsilon)\}}; \quad 0 < \epsilon < \epsilon_1$$

which must be bounded from above and below (see [**1**]). If Re $a_2(w) = 0$, then Re $a_n(w) = 0$ for all natural $n \geqq 3$, which contradicts the assumption that

$$w(z) + \overline{w(\bar{z})} \not\equiv 0.$$

Put

$$t(z) = (h(z) + \overline{h(\bar{z})})/2 \quad \text{and} \quad s(z) = (h(z) - \overline{h(\bar{z})})/2i.$$

Then $h(z)$ admits a unique representation $t(z) + is(z)$, where t and s are in $H(\bar{U})$ and have real coefficients.

Our main result gives necessary conditions and sufficient conditions on h and g so that the variation (3.1) is in $S \cap L$ for small ϵ.

THEOREM 3.1. A) *Let* $h \in H(\bar{U})$, $h(0) = 0$, Re $h'(0) > 0$. *Suppose that for* $\epsilon \in (0, \epsilon_0)$ *we have* $g(\cdot, \epsilon) \in H(\bar{\Delta}(R))$ *for some* $R > 1$ *such that* $g(\cdot, \epsilon)/\epsilon$ *converges uniformly to zero on* $\bar{\Delta}(R)$ *as* ϵ *tends to zero and that*

$$g(0, \epsilon) \equiv \frac{\partial g(0, \epsilon)}{\partial z} \equiv 0.$$

If

$$f_\epsilon(z) = [z - \epsilon h(z) + g(z, \epsilon)]/(1 - z)^2$$

is in $L \cap S$ *for all* $\epsilon \in (0, \epsilon_0)$, *then*

$$t(z) = [h(z) + \overline{h(\bar{z})}]/[2 \text{ Re } h'(0)] \in T_{\mathbf{R}}.$$

B) *Conversely, let* s *be any function in* $H(\bar{U})$ *with real coefficients,*

$u \in T_{\mathbf{R}}$, $t(z) = u(rz)/r$ for some $r \in (0, 1)$ and $h = t + is$. Then there is an $\epsilon_0 > 0$ such that

(3.2) $f_\epsilon(z) = z(1 - \epsilon h(z))/(1 - z)^2 \in L \cap S$

for all $\epsilon \in (0, \epsilon_0)$.

In proving Theorem 3.1 we shall study the equation

$$(3.3)\quad \frac{f_\epsilon(z) - f_\epsilon(\zeta)}{z - \zeta} = \frac{1 - z\zeta - \epsilon E_1(z, \zeta) + G_1(z, \zeta, \epsilon)}{(1 - z)^2(1 - \zeta)^2}$$

$$\equiv \frac{F_\epsilon(z, \zeta)}{(1 - z)^2(1 - \zeta)^2} = 0$$

in $U \times U$, where

$$(3.4)\quad \begin{aligned} E_1(z, \zeta) &= [(1 - \zeta)^2 z h(z) - (1 - z)^2 \zeta h(\zeta)]/(z - \zeta) \\ G_1(z, \zeta, \epsilon) &= [(1 - \zeta)^2 g(z, \epsilon) - (1 - z)^2 g(\zeta, \epsilon)/(z - \zeta) \text{ in A)} \end{aligned}$$

and

$$G(z, \zeta, \epsilon) \equiv 0 \text{ in B)}$$

are in $H(\overline{\Delta}^2(R))$ with respect to the variables z and ζ. From Cauchy's formula we conclude that $G_1(\cdot, \cdot, \epsilon)/\epsilon$ converge uniformly to zero in $\overline{\Delta}^2(\widetilde{R})$, $1 < \widetilde{R} < R$, as ϵ tends to zero.

For the proof of Theorem 3.1 we need some lemmas. First (Lemma 3.2) we show that for small ϵ and fixed ζ in a given annalus $A(R_1, R_2)$, $R_1 < 1 < R_2$, $F_\epsilon(\cdot, \zeta)$ has a unique zero in $\Delta(R_2)$. This zero, denoted by $z(\zeta, \epsilon)$, has the form

$$z(\zeta, \epsilon) = 1/\zeta + \epsilon c_1(\zeta) + o(\epsilon)$$

(Lemma 3.3) and is analytic in ζ (Lemma 3.5). Lemma 3.6 is purely computational and Lemmas 3.4 and 3.7 prepare the proof of statement B) of Theorem 3.1.

Let

$$E(z, \zeta) \in H(\overline{\Delta}^2(R))$$

and, for $\epsilon \in \Delta(\epsilon_0)$, let

$$G(\cdot, \cdot, \epsilon) \in H(\overline{\Delta}^2(R))$$

such that $G(\cdot, \cdot, \epsilon)/\epsilon$ converges uniformly to zero in $\overline{\Delta}^2(\widetilde{R})$, $1 < \widetilde{R} < R$, as ϵ tends to zero. Choose $R_2 \in (1, \widetilde{R})$ and define

$$0 < R_1 = (2R_2 - 1)^{-1/2} < 1.$$

For $n = 0, 1$ let

$$M_n > \max_{\overline{\Delta}^2(R_2)} |\partial^n E(z, \zeta)/\partial z^n|.$$

Then there is an $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\epsilon \in \Delta(\epsilon_1)$

$$\underset{\bar{\Delta}^2(R_2)}{\text{Max}} |\partial^n G(z, \zeta, \epsilon)/\partial z^n| \leqq \epsilon M_n.$$

LEMMA 3.2. *There is an* $\epsilon_2 \in (0, \epsilon_1)$ *such that for each*

$$\zeta \in A(R_1, R_2) = \{\zeta; R_1 < |\zeta| < R_2\}$$

and each $\epsilon \in \Delta(\epsilon_2) \subset \mathbf{C}$

$$(3.5) \quad F_\epsilon(z, \zeta) = 1 - z\zeta - \epsilon E(z, \zeta) + G(z, \zeta, \epsilon) = 0$$

has a unique solution $z(\zeta, \epsilon)$ *in* $\Delta(R_2)$.

