Canad. Math. Bull. 2025, pp. 1–16
http://dx.doi.org/10.4153/S0008439525101100

Absolute continuity in the reproducing kernel sense

Robert T. W. Martin and Edward J. Timko

Abstract. Given positive Radon measures, μ and λ, on the complex unit circle, we show that absolute continuity of μ with respect to λ is equivalent to their reproducing kernel Hilbert spaces of "analytic Cauchy transforms" in the complex unit disk having dense intersection in the space of μ-Cauchy transforms.

1 Introduction

This article establishes a function–theoretic characterization of absolute continuity for pairs of positive Radon measures on the complex unit circle. Throughout, $\mathbb{D}$ denotes the open unit disk centered at 0 in the complex plane and ζ denotes the independent variable or co-ordinate function on $\partial\mathbb{D}$. Let $L^2(\mu)$ denote the L^2-space on $\partial\mathbb{D}$ determined by μ. We write $\mathbb{C}[\zeta]$ for the ring of analytic polynomials in ζ, and denote by $H^2(\mu)$ the closure of $\mathbb{C}[\zeta]$ in $L^2(\mu)$. The linear operator of multiplication by the independent variable on $L^2(\mu)$ is denoted by M^μ_ζ. Evidently, M^μ_ζ is unitary on $L^2(\mu)$ and $H^2(\mu)$ is an invariant subspace for M^μ_ζ.

Given $h \in H^2(\mu)$, the μ-*Cauchy transform* of h on $\mathbb{D}$ is given by

$$(\mathscr{C}_\mu h)(z) = \int_{\partial\mathbb{D}} \frac{h}{1 - z\overline{\zeta}}\, d\mu .$$

It is clear that $\mathscr{C}_\mu h$ is holomorphic in the complex unit disk, and it is easily seen that $\mathscr{C}_\mu$ is an injective linear map from $H^2(\mu)$ into $\mathscr{O}(\mathbb{D})$, the algebra of holomorphic functions in $\mathbb{D}$. The *Herglotz space* of μ-Cauchy transforms, $\mathscr{H}^+(\mu) := \mathscr{C}_\mu H^2(\mu)$, is naturally equipped with the inner product

$$\langle \mathscr{C}_\mu g, \mathscr{C}_\mu h\rangle_{\mathscr{H}^+(\mu)} := \langle g, h\rangle_{L^2(\mu)}; \quad g, h \in H^2(\mu),$$

making $\mathscr{C}_\mu$ a surjective isometry from $H^2(\mu)$ onto $\mathscr{H}^+(\mu)$. Here, and in what follows, note that our inner product is conjugate linear in the first argument and linear in the second, following the convention used in [BMN24].

Received by the editors April 25, 2025; revised June 9, 2025; accepted June 10, 2025.
Published online on Cambridge Core August 27, 2025.
AMS subject classification: **47B32, 28A99**.
Keywords: Lebesgue decomposition, Cauchy transform, reproducing kernel Hilbert space.

The Herglotz space of μ-Cauchy transforms is a reproducing kernel Hilbert space (RKHS) of analytic functions in $\mathbb{D}$ with reproducing kernel

$$k^{\mu}(z,w) = \int_{\partial\mathbb{D}} \overline{\frac{1}{1-\overline{z}\zeta}} \frac{1}{1-\overline{w}\zeta} d\mu, \quad z,w \in \mathbb{D}.$$

It follows readily from this formula that domination of positive measures implies domination of the reproducing kernels for their Herglotz spaces: given positive Radon measures μ and λ on $\partial\mathbb{D}$, if $t > 0$ is such that $\mu \leq t^2\lambda$, then $k^{\mu} \leq t^2 k^{\lambda}$ as reproducing kernels. By Aronszajn's Inclusion Theorem for RKHS (see Section 2.1 for the statement), $\mathscr{H}^+(\mu)$ is then boundedly contained in $\mathscr{H}^+(\lambda)$, with the norm of the linear embedding bounded above by t.

More generally, absolute continuity can also be translated into a relationship between spaces of Cauchy transforms. A straightforward application of the Radon–Nikodym Theorem shows that μ is absolutely continuous with respect to λ, written $\mu \ll \lambda$, if and only if there is a monotonically increasing sequence of positive measures, μ_n, so that

$$0 \leq \mu_n \leq \mu, \quad \mu_n \uparrow \mu, \quad \text{and} \tag{1.1}$$

$$\mu_n \leq t_n^2 \lambda, \tag{1.2}$$

for some sequence of positive numbers, $t_n > 0$. Applying Aronszajn's Inclusion Theorem again and Lebesgue monotone convergence shows

$$\mathscr{H}^+(\mu_n) \subseteq \mathscr{H}^+(\mu), \qquad \text{and,} \qquad \left(\bigvee_{n=1}^{\infty} \mathscr{H}^+(\mu_n) \right)^{-\|\cdot\|_{\mathscr{H}^+(\mu)}} = \mathscr{H}^+(\mu), \tag{1.3}$$

by Equation (1.1). Here and throughout, $\bigvee$ is used to denote linear span. It follows from Equation (1.2) that

$$\mathscr{H}^+(\mu_n) \subseteq \mathscr{H}^+(\lambda). \tag{1.4}$$

The intersection of the spaces of μ- and λ-Cauchy transforms, viewed as a subspace of $\mathscr{H}^+(\mu)$, is

$$\cap^+(\mu,\lambda) := \mathscr{H}^+(\mu) \cap \mathscr{H}^+(\lambda).$$

Together, Equations (1.3 and 1.4) imply that $\cap^+(\mu,\lambda) \supseteq \mathscr{H}^+(\mu_n)$ for each $n \in \mathbb{N}$, and thus $\cap^+(\mu,\lambda)$ is norm dense in $\mathscr{H}^+(\mu)$.

These observations motivate the following definitions of domination and absolute continuity in the reproducing kernel sense: *μ is dominated by λ in the reproducing kernel (RK) sense*, written $\mu \leq_{RK} t^2\lambda$, for some $t > 0$, if $\mathscr{H}^+(\mu) \subseteq \mathscr{H}^+(\lambda)$, and the norm of the linear embedding is at most t; and μ is *reproducing kernel absolutely continuous with respect to* λ, written $\mu \ll_{RK} \lambda$, if the intersection space, $\cap^+(\mu,\lambda)$, is norm-dense in $\mathscr{H}^+(\mu)$. The argument of the preceding paragraph then shows that $\mu \ll \lambda$ implies $\mu \ll_{RK} \lambda$, that is, absolute continuity of measures implies absolute continuity in the reproducing kernel sense.

These ideas were first developed in [BMN24], where it was proven that domination in the reproducing kernel and classical senses are equivalent [BMN24, Theorem 4.1]. Theorem 4.12 of [BMN24] then claims that absolute continuity in the classical and reproducing kernel senses are equivalent. While this claim is true, the first author has discovered what is, unfortunately, a significant gap in the proof of this result. Namely, in the proof of [BMN24, Theorem 4.12], one finds contractive "co-embeddings," $\hat{E} : L^2(\mu+\lambda) \to L^2(\lambda)$ determined by $p + \overline{q} \overset{\hat{E}}{\longmapsto} p + \overline{q}$, for all $p, q \in \mathbb{C}[\zeta]$, as well as $E : H^2(\mu+\lambda) \to H^2(\mu)$ determined by $p \overset{E}{\longmapsto} p$ for all $p \in \mathbb{C}[\zeta]$. It follows from the assumption $\mu \ll_{RK} \lambda$ that E is injective, and it is stated in the proof that $\hat{E}$ is then injective as well. While this is true, it is not obvious. Indeed, by [BMN24, Lemma 4.7] or [Sim78, Section 1 and Remark 3, p. 381], injectivity of $\hat{E}$ is equivalent to absolute continuity of μ with respect to λ. The main goal of the present article is to bridge the aforementioned gap by establishing that absolute continuity in the reproducing kernel sense implies classical absolute continuity.

After developing preliminary material in Sections 2 and 3, we prove in Section 4 that $\mu \ll_{RK} \lambda$ implies $\mu \ll \lambda$, using reproducing kernel methods and Herglotz spaces of "extended" Cauchy transforms of holomorphic functions in $\mathbb{C} \setminus \partial\mathbb{D}$, in the spirit of [BMN24]. Spaces of extended Cauchy transforms of positive measures on the circle, or on the real line, have been considered previously, for example, in [GMR16, AMR13]. Our methods and intermediate results on spaces of extended Cauchy transforms may also be of independent interest, for example, in the extension of Lebesgue decomposition theory to $*$-representations of the Cuntz–Toeplitz C^*-algebra, that is, to the further development of the "non-commutative measure theory" initiated in [JM22a, JM22b, JMT23].