Proof. Let

$$\epsilon_2 = \text{Min}\left\{\epsilon_1, \frac{1}{2}, (1 - R_1)^2/(4R_1M_0)\right\}.$$

Then for $\zeta \in A(R_1, R_2)$, $\epsilon \in \Delta(\epsilon_2)$ and $|z| = R_2$ we have

$$|z\zeta - 1| \geqq |z\zeta| - 1 \geqq R_2R_1 - 1$$
$$= \frac{(1 - R_1)^2}{2R_1} \geqq 2\epsilon_2 M_0 > |\epsilon E(z, \zeta) - G(z, \zeta, \epsilon)|.$$

Observe also that $|\zeta| > R_1$ implies that $|1/\zeta| < R_2$. By Rouché's theorem we conclude that for fixed $\zeta \in A(R_1, R_2)$ and $\epsilon \in \Delta(\epsilon_2)$ the number of zeros of $F_\epsilon(z, \zeta)$ in $|z| < R_2$ is equal to one. We denote this zero by $z(\zeta, \epsilon)$.

The following estimate for $z(\zeta, \epsilon)$ from the above lemma holds. We have

$$|z(\zeta, \epsilon) - 1/\zeta| \leqq 2|\epsilon| \cdot M_0/R_1$$

and therefore

$$\left|z(\zeta, \epsilon) - \frac{1}{\zeta} + \frac{\epsilon E(1/\zeta, \zeta)}{\zeta}\right|$$
$$= |\{\epsilon[E(z(\zeta, \epsilon), \zeta) - E(1/\zeta, \zeta)] - G(z(\zeta, \epsilon), \zeta, \epsilon)\}|/|\zeta|$$
$$\leqq \frac{|\epsilon|}{R_1} \int_{1/\zeta}^{z(\zeta,\epsilon)} \left|\frac{\partial E(z, \zeta)}{\partial z}\right| |dz| + \frac{|G(z(\zeta, \epsilon), \zeta, \epsilon)|}{R_1}$$
$$\leqq \frac{2|\epsilon|^2}{R_1^2} M_1 M_0 + \frac{|G(z(\zeta, \epsilon), \zeta, \epsilon)|}{R_1}.$$

In other words, we get

LEMMA 3.3. *For* $\zeta \in A(R_1, R_2)$ *and* $\epsilon \in \Delta(\epsilon_2)$ *we have*

$$z(\zeta, \epsilon) = \frac{1}{\zeta} + \epsilon \cdot c_1(\zeta) + o(\epsilon)$$

where $c_1(\zeta) = -E(1/\zeta, \zeta)/\zeta$ *and* $o(\epsilon)/\epsilon$ *converges uniformly to zero in* $A(R_1, R_2)$.

LEMMA 3.4. *Let* $K \subset \{R_1 < r_0 \leqq |\zeta| \leqq 1\}$ *be a subset on which*

$$\text{Re}\{\zeta c_1(\zeta)\} \geqq \delta > 0.$$

Then there is an $\epsilon_7(\delta) \in (0, \epsilon_2)$ *such that* $|z(\zeta, \epsilon)| \geqq 1$ *for all* $\zeta \in K$ *and all* $\epsilon \in (0, \epsilon_7)$.

Proof. For $\zeta \in K$ and $\epsilon \in (0, \epsilon_2)$ we have from Lemma 3.3 that

$$|z(\zeta, \epsilon)|^2 \geqq |\zeta|^2 |z(\zeta, \epsilon)|^2 = 1 + 2\epsilon \,\text{Re}\{\zeta c_1(\zeta)\} + o(\epsilon)$$

where $o(\epsilon)/\epsilon$ converges uniformly to zero in K as ϵ tends to zero.

A local version of Lemma 3.3 is attained by the Implicit Function Theorem. Indeed, since

$$\left.\frac{\partial F_\epsilon(z, \zeta)}{\partial z}\right|_{z(\zeta,\epsilon)} = -\zeta - \epsilon \frac{\partial E(z, \zeta)}{\partial z} + \left.\frac{\partial G(z, \zeta, \epsilon)}{\partial z}\right|_{z(\zeta,\epsilon)} \neq 0$$

for each $\zeta \in A(R_1, R_2)$ and $\epsilon \in \Delta(\epsilon_3)$ with

$$\epsilon_3 = \text{Min}\{\epsilon_2, (4R_2 M_1)^{-1}\}$$

we have

LEMMA 3.5. *Let* $\zeta_0 \in A(R_1, R_2)$ *and* $\epsilon \in \Delta(\epsilon_3)$. *Then there is a neighborhood* $V_\epsilon(\zeta_0)$ *of* ζ_0 *in* $\mathbf{C}$, *where*

$$z(\cdot, \epsilon) \in H(V_\epsilon(\zeta_0)).$$

Furthermore, if for $(z, \zeta) \in \overline{\Delta}^2(R)$

$$G(z, \zeta, \cdot) \in H(\Delta(\epsilon_3)),$$

then there is a neighborhood $V(\zeta_0)$ *of* $(\zeta = \zeta_0, \epsilon = 0)$ *in* $\mathbf{C}^2$ *such that* $z(\zeta, \epsilon) \in H(V(\zeta_0))$ *and admits there the representation*

$$z(\zeta, \epsilon) = \frac{1}{\zeta_0} + \epsilon c_1(\zeta_0) + O(|\zeta - \zeta_0|) + O(\epsilon^2).$$

In the sequel, we confine our $E(z, \zeta)$ to the form (3.4) and prove

LEMMA 3.6. *Let* $E(z, \zeta)$ *be as in* (3.4). *Then we have*

$$\text{a)}\quad \text{Re}\{\zeta c_1(\zeta)\} = \begin{cases} \dfrac{2\zeta}{(1+\zeta)^2}\,\text{Re}\left\{\dfrac{(1-\zeta^2)}{\zeta} t(\zeta)\right\} & \textit{for } |\zeta| = 1, \zeta \neq -1, \\ 4\,\text{Re}\, h'(-1) = 4t'(-1) & \textit{if } \zeta = -1. \end{cases}$$

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If furthermore $G(z, \zeta, \epsilon) \equiv 0$, *then*

b) $z(1, \epsilon) \equiv 1$ *for* $\epsilon \in \Delta(\epsilon_2)$, *and*

c) *for* $\zeta_0 = 1$ *and* $(\zeta = 1 + \eta, \epsilon) \in V(1)$ (*see Lemma* 3.5) *we have*

$$(3.6)\quad (1 + \eta)z(1 + \eta, \epsilon) = 1 - \eta^2\epsilon h'(1) - \epsilon\eta^3 B_1(\eta, \epsilon) - \epsilon^2\eta^2 B_2(\eta, \epsilon)$$

where B_1 *and* B_2 *are analytic in* $V(1)$.

Proof. Let $\zeta \in A(R_1, R_2)$. Then

$$(3.7)\quad \zeta c_1(\zeta) = -E(1/\zeta, \zeta) = \frac{(1 - \zeta)}{(1 + \zeta)}[h(\zeta) - h(1/\zeta)]$$

and

$$\lim_{\zeta \to -1} \zeta c_1(\zeta) = 4h'(-1).$$

For $|\zeta| = 1$ and $\zeta \neq -1$ we have

$$\begin{aligned}\operatorname{Re}\{\zeta c_1(\zeta)\} &= \operatorname{Re}\left\{\frac{(1 - \zeta)}{(1 + \zeta)}[h(\zeta) - h(\bar{\zeta})]\right\} \\ &= \operatorname{Re}\left\{\frac{(1 - \zeta)}{(1 + \zeta)}[h(\zeta) + \overline{h(\bar{\zeta})}]\right\} \\ &= \frac{2\zeta}{(1 + \zeta)^2}\operatorname{Re}\left\{\frac{(1 - \zeta^2)}{\zeta}t(\zeta)\right\}.\end{aligned}$$

b) For $\epsilon \in \Delta(\epsilon_2)$ we have by Lemma 3.2 a unique solution $z(1, \epsilon)$ in $\Delta(R_2)$ of

$$F_\epsilon(1, z) = (1 - z) - \epsilon h(1)(1 - z) = 0$$

and therefore $z(1, \epsilon) \equiv 1$.

c) Since $c_1(1) = 0$, we need to know some higher terms of the development of $z(\zeta, \epsilon) \equiv z(1 + \eta, \epsilon)$ in $V(1)$ (Lemma 3.5). Consider

$$(1 + \eta)z(1 + \eta, \epsilon) = \sum_{k,j=0}^{\infty} a_{kj}\eta^k\epsilon^j.$$

Since

$$\zeta z(\zeta, \epsilon) = 1 - \epsilon E(z(\zeta, \epsilon), \zeta)$$

we have $a_{00} = 1$ and $a_{k0} = 0$ for all $k \in \mathbf{N}$. Furthermore we conclude from b) above that $a_{0j} = 0$ for all $j \in \mathbf{N}$. Consequently

$$(3.8)\quad (1 + \eta)z(1 + \eta, \epsilon) = 1 + \sum_{k,j=1}^{\infty} a_{kj}\eta^k\epsilon^j \equiv 1 + \sum_{k=1}^{\infty} d_k(\epsilon)\eta^k.$$

Next, we compare the coefficients a_{kj} and $d_k(\epsilon)$ from (3.8) in the equation

$$F_\epsilon(z(1 + \eta, \epsilon), 1 + \eta) = 0$$

to conclude that $a_{11} = 0$, $a_{12} = 0$, $a_{21} = -h'(1)$ and

$$d_1(\epsilon)(d_1(\epsilon) - 2)(1 - \epsilon h(1)) \equiv 0.$$

Since $d_1(\epsilon)$ converges to zero as ϵ tends to zero, we have $d_1(\epsilon) \equiv 0$.

In the next lemma we consider the case $G(z, \zeta, \epsilon) \equiv 0$. Let $\zeta \in A(R_1, R_2)$. Put $\zeta = re^{it}$ and

$$\phi(re^{it}, \epsilon) = |z(re^{it}, \epsilon)|^2 \quad \text{for } \epsilon \in \Delta(\epsilon_3).$$

LEMMA 3.7. *Let $E(z, \zeta)$ be as in* (3.4) *and $G \equiv 0$. Then there is a neighborhood W, $\bar{W} \subset V(1)$, of* $(\zeta = 1, \epsilon = 0)$ *such that*

$$\frac{\partial \phi}{\partial r}(re^{it}, \epsilon) < 0 \quad \textit{in } W.$$

Proof. Since $\epsilon \in \Delta(\epsilon_3)$ and $\phi \in C^\infty$ in a neighborhood of (1, 0), we have

$$\frac{\partial F_\epsilon(z(\zeta, \epsilon), \zeta)}{\partial r}$$
$$= \frac{-\partial z(re^{it}, \epsilon)}{\partial r} \cdot re^{it} - e^{it} z(re^{it}, \epsilon) - \frac{\epsilon \partial E(z(\zeta, \epsilon), \zeta)}{\partial r} \partial r \equiv 0.$$

But by Lemma 3.6. b), $z(1, \epsilon) \equiv 1$ for $\epsilon \in \Delta(\epsilon_3)$ and therefore we have

$$\left.\frac{\partial z(re^{it}, \epsilon)}{\partial r}\right|_{(1,0)} = -1$$

which implies that

$$\frac{\partial \phi(1, 0)}{\partial r} = 2 \operatorname{Re}\left\{ \bar{z}(re^{it}, \epsilon) \frac{\partial z(re^{it}, \epsilon)}{\partial r} \right\}\bigg|_{(1,0)} = -2.$$

The existence of W follows from the continuity of $\partial\phi/\partial r$.