2 Background

2.1 Reproducing kernel Hilbert spaces

A RKHS on a set, X, is a Hilbert space, $\mathcal{H}$, of functions on X, so that the linear functional of point evaluation at any point $x \in X$, $\ell_x(f) = f(x)$, $f \in \mathcal{H}$, is bounded. By the Riesz lemma, for each $x \in X$, there is a unique *point evaluation* or *kernel vector*, $k_x \in \mathcal{H}$, so that $\langle k_x, f \rangle_{\mathcal{H}} = \ell_x(f) = f(x)$ for all $f \in \mathcal{H}$. Here, we remind the reader that all quadratic forms and inner products in this article are conjugate linear in their first argument and linear in their second argument. From the kernel vectors, one can then obtain the *reproducing kernel* of $\mathcal{H}$ as the function $k : X \times X \to \mathbb{C}$ given by

$$k(x, y) := \langle k_x, k_y \rangle_{\mathcal{H}}, \qquad x, y \in X.$$

Any reproducing kernel is a *positive kernel function* on X, that is, given any finite subset, $\{x_1, \ldots, x_n\} \subseteq X$, the $n \times n$ matrix,

$$[k(x_i, x_j)]_{1 \le i, j \le n},$$

is positive semi-definite. By a classical theorem of Aronszajn and Moore, there is a bijection between positive kernel functions on a set X and RKHS on X. Namely, any reproducing kernel is a positive kernel function on X, and conversely, given a positive kernel function, $k : X \times X \to \mathbb{C}$, one can construct a RKHS on X with reproducing kernel k. This bijective correspondence motivates the notation, $\mathcal{H} = \mathcal{H}(k)$, if $\mathcal{H}$ is an RKHS on X with reproducing kernel k. We will also make frequent use of the following theorem of Aronszajn on bounded inclusions of RKHS [Aro50, Section 7] [PR16, Theorem 5.1].

Aronszajn's Inclusion Theorem *Let k, K be positive kernel functions on a set X. Then $\mathcal{H}(k) \subseteq \mathcal{H}(K)$ and the norm of the embedding is at most $t > 0$ if and only if $k \leq t^2 K$.*

Here and after, $k \leq t^2 K$ indicates that $t^2 K - k$ is a positive kernel function on X. This defines a natural partial order on positive kernel functions on X.

2.2 Closed and closeable operators

Standard references for closed or closeable linear maps are [RS80, Chapter VIII] and [AG93, Chapter IV]. Let A be a linear transformation from a (not necessarily closed) linear subspace $\mathrm{Dom}\, A \subseteq \mathcal{H}$ into a Hilbert space $\mathcal{J}$; more briefly, $A : \mathrm{Dom}\, A \subseteq \mathcal{H} \to \mathcal{J}$ is linear. Recall that A is called *closed* if its graph,

$$G(A) := \{(x, Ax) |\, x \in \mathrm{Dom}\, A\},$$

is closed as a subspace of $\mathcal{H} \oplus \mathcal{J}$, and A is *closeable* if it has a closed extension. Equivalently, A is closed if and only if, whenever there exists a sequence (x_n) in $\mathrm{Dom}\, A$ so that $x_n \to x \in \mathcal{H}$ and $Ax_n \to y \in \mathcal{J}$, it follows that $x \in \mathrm{Dom}\, A$ and $Ax = y$. Similarly, A is closeable if and only if, whenever (x_n) in $\mathrm{Dom}\, A$ obeys $x_n \to 0$ and $Ax_n \to y$, then $y = 0$. This ensures that the closure of $G(A)$ is the graph of a closed operator. If A is closeable, it has a minimal closed extension, $\overline{A}$, which can be constructed by taking the closure of $G(A)$. That is, $G(\overline{A}) = G(A)^{-\|\cdot\|_{\mathcal{H}\oplus\mathcal{J}}}$.

If a linear map $A : \mathrm{Dom}\, A \subseteq \mathcal{H} \to \mathcal{J}$ is densely defined, that is, $\mathrm{Dom}\, A^{-\|\cdot\|_{\mathcal{H}}} = \mathcal{H}$, then it has a closed Hilbert space adjoint A^*. One first defines

$$\mathrm{Dom}\, A^* := \{y \in \mathcal{J} |\, \exists\, y_* \in \mathcal{H} \text{ so that } \langle Ax, y\rangle_{\mathcal{J}} = \langle x, y_*\rangle_{\mathcal{J}}\ \forall\, x \in \mathrm{Dom}\, A\},$$

and then one defines $A^* : \mathrm{Dom}\, A^* \subseteq \mathcal{J} \to \mathcal{H}$ by $A^* y := y_*$. Here, the density of $\mathrm{Dom}\, A$ implies that y_* is unique when it exists. It is easily checked that A^* is always a closed linear operator. However, A^* is densely defined if and only if A is closeable, in which case $\overline{A} = (A^*)^* =: A^{**}$, the biadjoint of A.

A subset $\mathscr{C}$ of the domain of a closed linear map $A : \mathrm{Dom}\, A \subseteq \mathcal{H} \to \mathcal{J}$ is a *core* for A if

$$\{(x, Ax) |\, x \in \mathscr{C}\},$$

is norm-dense in the graph of A, that is, if A is the closure of its restriction to $\mathscr{C}$. In general, we say that an operator $\hat{A}$ is an *extension* of A, or that A is a *restriction* of $\hat{A}$, if $\mathrm{Dom}\, A \subseteq \mathrm{Dom}\, \hat{A}$ and $\hat{A}|_{\mathrm{Dom}\, A} = A$; we indicate this by writing $A \subseteq \hat{A}$. If A is a closed and densely defined linear operator, then A^*A is densely defined, self-adjoint (hence closed) and positive semi-definite, and $\mathrm{Dom}\, A^*A$ is a core for A. Any closed densely defined linear operator A then has a polar decomposition, $A = V\sqrt{A^*A}$, where V is a partial isometry with $\mathrm{Ran}\,(V^*V) = \mathrm{Ran}\,\sqrt{A^*A}^{-\|\cdot\|}$, $\mathrm{Ran}\, V = \mathrm{Ran}\, A^{-\|\cdot\|}$, and $\mathrm{Dom}\,\sqrt{A^*A} = \mathrm{Dom}\, A$. Here, $\sqrt{A^*A}$ is defined through the functional calculus for the self-adjoint and positive semi-definite operator A^*A.

2.3 Positive semi-definite quadratic forms

Let $\mathcal{H}$ be a separable complex Hilbert space. A *quadratic form* on $\mathcal{H}$ is a sesquilinear function $\mathfrak{q} : \mathrm{Dom}\,\mathfrak{q} \times \mathrm{Dom}\,\mathfrak{q} \to \mathbb{C}$ with *form domain* $\mathrm{Dom}\,\mathfrak{q}$ contained in $\mathcal{H}$. We work throughout this article with quadratic forms that are densely defined, meaning for $\mathfrak{q}$ above that $\mathrm{Dom}\,\mathfrak{q}$ is dense in $\mathcal{H}$. Such a quadratic form is *positive semi-definite* if $\mathfrak{q}(x, x) \geq 0$ for all $x \in \mathrm{Dom}\,\mathfrak{q}$. Standard references for the theory of sesquilinear forms in Hilbert space include [RS80, Section VIII.6] and [Kat95, Chapter 6].

Given a densely defined and positive semi-definite quadratic form $\mathfrak{q}$ we define $\hat{\mathcal{H}}(\mathfrak{q})$ as the Hilbert space completion of $\mathrm{Dom}\,\mathfrak{q}$ with respect to the inner product

$$\langle \cdot, \cdot \rangle_{\hat{\mathcal{H}}(\mathfrak{q})} := \langle \cdot, \cdot \rangle_{\mathcal{H}} + \mathfrak{q}(\cdot, \cdot).$$

Such a positive semi-definite form $\mathfrak{q}$ is said to be *closed* if $\mathrm{Dom}\,\mathfrak{q} = \hat{\mathcal{H}}(\mathfrak{q})$, that is, if $\mathrm{Dom}\,\mathfrak{q}$ is complete with respect to this inner product. Similarly, $\mathfrak{q}$ is *closeable* if $\mathfrak{q}$ has a closed extension, in which case $\overline{\mathfrak{q}}$, the *closure of* $\mathfrak{q}$, denotes the minimal closed extension of $\mathfrak{q}$. We let $\mathfrak{j} : \mathrm{Dom}\,\mathfrak{q} \hookrightarrow \hat{\mathcal{H}}(\mathfrak{q})$ denote the canonical linear embedding of $\mathrm{Dom}\,\mathfrak{q}$ into $\hat{\mathcal{H}}(\mathfrak{q})$ and we define the contractive linear *co-embedding* $E : \hat{\mathcal{H}}(\mathfrak{q}) \to \mathcal{H}$ by

$$E\mathfrak{j}(x) := x; \qquad x \in \mathrm{Dom}\,\mathfrak{q},$$

and extending by continuity to all of $\hat{\mathcal{H}}(\mathfrak{q})$. Note that E is a linear contraction with dense range. One can readily check that $\mathfrak{q}$ being closed, $\mathfrak{j}$ being closed, and E being injective are equivalent conditions (see [Sim78, Section 1] and [Mar25, Lemma 4]).