Proof of Theorem 3.1. A) Let f_ϵ satisfy the hypothesis of Theorem 3.1. A. Then

$$\frac{f_\epsilon(z) - f_\epsilon(\zeta)}{z - \zeta} = \frac{1 - z\zeta - \epsilon E_1(z, \zeta) - G_1(z, \zeta, \epsilon)}{(1 - z)^2(1 - y)^2} \neq 0 \quad \text{in } U \times U$$

for all $\epsilon \in (0, \epsilon_0)$, where E_1 and G_1 are of the form (3.4). Let $\tilde{R}$, R_2 and R_1 be as stated before Lemma 3.2. With no loss of generality we may assume that $\epsilon_0 \in (0, \epsilon_3)$, ϵ_3 being defined immediately before Lemma 3.5.

Fix $\zeta_0 \in \partial U$, $\zeta_0 \neq \pm 1$. We show that

$$|z(\zeta_0, \epsilon)| \geqq 1 \quad \text{for all } \epsilon \in (0, \epsilon_0).$$

Indeed, let $V_\epsilon(\zeta_0)$ as in Lemma 3.5 and consider the subset

$$V_1 = V_\epsilon(\zeta_0) \cap U.$$

Then for $\zeta \in V_1$ we have $|z(\zeta, \epsilon)| > 1$ which implies that $|z(\zeta_0, \epsilon)| \geqq 1$.
By Lemma 3.3 we have

$$|z(\zeta_0, \epsilon)|^2 = |\zeta_0 z(\zeta_0, \epsilon)|^2 = 1 + 2\epsilon \operatorname{Re}\{\zeta_0 c_1(\zeta_0)\} + o(\epsilon) \geqq 1$$

which, by Lemma 3.6.a), implies that

$$\operatorname{Re}\{\zeta_0 c_1(\zeta_0)\} = \frac{2\zeta_0}{(1 + \zeta_0)^2} \operatorname{Re}\left\{\frac{(1 - \zeta_0^2)}{\zeta_0} t(\zeta_0)\right\} \geqq 0 \quad (\zeta_0 \neq \pm 1)$$

and therefore

$$\operatorname{Re}\left\{\frac{(1 - \zeta_0^2)}{\zeta_0} t(\zeta_0)\right\} \geqq 0$$

on ∂U. Since

$$\frac{(1 - \zeta^2)}{\zeta} t(\zeta) \in H(\bar{U})$$

we conclude that

$$\operatorname{Re}\left\{\frac{(1 - \zeta^2)}{\zeta}\right\} t(\zeta) \geqq 0$$

in U. As we have observed in the beginning of this section that $\operatorname{Re} h'(0) = t'(0) > 0$, we conclude that

$$\operatorname{Re}\left\{\frac{(1 - \zeta^2)}{\zeta} t(\zeta)\right\} > 0$$

in U and

$$t(z)/t'(0) \in T_{\mathbf{R}}.$$

B) We shall make use of the following well-known result (see **[5]**):
If $F(z, \zeta) \in H(\bar{\Delta}^2(r))$, then $F(z, \zeta) \neq 0$ in $\bar{\Delta}^2(r)$ if and only if

I) $F(z, z) \neq 0$ in $\bar{\Delta}(r)$, and

II) $F(z, \zeta) \neq 0$ for all $(z, \zeta) \in \{(|z| = r) \times (|\zeta| = r)\}$.

Let $f_\epsilon(z)$ be as in (3.2). Then

$$\frac{f_\epsilon(z) - f_\epsilon(\zeta)}{z - \zeta} = \frac{1 - z\zeta - \epsilon E_1(z, \zeta)}{(1 - z)^2(1 - \zeta)^2} \equiv \frac{\hat{F}_\epsilon(z, \zeta)}{(1 - z)^2(1 - \zeta)^2}$$

where

$$E_1(z, \zeta) \in H(\bar{\Delta}^2(R_2)) \quad \text{for some } R_2 > 1$$

and has the form (3.4). Put

$$R_1 = (2R_2 - 1)^{-1/2}.$$

We show that there is an $\epsilon_0 > 0$ such that

$$\hat{F}_\epsilon(z, \zeta) \neq 0 \text{ in } U^2 \text{ for all } \epsilon \in (0, \epsilon_0).$$

This is done in three steps.

Step 1. We show that there is an $\epsilon_4 > 0$ such that

$$\hat{F}_\epsilon(z, z) = (1 - z)^4 f'_\epsilon(z) \neq 0$$

for all z in U and $\epsilon \in (0, \epsilon_4)$. Indeed we have

$$\begin{aligned}\hat{F}_\epsilon(z, z) &= 1 - z^2 - \epsilon(1 - z)[(1 - z)zh'(z) + (1 + z)h(z)] \\ &\equiv (1 - z) \cdot [1 + z - \epsilon T(z)], \quad T \in H(\bar{U}).\end{aligned}$$

Then the only possible zeros in U of $\hat{F}_\epsilon(z, z)$ satisfy

$$1 + z_\epsilon - \epsilon T(z_\epsilon) = 0$$

and are of the form

$$\begin{aligned}z_\epsilon &= -1 + \epsilon T(z_\epsilon) \\ &= -1 + \epsilon[(2 - \epsilon T(z_\epsilon))(-1 + \epsilon T(z_\epsilon)) \cdot h'(-1 + \epsilon T(z_\epsilon)) \\ &\qquad + \epsilon T(z_\epsilon) \cdot h(z_\epsilon)] \\ &= -1 - 2\epsilon h'(-1) + O(\epsilon^2)\end{aligned}$$

where $O(\epsilon^2)/\epsilon^2$ is uniformly bounded in $\bar{U}$. Since

$$\operatorname{Re} h'(-1) = u'(-r) > 0$$

there is an $\epsilon_4 > 0$ such that $\operatorname{Re} z_\epsilon < -1$ for $\epsilon \in (0, \epsilon_4)$ and therefore

$$\hat{F}_\epsilon(z, z) \neq 0 \quad \text{in } U.$$

Step 2. Next, we show that there is a $\rho_0 > 0$ and $\epsilon_5 > 0$ such that

$$\hat{F}_\epsilon(z, \zeta) \neq 0 \text{ in } U \times [\{|\zeta - 1| \leqq \rho_0\} \cap \bar{U}] \text{ for all } \epsilon \in (0, \epsilon_5).$$

For $\zeta \in A(R_1, R_2)$, let $z(\zeta, \epsilon)$ solve $\hat{F}_\epsilon(z, \zeta) = 0$ as in Lemma 3.2. Since

$$\operatorname{Re} h'(1) = u'(r) > 0,$$

let

$$\gamma = |\arg h'(1)| < \frac{\pi}{2}$$

and choose $d > 0$ and $\epsilon_6 > 0$, $\epsilon_6 < \epsilon_2$ such that

a) $W_1 = (\bar{U} \cap \{\text{Re } \zeta \geqq 1 - d\}) \times (0, \epsilon_6) \subset W$

(see Lemma 3.7), and

b) $d < 1 - \cos \dfrac{\pi - 2\gamma}{4}$.

Next, let $(\zeta = re^{it}, \epsilon) \in W_1$ and put $1 + \eta_0 = e^{it}$. Using Lemma 3.6. c) and Lemma 3.7, we get

$$\begin{aligned}\phi(re^{it}, \epsilon) > \phi(e^{it}, \epsilon) &= |z(1 + \eta_0, \epsilon)|^2 \\ &= |(1 + \eta_0)z(1 + \eta_0, \epsilon)|^2 \\ &= |1 - \epsilon\eta_0^2[h'(1) + \eta_0 B_1(\epsilon, \eta_0) + \epsilon B_2(\epsilon, \eta_0)]|^2 \\ &\geqq 1 - 2\epsilon \text{ Re}\{\eta_0^2[h'(1) + \eta_0 B_1(\epsilon, \eta_0) \\ &\qquad + \epsilon B_2(\epsilon, \eta_0)]\}.\end{aligned}$$

Since $B_1, B_2 \in H(\bar{W})$ we may choose $\rho_0 \in (0, d)$ and $\epsilon_5 \in (0, \epsilon_6)$ such that

$$(3.9) \quad |\arg[h'(1) + \eta B_1(\epsilon, \eta) + \epsilon B_2(\epsilon, \eta)] - \arg h'(1)| \leqq \frac{\pi - 2\gamma}{4};$$
$$\epsilon \in (0, \epsilon_5), \eta \in \Delta(\rho_0).$$

On the other hand $\zeta = 1 + \eta_0 \in \partial U$ and $|\eta_0| \leqq \rho_0 < d$ implies $\text{Re } \zeta > 1 - d$ and therefore, by b) above, η_0^2 lies in the sector

$$|\pi - \arg w| < \frac{\pi - 2\gamma}{4}$$

which in conjunction with (3.9) yields

$$\text{Re}\{\eta_0^2[h'(1) + \eta_0 B_1(\epsilon, \eta_0) + \epsilon B_2(\epsilon, \eta_0)]\} \leqq 0.$$

This in turn implies that

$$|z(1 + \eta, \epsilon)| \geqq 1 \text{ for } \epsilon \in (0, \epsilon_5) \text{ and } \eta \in \Delta(\rho_0).$$

In other words,

$$\hat{F}_\epsilon(z, \zeta) \neq 0 \text{ for all } \zeta \in \{|\zeta - 1| \leqq \rho_0\} \cap \bar{U},$$
$$z \in U \text{ and } \epsilon \in (0, \epsilon_5).$$

Step 3. We now show the existence of $\epsilon_7 > 0$ and an $r_0 \in (0, 1)$ such that

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$F_\epsilon(z, \zeta)$ does not vanish on $\{ |z| = r\} \times \{ |\zeta| = r\}$ for all $r \in (r_0, 1)$ and all $\epsilon \in (0, \epsilon_7)$. By assumption and Lemma 3.6. a), we have

$$\frac{2\zeta}{(1 + \zeta)^2} \operatorname{Re}\left\{\frac{(1 - \zeta^2)}{\zeta}\right\} t(\zeta) > 0$$

except for $\zeta = 1$. Therefore for $|\zeta| = 1$, $|\zeta - 1| \geqq \rho_0$; $\operatorname{Re}\{\zeta c_1(\zeta)\} \geqq \beta$ for some $\beta > 0$. By continuity there is an $r_0 \in (1 - \rho_0, 1)$ such that

$$\operatorname{Re}\{\zeta c_1(\zeta)\} \geqq \beta/2 > 0 \text{ on } \bar{A}(r_0, 1) \cap \{ |\zeta - 1| \geqq \rho_0\},$$

and by Lemma 3.4, there is an $\epsilon_7 > 0$, $\epsilon_7 < \epsilon_5$, such that $|z(\zeta, \epsilon)| \geqq 1$ for all

$$\epsilon \in (0, \epsilon_7) \quad \text{and} \quad \zeta \in A(r_0, 1) \cap \{ |\zeta - 1| \geqq \rho_0\}.$$