A positive semi-definite quadratic form, $\mathfrak{q}$, is closeable if and only if, given any sequence $(x_n) \subseteq \mathrm{Dom}\,\mathfrak{q}$ so that $x_n \to 0$ in $\mathcal{H}$ and $(\mathfrak{j}(x_n))$ is Cauchy in $\hat{\mathcal{H}}(\mathfrak{q})$, then $\mathfrak{j}(x_n) \to 0$. By a result of Kato [Kat95, Chapter VI, Theorems 2.1 and 2.23], a densely defined positive semi-definite form $\mathfrak{q}$ is closed if and only if there exists a unique self-adjoint positive semi-definite linear operator T with dense domain $\mathrm{Dom}\,T \subseteq \mathcal{H}$ such that $\mathrm{Dom}\,\mathfrak{q} = \mathrm{Dom}\,\sqrt{T}$ and

$$\mathfrak{q}(x, y) = \left\langle \sqrt{T}x, \sqrt{T}y \right\rangle_{\mathcal{H}} =: \mathfrak{i}_T(x, y); \qquad \forall\ x, y \in \mathrm{Dom}\,\mathfrak{q}.$$

This can be viewed as an extension of the Riesz lemma to densely defined closed positive semi-definite forms, and in this article, we will typically refer to this result as *Kato's Unbounded Riesz Lemma*. With $\mathfrak{i}(\cdot, \cdot) = \mathfrak{i}_{I_{\mathcal{H}}}$, the *identity form*, we refer to T as the *Radon–Nikodym derivative of* $\mathfrak{q}$ *with respect to* $\mathfrak{i}$. As with linear operators, given positive semi-definite forms $\hat{\mathfrak{q}}$ and $\mathfrak{q}$, we say that $\hat{\mathfrak{q}}$ is an *extension* of $\mathfrak{q}$, or that $\mathfrak{q}$ is a *restriction* of $\hat{\mathfrak{q}}$, if $\mathrm{Dom}\,\mathfrak{q} \subseteq \mathrm{Dom}\,\hat{\mathfrak{q}}$ and $\hat{\mathfrak{q}}(x, y) = \mathfrak{q}(x, y)$ for all $x, y \in \mathrm{Dom}\,\mathfrak{q}$. In this case, we write $\mathfrak{q} \subseteq \hat{\mathfrak{q}}$. A dense subset $\mathscr{D} \subseteq \mathrm{Dom}\,\mathfrak{q}$ is called a *form-core* for a closed form $\mathfrak{q}$ if $\mathfrak{j}(\mathscr{D})$ is dense in $\hat{\mathcal{H}}(\mathfrak{q})$. It is readily checked that $\mathscr{D}$ is a form-core for the closed form $\mathfrak{q} = \mathfrak{i}_T$ if and only if it is a core for $\sqrt{T}$.

Lemma 2.1 follows from the above definitions and the polar decomposition of a closed operator [Mar25, Lemma 2]. Recall here that the adjoint of a densely defined linear map is always a closed linear transformation, and thus $A^* = \overline{A}^*$.

Lemma 2.1 *Let* $A : \mathrm{Dom}\, A \subseteq \mathcal{H} \to \mathcal{H}$ *be a densely defined linear operator. Then the positive semi-definite quadratic form,* $\mathfrak{q} : \mathrm{Dom}\, A \times \mathrm{Dom}\, A \to \mathbb{C}$,

$$\mathfrak{q}(x, y) := \langle Ax, Ay \rangle_{\mathcal{H}}, \qquad x, y \in \mathrm{Dom}\, A,$$

is closeable or closed if and only if A is closeable or closed, respectively. If A is closeable, then $\overline{\mathfrak{q}} = \mathfrak{i}_{\overline{A}^* \overline{A}}$ *with form domain* $\mathrm{Dom}\sqrt{\overline{A}^*\overline{A}}$*, and* $\mathrm{Dom}\, A$ *is a core for* $\overline{A}$ *and a form-core for* $\mathfrak{i}_{\overline{A}^*\overline{A}}$*. If A is closed, then* $\mathfrak{q} = \mathfrak{i}_{A^*A}$ *is closed with form domain* $\mathrm{Dom}\sqrt{A^*A}$ *and*

$$\mathfrak{i}_{A^*A}(x, y) = \left\langle \sqrt{A^*A}x, \sqrt{A^*A}y \right\rangle_{\mathcal{H}} = \langle Ax, Ay \rangle_{\mathcal{H}}; \qquad x, y \in \mathrm{Dom}\sqrt{A^*A} = \mathrm{Dom}\, A.$$

2.4 Unbounded embeddings and multipliers

It will be useful to consider multipliers between RKHS that are (necessarily) closed but not necessarily bounded. Given two RKHS $\mathcal{H}(k)$ and $\mathcal{H}(K)$ on a set X, let $\mathscr{D} = \mathcal{H}(k) \cap \mathcal{H}(K)$, the set of functions on X that are in both $\mathcal{H}(k)$ and $\mathcal{H}(K)$. Viewing $\mathscr{D}$ as a subspace of $\mathcal{H}(k)$, define the *embedding*, $\mathrm{e} : \mathscr{D} \hookrightarrow \mathcal{H}(K)$, by setting $\mathrm{e}(f) = f$ for each $f \in \mathscr{D}$. It is readily verified that e is closed, by definition. When $\mathscr{D}$ is dense, then we can think of e as being a densely defined multiplication operator from $\mathcal{H}(k)$ into $\mathcal{H}(K)$, the symbol of the multiplication operator in this case being the function identically equal to 1.

More generally, a function h on X is a *densely defined multiplier of* $\mathcal{H}(k)$ *into* $\mathcal{H}(K)$ if

$$\mathscr{D}_{\max}(h) := \{f \in \mathcal{H}(k) |\, h \cdot f \in \mathcal{H}(K)\}$$

is dense in $\mathcal{H}(k)$. This is, clearly, the maximal domain in $\mathcal{H}(k)$ on which the linear map $M_h^{K,k}$ of multiplication by h from $\mathcal{H}(k)$ into $\mathcal{H}(K)$ can be defined. Explicitly, $(M_h^{K,k} f)(x) = h(x)f(x)$ for all $x \in X, f \in \mathscr{D}_{\max}(h)$. It is elementary to check that densely defined multipliers between RKHS are always closed on their maximal domains [BMN24, Proposition 2.1].

Proposition 2.2 *Let k and K be positive kernel functions on X, and let h be a function on X. Suppose* $\mathscr{X}$ *is a dense subspace of* $\mathcal{H}(k)$*. The operator* $A : \mathscr{X} \to \mathcal{H}(K)$ *given by* $(Af)(x) = h(x)f(x)$*,* $f \in \mathscr{X}$ *and* $x \in X$*, is then closeable. The maximal closed extension of A is* $M_h^{K,k}$, $\mathrm{Dom}\, M_h^{K,k} = \mathscr{D}_{\max}(h)$*. Moreover,* $(M_h^{K,k})^* K_x = \overline{h(x)} k_x$ *for all* $x \in X$*, and* $\bigvee_{x \in X} K_x$ *is a core for* $(M_h^{K,k})^*$*.*

In particular, if e is a densely defined embedding with dense domain $\mathcal{H}(k) \cap \mathcal{H}(K)$ in $\mathcal{H}(k)$, as defined at the start of this subsection, we have $\mathrm{e} = M_1^{K,k}$ and $\mathrm{e}^* K_x = k_x$ for all $x \in X$.

3 Analytic Cauchy transforms of H^2 spaces

In this section, we describe Cauchy transforms of H^2 spaces and provide an operator–theoretic characterization of $\ll_{RK}$ in Proposition 3.3. Given a positive Radon measure μ on $\partial\mathbb{D}$, we define the Herglotz space of analytic μ-Cauchy transforms to be

$\mathscr{H}^+(\mu) := \mathscr{C}_\mu H^2(\mu)$. As in the introduction, given any $h \in H^2(\mu)$, we define h_μ^+ in $\mathscr{O}(\mathbb{D})$, the holomorphic functions on $\mathbb{D}$, by

$$h_\mu^+(z) := (\mathscr{C}_\mu h)(z) = \int_{\partial\mathbb{D}} \frac{h}{1-\overline{\zeta}z}\, d\mu$$
$$= \langle k_z, h\rangle_{L^2(\mu)},$$

where $k_z(\zeta) := \frac{1}{1-\overline{z}\zeta}$ is a *Szegö kernel vector* at the point $z \in \mathbb{D}$. We refer the reader to [CMR06] for more analytic details on the Cauchy Transform. More generally, we also require Szegö kernel vectors, k_z, for $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. The following lemma is readily established and we omit the proof. In the statement below, let $\|\cdot\|_\infty$ denote the supremum norm for functions on $\partial\mathbb{D}$, and let $\mathrm{C}(\partial\mathbb{D})$ denote the unital and commutative C^*-algebra of continuous functions on $\partial\mathbb{D}$. Recall that $\bigvee$ is used in this article to indicate linear span.

Lemma 3.1 *The following equalities hold:*

$$\mathbb{C}[\zeta]^{-\|\cdot\|_\infty} = \left(\bigvee_{z\in\mathbb{D}} k_z\right)^{-\|\cdot\|_\infty}, \qquad \left(\bigvee_{n=1}^{\infty} \zeta^{-n}\right)^{-\|\cdot\|_\infty} = \left(\bigvee_{z\in\mathbb{C}\backslash\overline{\mathbb{D}}} k_z\right)^{-\|\cdot\|_\infty},$$

$$\textit{and} \quad \mathrm{C}(\partial\mathbb{D}) = \left(\bigvee_{z\in\mathbb{C}\backslash\partial\mathbb{D}} k_z\right)^{-\|\cdot\|_\infty}.$$

Remark 3.2 It follows from Lemma 3.1 that if μ is any positive Radon measure on $\partial\mathbb{D}$, then

$$H^2(\mu) = \mathbb{C}[\zeta]^{-\|\cdot\|_{L^2(\mu)}} = \left(\bigvee_{z\in\mathbb{D}} k_z\right)^{-\|\cdot\|_{L^2(\mu)}}.$$

$$\text{Similarly,} \quad \overline{H_0^2(\mu)} := \left(\bigvee_{n=1}^{\infty} \zeta^{-n}\right)^{-\|\cdot\|_{L^2(\mu)}} = \left(\bigvee_{z\in\mathbb{C}\backslash\overline{\mathbb{D}}} k_z\right)^{-\|\cdot\|_{L^2(\mu)}},$$

$$\text{and hence,} \quad L^2(\mu) = \left(\bigvee_{z\in\mathbb{C}\backslash\partial\mathbb{D}} k_z\right)^{-\|\cdot\|_{L^2(\mu)}}.$$

By construction of $\mathscr{H}^+(\mu)$, the analytic Cauchy transform $\mathscr{C}_\mu$ is a surjective isometry from $H^2(\mu)$ onto $\mathscr{H}^+(\mu)$. As described in the introduction, the point evaluation vector k_z^μ at $z \in \mathbb{D}$ in $\mathscr{H}^+(\mu)$ is the μ-Cauchy transform of the Szegö kernel vector $k_z = (1-\overline{z}\zeta)^{-1}$, so that

$$h_\mu^+(z) = \int_{\partial\mathbb{D}} \overline{k_z(\zeta)} h\, d\mu$$
$$= \sum_{n=0}^{\infty} \underbrace{\left(\int_{\partial\mathbb{D}} \zeta^{-n} h\, d\mu\right)}_{=\hat{h}_n} z^n$$
$$= \sum_{n=0}^{\infty} \hat{h}_n z^n,$$

is the Taylor series expansion of $h_\mu^+ = \mathscr{C}_\mu h$ at 0. It further follows that the *coefficient evaluation vector*, $\hat{k}_n^\mu := \mathscr{C}_\mu \zeta^n$, obeys

$$\left\langle \hat{k}_n^\mu, h_\mu^+ \right\rangle_{\mathscr{H}^+(\mu)} = \langle \zeta^n, h \rangle_{H^2(\mu)} = \hat{h}_n,$$

for all $n \in \mathbb{N} \cup \{0\}$. The *Herglotz–Riesz transform* of μ is

$$H_\mu(z) := \int_{\partial\mathbb{D}} \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} d\mu.$$

A Herglotz function is a holomorphic function in the complex unit disk with positive semi-definite real part. By the Herglotz Representation Theorem, every Herglotz function is, up to an imaginary constant, the Herglotz–Riesz transform of a uniquely-determined positive Radon measure on $\partial\mathbb{D}$. The reproducing kernel for $\mathscr{H}^+(\mu)$ can be expressed in terms of H_μ,

$$k^\mu(z, w) = \int_{\partial\mathbb{D}} \overline{\frac{1}{1 - \overline{z}\zeta}} \frac{1}{1 - \overline{w}\zeta} d\mu = \frac{1}{2} \frac{H_\mu(z) + \overline{H_\mu(w)}}{1 - z\overline{w}}.$$

Let μ and λ be positive Radon measures on $\partial\mathbb{D}$. Set

$$\cap^+(\mu, \lambda) := \mathscr{H}^+(\mu) \cap \mathscr{H}^+(\lambda)$$

and view it as a subspace of $\mathscr{H}^+(\mu)$. The embedding $\mathrm{e} : \cap^+(\mu, \lambda) \hookrightarrow \mathscr{H}^+(\lambda)$, given by $\mathrm{e}(f) = f$ for each $f \in \cap^+(\mu, \lambda)$, is automatically closed by Proposition 2.2. Should we need to emphasize the role of μ and λ, we will write $\mathrm{e}_{\lambda,\mu}$ in place of e. Suppose that $\mu \ll_{RK} \lambda$, so that e is a densely defined. Since e is closed, its adjoint $\mathrm{e}^* : \mathrm{Dom}\, \mathrm{e}^* \to \mathscr{H}^+(\mu)$ is densely defined and closed. Moreover, the linear span of point evaluation vectors, $\bigvee_{z \in \mathbb{D}} k_z^\lambda$, is a core for e^* and $\mathrm{e}^* k_z^\lambda = k_z^\mu$. Since Taylor coefficient evaluations are bounded on both $\mathscr{H}^+(\lambda)$ and $\mathscr{H}^+(\mu)$, and e acts as multiplication by 1, it further follows that $\bigvee_{n=0}^\infty \hat{k}_n^\lambda$ is a core for e^*, and $\mathrm{e}^* \hat{k}_n^\lambda = \hat{k}_n^\mu$ (see [BMN24, Remark 2.2]). Hence, if we define the closed linear map,

$$E := \mathscr{C}_\mu^* \mathrm{e}^* \mathscr{C}_\lambda$$

from $\mathscr{C}_\lambda^* \mathrm{Dom}\, \mathrm{e}^* \subseteq H^2(\lambda)$ into $H^2(\mu)$, then $\mathrm{Dom}\, E = \mathscr{C}_\lambda^* \mathrm{Dom}\, \mathrm{e}^*$ is dense in $H^2(\lambda)$, and both

$$\bigvee_{z \in \mathbb{D}} k_z = \bigvee_{z \in \mathbb{D}} \mathscr{C}_\lambda^* k_z^\lambda, \quad \text{and} \quad \mathbb{C}[\zeta] = \bigvee_{n=0}^\infty \zeta^n = \bigvee_{n=0}^\infty \mathscr{C}_\lambda^* \hat{k}_n^\lambda,$$

are cores for E. Moreover, for any $z \in \mathbb{D}$, the Szegö kernel k_z is of class $H^2(\lambda)$, and

$$\begin{aligned} E k_z &= \mathscr{C}_\mu^* \mathrm{e}^* \mathscr{C}_\lambda k_z \\ &= \mathscr{C}_\mu^* \mathrm{e}^* k_z^\lambda \\ &= \mathscr{C}_\mu^* k_z^\mu = k_z. \end{aligned}$$

Similarly, for any $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} E\zeta^n &= \mathscr{C}_\mu^* \mathrm{e}^* \mathscr{C}_\lambda \zeta^n \\ &= \mathscr{C}_\mu^* \mathrm{e}^* \hat{k}_n^\lambda \\ &= \mathscr{C}_\mu^* \hat{k}_n^\mu = \zeta^n \in H^2(\mu). \end{aligned}$$

We call the E the *co-embedding* into $H^2(\mu)$. Should we need to emphasize the role of μ and λ, we write $E_{\mu,\lambda}$ for E.