In particular we have shown that for all ϵ,

$$0 < \epsilon < \epsilon_0 = \operatorname{Min}(\epsilon_4, \epsilon_5, \epsilon_7)$$

and for all $r \in (r_0, 1)$

$\alpha)$ $\hat{F}_\epsilon(z, z) \neq 0$ in $\bar{\Delta}(r)$, and

$\beta)$ $\hat{F}_\epsilon(z, \zeta) \neq 0$ for all $(z, \zeta) \in \{ |z| = r\} \times \{ |\zeta| = r\}$,

and therefore $\hat{F}_\epsilon$ does not vanish in U^2 for all $\epsilon \in (0, \epsilon_0)$.

For $S_{\mathbf{R}} \cap L$ we have even more:

THEOREM 3.8. *Let $f \in T_{\mathbf{R}} \cap L$ and let h be defined by*

$$f(z) = z(1 - [2 - a_2(f)]h(z))/(1 - z)^2.$$

Then either $h \equiv 0$ or $h \in T_{\mathbf{R}}$.

Proof. Suppose $f(z) \neq z/(1 - z)^2$. Then $(2 - a_2(f)) > 0$. Now

$$h(z) = z + \sum_{k=2}^{\infty} h_k z^k \in H(\bar{U}),$$

$h_k \in \mathbf{R}$ for all $k \geqq 2$, and for $z \in \partial U$, $z \neq \mp 1$ we have

$$\operatorname{Re}\left\{\frac{(1 - z^2)}{z} h(z)\right\} = \frac{-1}{2 - a_2(f)} \operatorname{Re}\left\{\left[\frac{f(z)}{k_1(z)} - 1\right]\left[\frac{1 - z^2}{z}\right]\right\}$$

$$= \frac{-2 \operatorname{Im}\{z\} \cdot \operatorname{Im}\{f(z)/k_1(z)\}}{2 - a_2(f)}$$

$$= -2[(1 - z)^2/z] \operatorname{Im}\{z\}$$

$$\times \operatorname{Im}\{f(z)\}/(2 - a_2(f)).$$

Since $(1 - z^2)h(z)/z \in H(\bar{U})$, and $h'(0) = 1$, we conclude that

$$\operatorname{Re}\left\{\frac{(1 - z^2)}{z}h(z)\right\} > 0$$

in U and so $h \in T_{\mathbf{R}}$.

COROLLARY 3.9. *Theorem* 3.8 *holds in particular for* $f \in S_{\mathbf{R}} \cap L$.

4. Applications. We give now two examples to show how these variations can be used. Our first application is

THEOREM 4.1. *Let*

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in S_{\mathbf{R}}, \quad f \neq k_1 \text{ and } k_{-1}.$$

Then for all m *and* $n \geqq 2$ *we have*

$$(4.1) \quad c(n, m) = \inf_{\theta \in [0,2\pi]} s_n(\theta)/s_m(\theta) \leqq \frac{n - a_n(f)}{m - a_m(f)} \leqq \sup_{\theta \in [0,2\pi]} s_n(\theta)/s_m(\theta) = d(n, m)$$

where

$$s_n(\theta) = \sin(n\theta) - n \sin \theta.$$

The given bounds are best possible in $S_{\mathbf{R}}$.

For the proof we shall use the following lemma. Let $\Lambda_1, \Lambda_2 \in H'(U)$ satisfy the following property:

(4.2) $\Lambda_1, \Lambda_2 \geqq 0$ on $T_{\mathbf{R}}$ and both vanish only for one function $\hat{h} \in T_{\mathbf{R}}$. Let also

$$E(T_{\mathbf{R}}) = \left\{\frac{z}{(1 - \eta z)(1 - \bar{\eta} z)}; |\eta| = 1\right\}$$

be the set of extreme points of $T_{\mathbf{R}}$. Note, that $\hat{h} \in E(T_{\mathbf{R}})$ and put

$$M = \sup\{\Lambda_1(h)/\Lambda_2(h), h \neq \hat{h}, h \in T_{\mathbf{R}}\}.$$

We have

LEMMA 4.2. *Let* Λ_1 *and* Λ_2 *be in* $H'(U)$ *satisfying the property* (4.2). *Then*

$$M = \sup\{\Lambda_1(h)/\Lambda_2(h); h \neq \hat{h}, h \in E(T_{\mathbf{R}})\}.$$

Proof. Without loss of generality we may assume that Λ_1/Λ_2 is not constant on $T_{\mathbf{R}}\backslash\{\hat{h}\}$.

Step 1. We first show that the set $\{\Lambda_1(h)/\Lambda_2(h);\ h \in T_{\mathbf{R}},\ h \neq \hat{h}\}$ is connected. Indeed, fix $h \in T_{\mathbf{R}}$, $h \neq \hat{h}$ and let

$$h_r(z) = \begin{cases} h(rz)/r & \text{for } 0 < r \leqq 1 \\ z & \text{for } r = 0. \end{cases}$$

Since $\hat{h} \in E(T_{\mathbf{R}})$, we conclude that $h_r \neq \hat{h}$ for all $r \in [0, 1]$. Therefore the function

$$\phi(r) = \Lambda_1(h_r)/\Lambda_2(h_r)$$

is continuous in $[0, 1]$ and its values connect $\Lambda_1(h)/\Lambda_2(h)$ with $\Lambda_1(z)/\Lambda_2(z)$.

Step 2. If $M < \infty$ let $g_j \in T_{\mathbf{R}}$ such that

$$\Lambda_1(g_j)/\Lambda_2(g_j) = M - 1/(j + 1)$$

and $g_j \neq \hat{h}$ for all $j \geqq j_0$, $j \in \mathbf{N}$. Then we have for $j \geqq j_0$

$$\Lambda_1(g_j) - [M - 1/j]\Lambda_2(g_j) = \frac{1}{j(j + 1)}\Lambda_2(g_j) > 0.$$

Put

$$L_j = \Lambda_1 - [M - 1/j]\Lambda_2 \in H'(U).$$

Since $L_j(g_j) > 0$, there is a $t_j \in E(T_{\mathbf{R}})$ such that

$$L_j(t_j) \geqq L_j(g_j) > 0.$$

Observe that $t_j \neq \hat{h}$ since $L_j(\hat{h}) = 0$.

Step 3. If $M = \infty$, there are $g_j \in T_{\mathbf{R}}$ such that

$$\Lambda_1(g_j)/\Lambda_2(g_j) = j + 1;\ j \geqq j_0,\ j \in \mathbf{N}$$

and $g_j \neq \hat{h}$ for all $j \geqq j_0$. Since

$$L_j(g_j) \equiv \Lambda_1(g_j) - j\Lambda_2(g_j) = \Lambda_2(g_j) > 0,$$

there is a $t_j \in E(T_{\mathbf{R}})$, $t_j \neq \hat{h}$, such that

$$L_j(t_j) \geqq L_j(g_j) > 0.$$

Step 4. In Step 2 and Step 3, for arbitrary $M \in (0, \infty]$, we have found $t_j \in E(T_{\mathbf{R}})$, $t_j \neq \hat{h}$, such that

$$\lim_{j \to \infty} \Lambda_1(t_j)/\Lambda_2(t_j) = M.$$

COROLLARY 4.3. *Let $\Lambda_1, \Lambda_2 \in H'(U)$ satisfy the property* (4.2). *Then*

$$\begin{aligned} m &\equiv \inf\{\Lambda_1(h)/\Lambda_2(h),\ h \neq \hat{h},\ h \in T_{\mathbf{R}}\} \\ &= \inf\{\Lambda_1(h)/\Lambda_2(h),\ h \neq \hat{h},\ h \in E(T_{\mathbf{R}})\}. \end{aligned}$$

Proof of Theorem 4.1. Step 1. It is enough to proof the theorem for functions in $L \cap S_{\mathbf{R}}$. Indeed, we have seen (Theorem 2.2) that

$$\overline{L \cap S_{\mathbf{R}}} = S_{\mathbf{R}}.$$

If $f \in S_{\mathbf{R}}, f \neq k$, and k_{-1}, then there are $f_j \in L \cap S_{\mathbf{R}}, f_j \neq k_1$ and k_{-1} converging normally to f. In particular, since $|a_n(f)| < n$ for all $n \geqq 2$, we have

$$\lim_{j\to\infty} \frac{n - a_n(f_j)}{m - a_m(f_j)} = \frac{n - a_n(f)}{m - a_m(f)}; \quad n, m \geqq 2.$$

Step 2. Let $f \in L \cap S_{\mathbf{R}}$, $f \neq k_1$ and k_{-1}. We show that for n, $m \geqq 2$,

$$(4.3) \quad \frac{n - a_n(f)}{m - a_m(f)} = \frac{a_n[zh(z)/(1 - z)^2]}{a_m[zh(z)/(1 - z)^2]} = \frac{\Lambda_1(h)}{\Lambda_2(h)}, \quad h \in T_{\mathbf{R}},$$

where $\Lambda_1, \Lambda_2 \in H'(U)$ are nonnegative on $T_{\mathbf{R}}$. Furthermore $\Lambda_2(h)(\Lambda_1(h)$ respectively) $= 0$ on $T_{\mathbf{R}}$ if and only if $h = \hat{h} = k_{-1}$ and m (n respectively) is odd. Indeed, by Theorem 3.8 we have

$$f(z) = k_1(z) - (2 - a_2(f))h(z) \cdot z/(1 - z)^2, \quad h \in T_{\mathbf{R}}.$$

In particular

$$n - a_n(f) = (2 - a_2(f))a_n(h(z) \cdot z/(1 - z^2)), \quad n \geqq 2$$

and (4.3) follows. To see that Λ_2 is nonnegative on $T_{\mathbf{R}}$, just note that

$$\Lambda_2\left(\frac{z}{(1 - \eta z)(1 - \bar{\eta} z)}\right) = \begin{cases} \dfrac{\eta}{(1 - \eta)^2} \dfrac{\operatorname{Im}\{\eta^m - m\eta\}}{\operatorname{Im} \eta} \\ \quad = \dfrac{\eta}{(1 - \eta)^2} \dfrac{s_m(\theta)}{\operatorname{Im} \eta} > 0; \ |\eta| = 1, \eta \neq \mp 1, \\ m(m^2 - 1)/6 \quad ; \eta = 1 \quad , \\ m/2 \quad ; \eta = -1 \quad , m \text{ even}, \\ 0 \quad ; \eta = -1 \quad , m \text{ odd}. \end{cases}$$

It remains to show that $\Lambda_2(h)$ ($\Lambda_1(h)$ respectively) $= 0$ if and only if $h = k_{-1}$ and m is odd. Indeed, for $h \in T_{\mathbf{R}}$ there is a probability measure μ on the Borel σ-algebra of ∂U such that

$$h(z) = \int_{|\eta|=1} \frac{z}{(1 - \eta z)(1 - \bar{\eta} z)} d\mu.$$

Since $\Lambda_2 \in H'(U)$, we have

$$\Lambda_2(h) = \int_{|\eta|=1} \Lambda_2\left(\frac{z}{(1 - \eta z)(1 - \bar{\eta} z)}\right) d\mu \geqq 0$$

where equality holds if and only if μ is concentrated at the point $\eta = -1$, i.e., $h = k_{-1}$.