Proposition 3.3 *Given positive Radon measures μ and λ on $\partial\mathbb{D}$, the following conditions are equivalent:*

(i) $\mu \ll_{RK} \lambda$.
(ii) $\mathrm{e}_{\lambda,\mu}$ *is densely defined.*
(iii) *If $\mathscr{K}_\mathbb{D}$ denotes the linear span of the Szegö kernel vectors, k_z, for $z \in \mathbb{D}$, then there exists a well-defined and closeable linear map $B : \mathscr{K}_\mathbb{D} \subseteq H^2(\lambda) \to H^2(\mu)$ satisfying $Bk_z = k_z \in H^2(\mu)$ for every $z \in \mathbb{D}$.*

If the above conditions hold, then the linear spans $\mathcal{K}_\mathbb{D} = \bigvee_{z\in\mathbb{D}} k_z^\lambda$ and $\bigvee_{n=0}^\infty \hat{k}_n^\lambda$ are cores for $\mathrm{e}_{\lambda,\mu}^*$, $\overline{B} = E_{\mu,\lambda}$, and $\mathrm{e}_{\lambda,\mu} = \mathscr{C}_\lambda E_{\mu,\lambda}^* \mathscr{C}_\mu^*$.

Proof Equivalence of items (i) and (ii) is evident from the definitions above. We have also shown above that $\mu \ll_{RK} \lambda$ implies that $E = E_{\mu,\lambda}$ exists as a densely defined closed linear map with $\bigvee_{z\in\mathbb{D}} k_z \subseteq H^2(\lambda)$ and $\bigvee_{n=0}^\infty \zeta^n$ both being cores for E so that (i) and (ii) both imply (iii) with $B := E|_{\mathscr{K}_\mathbb{D}}$.

Conversely, suppose (iii) holds so that such a closeable B exists and set $\mathrm{e} = \mathrm{e}_{\lambda,\mu}$. Because $\mathrm{Dom}\, B = \mathscr{K}_\mathbb{D} = \bigvee_{z\in\mathbb{D}} k_z$, we see that B is densely defined as well. Thus, $\mathrm{b} := \mathscr{C}_\lambda B^* \mathscr{C}_\mu^*$ is a closed and densely defined linear operator. For any $z \in \mathbb{D}$ and $f \in \mathrm{Dom}\,\mathrm{b}$,

$$\begin{aligned} (\mathrm{b}f)(z) &= \left\langle k_z^\lambda, \mathrm{b}f \right\rangle_{\mathscr{H}^+(\lambda)} \\ &= \left\langle \mathrm{b}^* k_z^\lambda, f \right\rangle_{\mathscr{H}^+(\mu)} \\ &= \left\langle k_z^\mu, f \right\rangle_{\mathscr{H}^+(\mu)} = f(z). \end{aligned}$$

This proves that $f \in \cap^+(\mu,\lambda) = \mathrm{Dom}\,\mathrm{e}$ and $\mathrm{b} \subseteq \mathrm{e}$. In particular, $\mathrm{Dom}\,\mathrm{b} \subseteq \mathrm{Dom}\,\mathrm{e}$, from which we see that e is densely defined. The containment $\mathrm{b} \subseteq \mathrm{e}$ also implies that $\mathrm{e}^* \subseteq \mathrm{b}^*$. However,

$$\mathrm{b}^* = \left(\mathscr{C}_\lambda B^* \mathscr{C}_\mu^*\right)^* = \mathscr{C}_\mu B^{**} \mathscr{C}_\lambda^*,$$

where $B^{**} = \overline{B}$ so that $\mathrm{Dom}\, B = \mathscr{K}_\mathbb{D}$ is a core for $\overline{B}$ and hence $\mathscr{K}_\mathbb{D}^\lambda := \bigvee_{z\in\mathbb{D}} k_z^\lambda = \mathscr{C}_\lambda \mathscr{K}_\mathbb{D}$ is a core for b^*. By Proposition 2.2, $\mathscr{K}_\mathbb{D}^\lambda$ is also a core for e^*. Since both $\mathrm{e}^*, \mathrm{b}^*$ are necessarily closed, and

$$\mathrm{b}^* k_z^\lambda = k_z^\mu = \mathrm{e}^* k_z^\lambda$$

for all $z \in \mathbb{D}$, we obtain that

$$\mathrm{b}^* = \overline{\mathrm{b}^*|_{\mathscr{K}_\mathbb{D}^\lambda}} \subseteq \overline{\mathrm{e}^*|_{\mathscr{K}_\mathbb{D}^\lambda}} = \mathrm{e}^*.$$

We conclude that $\mathrm{b}^* \subseteq \mathrm{e}^*$ and hence $\mathrm{b}^* = \mathrm{e}^*$. This further implies, immediately, that $E = \mathscr{C}_\mu^* \mathrm{e}^* \mathscr{C}_\lambda = \overline{B}$ and that $\mathrm{e} = \mathscr{C}_\lambda B^* \mathscr{C}_\mu^*$. ■

4 Extended Cauchy transforms of L^2 spaces

Let μ be a positive Radon measure on $\partial\mathbb{D}$. Given any $f \in L^2(\mu)$ and $z \in \mathbb{C}\backslash\partial\mathbb{D}$, we define the *extended Cauchy transform* of f as

$$f^\mu(z) := (\hat{\mathscr{C}}_\mu f)(z) := \int_{\partial\mathbb{D}} \frac{1}{1 - \overline{\zeta} z} f \, d\mu.$$

It is readily checked that $f^\mu \in \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D})$, and that $\hat{\mathscr{C}}_\mu$ is injective and linear. By Lemma 3.1, $\bigvee_{z \in \mathbb{C}\backslash\partial\mathbb{D}} \frac{1}{1-\overline{\zeta}z}$ is supremum-norm dense in $\mathrm{C}(\partial\mathbb{D})$, and thus it is norm-dense in $L^2(\mu)$. Equip $\mathscr{H}(\mu) := \hat{\mathscr{C}}_\mu L^2(\mu)$ with the inner product

$$\langle f^\mu, g^\mu \rangle_{\mathscr{H}(\mu)} := \langle f, g \rangle_{L^2(\mu)}$$

for each $f, g \in L^2(\mu)$. With this inner product, $\mathscr{H}(\mu)$ is an RKHS on $\mathbb{C}\backslash\partial\mathbb{D}$, with point evaluation vectors given by

$$K_z^\mu := \hat{\mathscr{C}}_\mu k_z, \quad z \in \mathbb{C}\backslash\partial\mathbb{D},$$

and the reproducing kernel by

$$K^\mu(z, w) := \langle k_z, k_w \rangle_{L^2(\mu)} = \int_{\partial\mathbb{D}} \frac{1}{1 - \overline{\zeta} z} \frac{1}{1 - \zeta \overline{w}} d\mu.$$

With this added Hilbert space structure, the extended Cauchy transform, $\hat{\mathscr{C}}_\mu$, is an isometry from $L^2(\mu)$ onto $\mathscr{H}(\mu)$. The restriction of K^μ to $\mathbb{D} \times \mathbb{D}$ is $k^\mu(z, w)$, the reproducing kernel for $\mathscr{H}^+(\mu)$. We can also extend the Herglotz function of μ to $\mathbb{C}\backslash\partial\mathbb{D}$ via

$$H_\mu(z) := \int_{\partial\mathbb{D}} \frac{1 + \overline{\zeta} z}{1 - \overline{\zeta} z} d\mu, \qquad z \in \mathbb{C}\backslash\partial\mathbb{D},$$

so that

$$H_\mu(z^{-1}) = \int_{\partial\mathbb{D}} \frac{z\zeta + 1}{z\zeta - 1} d\mu = -\overline{H_\mu(\overline{z})}.$$

It is also easy to check that $\mathrm{Re}\, H_\mu(z) \geq 0$ for all $z \in \mathbb{C}\backslash\partial\mathbb{D}$. With this definition, we have

$$K^\mu(z, w) = \frac{1}{2} \frac{H_\mu(z) + \overline{H_\mu(w)}}{1 - z\overline{w}}$$

whenever $z, w \in \mathbb{C}\backslash\partial\mathbb{D}$ with $z \neq \overline{w}^{-1}$. We remark that the case of $z^{-1} = \overline{w}$ is handled with a limit:

$$K^\mu(z, w) = \lim_{\varepsilon \to 0} K^\mu(z + \varepsilon, w) = \frac{z}{2} \lim_{\varepsilon \to 0} \frac{H_\mu(z + \varepsilon) - H_\mu(z)}{z - (z + \varepsilon)} = -\frac{z}{2} H_\mu'(z).$$

We now examine the structure of the elements of $\mathscr{H}(\mu)$. Set

$$\hat{K}_j^\mu = \begin{cases} \hat{\mathscr{C}}_\mu(\zeta^j), & \text{if } j \geq 0, \\ -\hat{\mathscr{C}}_\mu(\zeta^j), & \text{if } j < 0, \end{cases}$$

for each integer j. For $|z| < 1$, we have

$$K_z^\mu = \sum_{j=0}^{\infty} \overline{z}^j \hat{\mathscr{C}}_\mu(\zeta^j) = \sum_{j=0}^{\infty} \overline{z}^j \hat{K}_j^\mu .$$