Step 3. We prove now the upper bound inequality and distinguish among the following cases

a) n even, m odd: In this case we have

$$M = \infty \quad \text{and} \quad \sup_{\theta \in [0,2\pi]} s_n(\theta)/s_m(\theta) = \infty.$$

b) m even: In this case $\Lambda_2(h) > 0$ for all $h \in T_{\mathbf{R}}$, and therefore $\Lambda_1(h)/\Lambda_2(h)$ is a continuous functional on $T_{\mathbf{R}}$. Hence $M < \infty$. There exist $g_j \in T_{\mathbf{R}}$ such that

$$\Lambda_1(g_j)/\Lambda_2(g_j) \geqq M - 1/j, \quad j \in \mathbf{N},$$

and therefore

$$L_j(g_j) \equiv \Lambda_1(g_j) - \left(M - \frac{1}{j}\right)\Lambda_2(g_j) > 0.$$

Then, there is a $t_j \in E(T_{\mathbf{R}})$ such that

$$L_j(t_j) \geqq L_j(g_j) > 0$$

and so for some $\eta_j = e^{i\theta_j}$, $\eta_j \neq -1$,

$$M - 1/j < \frac{\Lambda_1(t_j)}{\Lambda_2(t_j)} = \frac{\operatorname{Im}(\eta_j^n - n\eta_j)}{\operatorname{Im}(\eta_j^m - m\eta_j)} = \frac{s_n(\theta_j)}{s_m(\theta_j)} \leqq d(m, n).$$

Let us remark that this case is contained in a very general theorem of Ruscheweyh [**4**].

c) n odd, m odd: In this case Λ_1 and Λ_2 satisfy the condition (4.2) where $\hat{h} = k_{-1}$. By Lemma 4.2

$$M = \sup_{\theta \in [0,2\pi]} \operatorname{Im}(\eta^n - n\eta)/\operatorname{Im}(\eta^m - m\eta) = d(n, m).$$

d) Finally we show that the upper bound is best possible for functions in $S_{\mathbf{R}}$. To do so, let $t \in E(T_{\mathbf{R}})$, $t \neq k_{-1}$ and $r \in (0, 1)$. Then by Theorem 3.1. B, there is an $\epsilon_0(r)$ such that

$$f_{\epsilon,r}(z) = k(z)(1 - \epsilon t(rz)/r) \in S_{\mathbf{R}} \cap L$$

for all $\epsilon \in (0, \epsilon_0(r))$. Pick any of such ϵ, then

$$\frac{n - a_n(f_{\epsilon,r})}{m - a_m(f_{\epsilon,r})} = \frac{\Lambda_1(t(rz)/r)}{\Lambda_2(t(rz)/r}$$

which converges to $\Lambda_1(t)/\Lambda_2(t)$ as r tends to one. Choosing

$$t_j \in E(T_{\mathbf{R}}),\ t_j \neq k_{-1} \quad \text{and} \quad \lim_{j\to\infty} \Lambda_1(t_j)/\Lambda_2(t_j) = M$$

we conclude that $d(n, m)$ is best possible.

Step 4. The lower bound inequality follows by the same arguments using Corollary 4.3 and it is also best possible in $S_{\mathbf{R}}$ by Theorem 3.1. B.

In our next application we show that the Bombieri conjecture (see introduction) is valid for variations of k_1 discussed in Theorem 3.1. A.

THEOREM 4.4. *Let f_ϵ satisfy the same hypothesis of Theorem* 3.1. A. *Then, for all m, $n \geqq 2$, we have*

$$c(n, m) \leqq \lim_{\epsilon \to 0} \frac{n - \operatorname{Re} a_n(f_\epsilon)}{m - \operatorname{Re} a_m(f_\epsilon)} \leqq d(n, m)$$

where $c(n, m)$ and $d(n, m)$ are defined in Theorem 4.1. *The bounds are best possible.*

Proof. Since f_ϵ is of the form (3.1), we have

$$\operatorname{Re} h'(0) > 0 \quad \text{and } (h(z) + \overline{h(\bar{z})})/2 \operatorname{Re} h'(0) = t(z)/t'(0) \in T_{\mathbf{R}}.$$

Hence

$$\frac{n - \operatorname{Re} a_n(f_\epsilon)}{m - \operatorname{Re} a_m(f_\epsilon)} = \frac{\operatorname{Re} a_n[(zt(z) + t'(0)g(z, \epsilon)/\epsilon)/(1 - z)^2]}{\operatorname{Re} a_m[(zt(z) + t'(0)g(z, \epsilon)/\epsilon)/(1 - z)^2]}$$

converges to $\Lambda_1(t)/\Lambda_2(t) \in [c(n, m), d(n, m)]$. We show that these bounds are best possible. For every $t(z) = u(rz)/r$, $0 < r < 1$ and $u \in T_{\mathbf{R}}$, there is an $\epsilon_0(t)$ (Theorem 3.1. B) such that for all $\epsilon \in (0, \epsilon_0(t))$ f_ϵ defined by (3.2) is in S. Then

$$\frac{n - \operatorname{Re} a_n(f_\epsilon)}{m - \operatorname{Re} a_m(f_\epsilon)} = \frac{a_n(k_1(z) \cdot u(rz)/r)}{a_m(k_1(z) \cdot u(rz)/r)}$$

is independent of ϵ. Therefore, every value of

$$\{\Lambda_1(u(rz)/r)/\Lambda_2(u(rz)/r),\ u \in T_{\mathbf{R}}\}$$

is attained, so that we can get as close as we please to $c(n, m)$ or $d(n, m)$.

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