For $|z| > 1$, using the fact that $k_z = -\frac{1/(\zeta\overline{z})}{1-1/(\zeta\overline{z})}$, we find

$$K_z^\mu = \sum_{j=1}^{\infty} (1/\overline{z})^j \cdot (-\hat{\mathscr{C}}_\mu(\zeta^{-j})) = \sum_{j=1}^{\infty} \overline{z}^{-j} \hat{K}_{-j}^\mu .$$

For all $f \in L^2(\mu)$ and any integer $j \in \mathbb{Z}$, set

$$\hat{f}_j^\mu = \langle \hat{K}_j^\mu, f^\mu \rangle_{\mathscr{H}(\mu)} .$$

Note that $\hat{f}_j^\mu = \int_{\partial\mathbb{D}} \overline{\zeta}^j f(\zeta)\mu(d\zeta)$ when $j \geq 0$ and $= -\int_{\partial\mathbb{D}} \overline{\zeta}^j f(\zeta)\mu(d\zeta)$ when $j < 0$. For $|z| < 1$,

$$f^\mu(z) = \left\langle K_z^\mu, f \right\rangle_{\mathscr{H}(\mu)} = \sum_{j=0}^{\infty} \hat{f}_j^\mu z^j ,$$

while for $|z| > 1$,

$$f^\mu(z) = \left\langle K_z^\mu, f \right\rangle_{\mathscr{H}(\mu)} = \sum_{j=1}^{\infty} \hat{f}_{-j}^\mu z^{-j} .$$

Thus, $\hat{K}_j^\mu$ acts through the inner product on an $f \in \mathscr{H}(\mu)$ by extracting the jth coefficient of its power series expansion about 0 (when j is nonnegative) or about ∞ (when $j < 0$). We now reinterpret Proposition 2.2 in terms of these coefficient evaluation vectors.

Lemma 4.1 *Let μ, λ be positive Radon measures on $\partial\mathbb{D}$, and let h be a function on $\mathbb{C}\backslash\partial\mathbb{D}$. Assume that h is a densely defined multiplier of $\mathscr{H}(\mu)$ into $\mathscr{H}(\lambda)$, so that $M_h^{\lambda,\mu} : \mathscr{D}_{\max}(h) \subseteq \mathscr{H}(\mu) \to \mathscr{H}(\lambda)$ is densely defined and closed, on its maximal domain, $\mathscr{D}_{\max}(h) \subseteq \mathscr{H}(\mu)$. Then, $h \in \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D})$. If h is given by the series,*

$$h(z) = \begin{cases} \sum_{j=0}^{\infty} \hat{h}_j z^j, & \text{if } |z| < 1, \\ \sum_{j=1}^{\infty} \hat{h}_{-j} z^{-j}, & \text{if } |z| > 1, \end{cases} \quad \textit{then,}$$

$$(M_h^{\lambda,\mu})^* \hat{K}_n^\lambda = \begin{cases} \sum_{\ell=0}^{n} \overline{\hat{h}_{n-\ell}} \hat{K}_\ell^\mu, & \text{if } n \in \mathbb{N} \cup \{0\}, \\ \sum_{\ell=1}^{-n} \overline{\hat{h}_{\ell+n}} \hat{K}_{-\ell}^\mu, & \text{if } n \in -\mathbb{N}, \end{cases}$$

and $\bigvee_{n\in\mathbb{Z}} \hat{K}_n^\lambda$ is a core for $(M_h^{\lambda,\mu})^*$.

Proof Setting $k = K^\mu$ and $K = K^\lambda$ in Proposition 2.2 gives us that $M_h^{\lambda,\mu}$ is densely defined and closed, and thus that the adjoint of $M_h^{\lambda,\mu}$ is a closed and densely defined linear operator. Next, we note that $K_z^\mu(z) = \int_{\partial\mathbb{D}} |1 - \overline{z}\zeta|^{-2} d\mu > 0$ for all $z \in \mathbb{C}\backslash\mathbb{D}$, and thus $K_z^\mu \neq 0$ for any $z \in \mathbb{C}\backslash\partial\mathbb{D}$. That is, there is no $w \in \mathbb{C}\backslash\partial\mathbb{D}$ so that $h(w) = 0$ for all $h \in \mathscr{H}(\mu)$. Let $\mathscr{D} := \mathrm{Dom}\, M_h^{\lambda,\mu} = \mathscr{D}_{\max}(h)$, a dense linear subspace of $\mathscr{H}(\mu)$. Since $\mathscr{D}$ is dense in $\mathscr{H}(\mu) \subseteq \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D})$ and point evaluation is norm continuous, it follows that for any $w \in \mathbb{C}\backslash\partial\mathbb{D}$, we can also find an $f \in \mathscr{D}$ so that $f(z) \neq 0$ in some open neighborhood of w in $\mathbb{C}\backslash\partial\mathbb{D}$. For any point $w \in \mathbb{C}\backslash\partial\mathbb{D}$, select such an $f \in \mathcal{D}$ so that $\frac{1}{f}$ is holomorphic in an open neighborhood of w. Hence, since $hf \in \mathscr{H}(\lambda) \subseteq \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D})$, and $\frac{1}{f}$ is also holomorphic in an open neighborhood of w, it follows that $h = (hf)/f$ is holomorphic in an open neighborhood of w in $\mathbb{C}\backslash\partial\mathbb{D}$. As w is arbitrary, we conclude that $h \in \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D})$.

Write $h(z) = \sum_{j=0}^\infty \hat{h}_j z^j$ for $|z| < 1$ and $h(z) = \sum_{j=0}^\infty \hat{h}_{-j} z^{-j}$ for $|z| > 1$. From Proposition 2.2, $(M_h^{\lambda,\mu})^* K_z^\lambda = \overline{h(z)} K_z^\mu$. Let $f \in \mathscr{D}$, and note then that

$$\left\langle K_z^\lambda, hf \right\rangle_{\mathscr{H}(\lambda)} = h(z) \left\langle K_z^\mu, f \right\rangle_{\mathscr{H}(\mu)} . \tag{4.1}$$

Assuming $|z| < 1$ in (4.1), we find

$$\sum_{n=0}^\infty \left\langle \hat{K}_n^\lambda, hf \right\rangle z^n = \sum_{n=0}^\infty \left(\sum_{\ell=0}^n \hat{h}_{n-\ell} \left\langle \hat{K}_\ell^\mu, f \right\rangle \right) z^n .$$

Matching terms and using the fact that $\mathscr{D}$ is dense gives us

$$(M_h^{\lambda,\mu})^* \hat{K}_n^\lambda = \sum_{\ell=0}^n \overline{\hat{h}_{n-\ell}} \hat{K}_\ell^\mu$$

for $n \in \mathbb{N} \cup \{0\}$. Assuming now that $|z| > 1$ in (4.1), we have

$$\sum_{n=1}^\infty \left\langle \hat{K}_{-n}^\lambda, hf \right\rangle z^{-n} = \sum_{n=0}^\infty \sum_{m=0}^\infty \hat{f}_{-m-1}^\mu \hat{h}_{-n} z^{-(n+m+1)} = \sum_{n=1}^\infty \left(\sum_{\ell=0}^{n-1} \hat{f}_{-\ell-1}^\mu \hat{h}_{-(n-1-\ell)} \right) z^{-n}$$

and thus

$$(M_h^{\lambda,\mu})^* \hat{K}_{-n}^\lambda = \sum_{\ell=1}^n \overline{\hat{h}_{\ell-n}} \hat{K}_{-\ell}^\mu$$

for $n \in \mathbb{N}$.

It remains to show that $\bigvee_{n\in\mathbb{Z}} \hat{K}_n^\mu$ is a core for $(M_h^{\lambda,\mu})^*$. Let A be the restriction of $(M_h^{\lambda,\mu})^*$ to $\bigvee_{n\in\mathbb{Z}} \hat{K}_n^\mu$. It follows from the preceding work that for any $g \in \mathrm{Dom}\, A^*$ and for any $|z| < 1$,

$$\left\langle K_z^\lambda, A^* g \right\rangle_{\mathscr{H}(\lambda)} = \sum_{n=0}^\infty \left\langle \hat{K}_n^\lambda, A^* g \right\rangle_{\mathscr{H}(\lambda)} z^n = \sum_{n=0}^\infty \left\langle (M_h^{\lambda,\mu})^* \hat{K}_n^\lambda, g \right\rangle_{\mathscr{H}(\mu)} z^n = h(z) \cdot g(z).$$

An analogous computation shows that $\left\langle K_z^\lambda, A^* g \right\rangle_{\mathscr{H}(\lambda)} = h(z) g(z)$ when $|z| > 1$ as well. Because $M_h^{\lambda,\mu}$ is closed and $h(z)g(z) = \left\langle (M_h^{\lambda,\mu})^* K_z^\lambda, g \right\rangle_{\mathscr{H}(\mu)}$ for any z, it follows that $g \in \operatorname{Dom} M_h^{\lambda,\mu} = \mathscr{D}$ and $A^* g = M_h^{\lambda,\mu} g$. We then have $A^* \subseteq M_h^{\lambda,\mu}$, and therefore $(M_h^{\lambda,\mu})^* \subseteq \overline{A}$. From the definition of A, we have $A \subseteq (M_h^{\lambda,\mu})^*$. Because $(M_h^{\lambda,\mu})^*$ is closed, we have $\overline{A} = (M_h^{\lambda,\mu})^*$. ∎

Remark 4.2 In the case where $h = 1$, we have $\hat{h}_0 = 1$ and $\hat{h}_j = 0$ when $j \in \mathbb{Z}\backslash\{0\}$. Thus, when

$$\cap(\mu,\lambda) := \mathscr{H}(\mu) \cap \mathscr{H}(\lambda)$$

is dense in $\mathscr{H}(\mu)$, the operator $\hat{e} = M_1^{\mu,\lambda}$ is a densely defined and closed embedding, and $\hat{e}^* \hat{K}_n^\lambda = \hat{K}_n^\mu$ for all $n \in \mathbb{Z}$.

4.1 Backward shifts

Define the linear map $\mathfrak{B} : \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D}) \to \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D})$, the *backward shift*, by

$$(\mathfrak{B}h)(z) := \begin{cases} \frac{h(z)-h(0)}{z} & z \neq 0 \\ h'(0) & z = 0 \end{cases}, \qquad h \in \mathscr{O}(\mathbb{C}\backslash\partial\mathbb{D}).$$

Here, h' is the derivative of h. Let $\hat{V}_\mu := \hat{\mathscr{C}}_\mu M_\zeta^\mu \hat{\mathscr{C}}_\mu^*$, the image of M_ζ^μ under extended Cauchy transform. Note that $\hat{V}_\mu$ is unitary.

Lemma 4.3 *For any $h \in L^2(\mu)$, $\hat{\mathscr{C}}_\mu(\overline{\zeta} \cdot h) = \mathfrak{B}\hat{\mathscr{C}}_\mu h$. That is, $\mathfrak{B}|_{\mathscr{H}(\mu)} = \hat{V}_\mu^*$.*

Proof If $z \in \mathbb{C}\backslash\{0\}$, then

$$\zeta k_z(\zeta) = \frac{\zeta}{1-\overline{z}\zeta} = \overline{z}^{-1} \cdot \frac{\overline{z}\zeta - 1 + 1}{1-\overline{z}\zeta} = \overline{z}^{-1} \cdot (k_z(\zeta) - k_0(\zeta)).$$

Thus,

$$\begin{aligned}(\hat{\mathscr{C}}_\mu \overline{\zeta} h)(z) &= \langle \zeta k_z^\mu, h \rangle_{L^2(\mu)} \\ &= \frac{1}{z} \langle k_z - k_0, h \rangle_{L^2(\mu)} \\ &= \frac{(\hat{\mathscr{C}}_\mu h)(z) - (\hat{\mathscr{C}}_\mu h)(0)}{z} = (\mathfrak{B}\hat{\mathscr{C}}_\mu h)(z).\end{aligned}$$

Since $h^\mu = \hat{\mathscr{C}}_\mu h$ is holomorphic in $\mathbb{D}$, we can then take the limit in the above as $z \to 0$ to obtain that

$$(\hat{\mathscr{C}}_\mu \overline{\zeta} h)(0) = (\hat{\mathscr{C}}_\mu h)'(0) = (\mathfrak{B}h^\mu)(0).$$

∎

4.2 Proof of the main result

For each $n \in \mathbb{N}$, set $H^2_{-n}(\lambda) := \left(\bigvee_{j=-n}^{+\infty} \zeta^j\right)^{-\|\cdot\|_{L^2(\lambda)}}$, and similarly define $H^2_{-n}(\mu)$. We also define $\mathscr{H}_{-n}(\mu) := \hat{\mathscr{C}}_\mu H^2_{-n}(\lambda) \subseteq \mathscr{H}(\mu)$, so that $\mathscr{H}_{-n}(\mu)$ is a closed Hilbert subspace of the RKHS $\mathscr{H}(\mu)$, with the restriction of the inner product, for each $n \in \mathbb{N}$. The following is a corollary of Lemma 4.3.

Corollary 4.4 *For each $n \in \mathbb{N} \cup \{0\}$, $\mathscr{H}_{-n}(\mu) = \hat{V}_{\mu}^{*n} \mathscr{H}_0(\mu) = \mathfrak{B}^n \mathscr{H}_0(\mu)$ and*

$$\mathscr{H}_{-(n+1)}(\mu) \supseteq \mathscr{H}_{-n}(\mu).$$

In the above, $\mathscr{H}_0(\mu)$ is the space of extended Cauchy transforms of $H^2(\mu)$, and can be identified, isometrically, with $\mathscr{H}^+(\mu)$ by restriction to $\mathbb{D}$.

Corollary 4.5 *If $\mu \ll_{RK} \lambda$, then $\cap(\mu, \lambda)$ is norm dense in $\mathscr{H}(\mu)$.*

Proof If $\mu \ll_{RK} \lambda$, then the intersection space of Cauchy transforms $\cap^+(\mu, \lambda)$ is norm-dense in $\mathscr{H}^+(\mu)$, by definition. Hence, identifying $\mathscr{H}^+(\mu)$ with $\mathscr{H}_0(\mu)$, we have that

$$\cap_0(\mu, \lambda) := \mathscr{H}_0(\mu) \cap \mathscr{H}_0(\lambda),$$

is norm-dense in $\mathscr{H}_0(\mu)$. By Lemma 4.3, $\mathfrak{B}$ is a unitary operator on both $\mathscr{H}(\mu)$ and on $\mathscr{H}(\lambda)$. It follows that, for any $n \in \mathbb{N}$,

$$\cap_{-n}(\mu, \lambda) := \mathscr{H}_{-n}(\mu) \cap \mathscr{H}_{-n}(\lambda) = \mathfrak{B}^n \mathscr{H}_0(\mu) \cap \mathfrak{B}^n \mathscr{H}_0(\lambda) = \mathfrak{B}^n \cap_0 (\mu, \lambda),$$

is norm-dense in $\mathscr{H}_{-n}(\mu)$. Since

$$\mathscr{H}(\mu) = \left(\bigvee_{n=0}^{\infty} \mathscr{H}_{-n}(\mu) \right)^{-\|\cdot\|_{\mathscr{H}(\mu)}},$$

it follows that

$$\cap(\mu, \lambda) = \mathscr{H}(\mu) \cap \mathscr{H}(\lambda) \supseteq \bigvee_{n=0}^{\infty} \cap_{-n}(\mu, \lambda),$$

is norm-dense in $\mathscr{H}(\mu)$. ■

We are now prepared to prove our main result.

Theorem 4.6 *$\mu \ll_{RK} \lambda$ implies that $\mu \ll \lambda$.*

Proof Since $\cap(\mu, \lambda)$ is norm-dense in $\mathscr{H}(\mu)$, the embedding

$$\hat{\mathrm{e}} : \cap(\mu, \lambda) \hookrightarrow \mathscr{H}(\lambda),$$

is densely defined. By Proposition 2.2, $\hat{\mathrm{e}}$ is closed, and therefore $\hat{\mathrm{e}}^*$ is closed and densely defined. We then define $\hat{E} : \mathrm{Dom}\, \hat{E} \subseteq L^2(\lambda) \to L^2(\mu)$ by $\hat{E} = \hat{\mathscr{C}}_{\mu}^* \hat{\mathrm{e}}^* \hat{\mathscr{C}}_{\lambda}$. This operator is also closed and densely defined. By Proposition 2.2, $\bigvee_{z \in \mathbb{C} \setminus \partial\mathbb{D}} K_z^{\lambda}$ is a core for $\hat{\mathrm{e}}^*$. Since $\hat{\mathscr{C}}_{\lambda} k_z = K_z^{\lambda}$, for any $z \in \mathbb{C} \setminus \partial\mathbb{D}$, it follows that

$$\bigvee_{z \in \mathbb{C} \setminus \partial\mathbb{D}} k_z \subseteq \mathrm{Dom}\, \hat{E},$$

is a core for $\hat{E}$. Similarly, since $\hat{\mathscr{C}}_\lambda \zeta^n = \hat{K}_n^\lambda$, and $\bigvee_{n\in\mathbb{Z}} \hat{K}_n^\lambda$ is a core for $\hat{\mathrm{e}}^*$, it follows from Lemma 4.1 that $\overline{\mathbb{C}[\zeta]} + \mathbb{C}[\zeta]$ is a core for $\hat{E}$. Moreover, for any $z \in \mathbb{C}\backslash\partial\mathbb{D}$,

$$\begin{aligned}\hat{E}k_z = \hat{\mathscr{C}}_\mu^* \hat{\mathrm{e}^*} \hat{\mathscr{C}}_\lambda k_z &= \hat{\mathscr{C}}_\mu^* \hat{\mathrm{e}}^* K_z^\lambda \\ &= \hat{\mathscr{C}}_\mu^* K_z^\mu = k_z \in L^2(\mu).\end{aligned}$$

Similarly, for any $n \in \mathbb{Z}$,

$$\hat{E} \underbrace{\zeta^n}_{\in L^2(\lambda)} = \zeta^n \in L^2(\mu).$$

Hence, $\hat{E}(g) = g$ for any $g \in \mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}$, and $\mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}$ is a core for $\hat{E}$. Because $M_\zeta^\lambda(\mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}) \subseteq \mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}$, it follows that $\hat{E}M_\zeta^\lambda \subseteq M_\zeta^\mu \hat{E}$, that is, that $M_\zeta^\lambda \operatorname{Dom} \hat{E} \subseteq \hat{E}$ and $\hat{E}M_\zeta^\lambda x = M_\zeta^\mu \hat{E}x$ for all $x \in \operatorname{Dom} \hat{E}$. Setting $\hat{T} := \hat{E}^*\hat{E}$, it follows that, for all $x, y \in \operatorname{Dom} \hat{E} = \operatorname{Dom} \mathfrak{i}_{\hat{T}}$,

$$\begin{aligned}\mathfrak{i}_{\hat{T}}(M_\zeta^\lambda x, M_\zeta^\lambda y) &= \left\langle \hat{T}^{\frac{1}{2}} x, \hat{T}^{\frac{1}{2}} y \right\rangle_{L^2(\lambda)} \\ &= \left\langle \hat{E}M_\zeta^\lambda x, \hat{E}M_\zeta^\lambda y \right\rangle_{L^2(\mu)} \\ &= \left\langle \hat{E}x, \hat{E}y \right\rangle_{L^2(\mu)} = \mathfrak{i}_{\hat{T}}(x, y).\end{aligned}$$

Hence the positive quadratic forms of $\hat{T}$ and of $M_\zeta^{\lambda;*} \hat{T} M_\zeta^\lambda$ are the same, and Kato's unbounded Riesz lemma, as described in Section 2.3, then implies that

$$\hat{T} = M_\zeta^{\lambda;*} \hat{T} M_\zeta^\lambda.$$

It follows from this that the bounded, positive operator, $(I + \hat{T})^{-1}$, commutes with the cyclic unitary M_ζ^λ and hence must act as multiplication by the function $h := (I + \hat{T})^{-1}1 \in L^\infty(\lambda)$, where $h > 0$, λ-almost everywhere. Moreover, $\hat{T}$ and hence $\hat{T}^{\frac{1}{2}} \geq 0$ must then act as multiplication by some (generally unbounded) λ-measurable functions, g^2 and g, respectively, so that $g \geq 0$, λ-almost everywhere. By polar decomposition, $\operatorname{Dom} \hat{T}^{\frac{1}{2}} = \operatorname{Dom} \hat{E} \supseteq \mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}$. As the constant function, 1, is in the domain of $\hat{T}^{\frac{1}{2}}$, it follows that $\hat{T}^{\frac{1}{2}} = M_g$ where,

$$g = \hat{T}^{\frac{1}{2}} 1 \in L^2(\lambda).$$

Hence, for any $f \in \mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}$,

$$\int_{\partial\mathbb{D}} f d\mu = \langle 1, f \rangle_{L^2(\mu)} = \left\langle \hat{E}(1), \hat{E}(f) \right\rangle_{L^2(\mu)} = \left\langle \hat{T}^{\frac{1}{2}} 1, \hat{T}^{\frac{1}{2}} f \right\rangle_{L^2(\lambda)} = \int_{\partial\mathbb{D}} f \cdot g^2 d\lambda.$$

It then follows, by Weierstrass approximation, that $\mu \ll \lambda$ with

$$\mu = g^2 \cdot \lambda, \qquad g^2 = \frac{d\mu}{d\lambda} \in L^1(\lambda). \qquad \blacksquare$$

Remark 4.7 The von Neumann algebra generated by the cyclic unitary, M_ζ^λ, can be identified with $L^\infty(\lambda)$. Since $\hat{T} = M_\zeta^{\lambda;*} \hat{T} M_\zeta^\lambda$, a straightforward argument then shows that $\hat{T}$ is *affiliated* to $L^\infty(\lambda)$, meaning that $\hat{T}$ is a closed, densely -defined operator in $L^2(\lambda)$, $M_F \text{Dom}\,\hat{T} \subseteq \text{Dom}\,\hat{T}$, where M_F denotes multiplication by F, for any $F \in L^\infty(\lambda)$, and $\hat{T} M_F f = M_F \hat{T} f$ for any $f \in \text{Dom}\,\hat{T}$. This gives an alternative way of showing that $\hat{T}^{\frac{1}{2}}$ acts as multiplication by an $L^2(\lambda)$ function. Indeed, if $\hat{T}$ is affiliated to $L^\infty(\lambda)$, so is $\hat{T}^{\frac{1}{2}}$, and since $1 \in \mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]} \subseteq \text{Dom}\,\hat{T}^{\frac{1}{2}}$, it follows that $g := \hat{T}^{\frac{1}{2}} 1 \in L^2(\lambda)$. Since $\hat{T}^{\frac{1}{2}}$ is affiliated to $L^\infty(\lambda)$, $F = M_F 1 \in \text{Dom}\,\hat{T}^{\frac{1}{2}}$ for any $F \in L^\infty(\lambda)$ and

$$\hat{T}^{\frac{1}{2}} F = \hat{T}^{\frac{1}{2}} M_F 1 = M_F \hat{T}^{\frac{1}{2}} 1 = M_F g = M_g F.$$

One can further show that M_g is densely defined and closed on its maximal domain in $L^2(\lambda)$ for any $g \in L^2(\lambda)$ and that $\mathbb{C}[\zeta] + \overline{\mathbb{C}[\zeta]}$ is a core for M_g, so that $M_g = \hat{T}^{\frac{1}{2}}$.

References

[AG93] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Dover Publications, New York, NY, 1993.

[AMR13] A. Aleman, R. T. W. Martin, and W. T. Ross, *On a theorem of Livsic*. J. Funct. Anal. **264**(2013), 999–1048.

[Aro50] N. Aronszajn, *Theory of reproducing kernels*. Trans. Am. Math. Soc. **68**(1950), 337–404.

[BMN24] J. Bal, R. T. W. Martin, and F. Naderi, *A reproducing kernel approach to Lebesgue decomposition*. Can. J. Math. 77(2025), 1570–1610.

[CMR06] J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy transform*, Vol. 125, American Mathematical Society, Providence, RI, 2006.

[GMR16] S. R. Garcia, R. T. W. Martin, and W. T. Ross, *Partial orders on partial isometries*. J. Oper. Theory 75(2016), 409–442.

[JM22a] M. T. Jury and R. T. W. Martin, *Fatou's theorem for non-commutative measures*. Adv. Math. **400**(2022), 108293.

[JM22b] M. T. Jury and R. T. W. Martin, *Lebesgue decomposition of non-commutative measures*. Int. Math. Res. Not. **2022**(2022), 2968–3030.

[JMT23] M. T. Jury, R. T. W. Martin, and E. J. Timko, *An F.&M. Riesz theorem for non-commutative measures*. J. Oper. Theory **90**(2023), 569–604.

[Kat95] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1995.

[Mar25] R. T. W. Martin, *On log-integrability of Radon–Nikodym derivatives*. In: D. Alpay and J. Mashreghi (eds.), Operator theory, Springer, Basel, 2025, pp. 1–30.

[PR16] V. I. Paulsen and M. Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2016.

[RS80] M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. 1, Functional Analysis, Academic Press, San Diego, CA, 1980.

[Sim78] B. Simon, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems*. J. Funct. Anal. **28**(1978), 377–385.

Department of Mathematics, University of Manitoba, Winnipeg, MB, Canada
e-mail: Robert.Martin@umanitoba.ca

Department of Mathematics and Statistics, University of Windsor, Windsor, ON, Canada
e-mail: Edward.Timko@uwindsor.ca