A STABILITY THEOREM FOR THE NONLINEAR DIFFERENTIAL EQUATION $x''+p(t)g(x)h(x')=0$

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K. W. Chang generalizing a result of Lazer [3], proved in [4] the following

THEOREM 1. *Suppose that* $f : I \to \mathbf{R}_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqq 0$, *is a non-decreasing function whose derivatives of orders* $\leqq 3$ *exist and are continuous on* $[t_0, +\infty)$. *Moreover,* $\lim_{t\to+\infty} f(t) = +\infty$ *and for some* α, $1 \leqq \alpha \leqq 2$, *and* $F = f^{-1/\alpha}$

$$\int_{t_0}^{+\infty} |F'''(t)|dt < +\infty;$$

then every solution $x(t)$ *of the equation*

$$(*) \qquad x''+f(t)x = 0$$

tends to zero as $t \to +\infty$.

Here we extend the above theorem to a nonlinear equation of the form:

$$(**) \qquad x''+p(t)g(x)h(x') = 0.$$

As solutions of (**) we consider only functions $x(t) \in C^2[t_0, +\infty)$, $t_0 \geqq 0$, which satisfy (**) on the whole interval $[t_0, +\infty)$. By an oscillatory solution of (**) we mean a solution with arbitrarily large zeros. We suppose also that the only solution $y(t)$ of (**) satisfying the initial conditions $y(a) = 0$, $y'(a) = 0$ for any $a \geqq t_0$ is the trivial solution $y(t) \equiv 0$, $t \in [t_0, +\infty)$.

We prove the following:

THEOREM 2. *Consider* (**) *with the assumptions*:

(i) $p : I \to \mathbf{R}_+$, $I = [t_0, +\infty)$, $t_0 \geqq 0$, *non-decreasing with continuous derivatives of orders* $\leqq 3$ *on* $[t_0, +\infty)$. *Moreover,* $\lim_{t\to+\infty} p(t) = +\infty$, *and*

$$\int_{t_0}^{+\infty} |P'''(t)|dt < +\infty,$$

with $P(t) = [p(t)]^{-1/\alpha}$, α *a positive constant greater than* 1;

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(ii) $g : \mathbf{R} \to \mathbf{R}$, $g'(x)$ *exists and is continuous on* $\mathbf{R}$, $xg(x) > 0$ *for* $x \neq 0$, $g(-x) = -g(x)$, *and* $\lim_{|x|\to+\infty} G(x) = +\infty$, *where*

$$G(x) = \int_0^x g(u)du;$$

(iii) $h : \mathbf{R} \to \mathbf{R}_+$, *continuous, even and such that*

$$\text{(S)} \qquad 2H(y)/(\alpha-1)+p(t)(g^2(x)h(y)-2G(x))-g'(x)y^2 \geqq 0, \quad (t, x, y) \in I\times\mathbf{R}^2,$$

where
$$H(y) = \int_0^y udu/h(u)$$

($H(y)$ *is non-negative and finite valued*); *then if* $x(t)$ *is a nontrivial solution of* (**), *we have* $\lim_{t\to+\infty} x(t) = 0$.

Proof. For the sake of completeness we shall give the whole proof of the theorem, although the boundedness of the solutions can be traced in Bihari's Theorem 1, in [1].

First we show that all solutions of (**) are bounded. In fact, by differentiation of the function

$$V = V(t) = H(y(t))+p(t)G(x(t)) \qquad (y(t) = x'(t)) \tag{1}$$

where $x(t)$ is a solution of (**), we find

$$\begin{aligned} V'(t) &= p'(t)G(x(t)) \\ &\leqq [p'(t)/p(t)]V(t) \end{aligned} \tag{2}$$

which by integration from t_0 to t ($t \geqq t_0$) and application of a well known inequality gives

$$V(t) \leqq V(t_0)+\int_{t_0}^t [p'(s)/p(s)]V(s)ds$$

and

$$\begin{aligned} V(t) &\leqq V(t_0)\exp\int_{t_0}^t [p'(s)/p(s)]ds \\ &= V(t_0)[p(t)/p(t_0)]. \end{aligned} \tag{3}$$

Thus, $G(x(t)) \leqq V(t_0)/p(t_0)$, and consequently, $x(t)$ is bounded on $[t_0, +\infty)$.

Now we prove that all solutions of (**) are oscillatory. The proof is by contradiction. Let $x(t)$, $t \in [t_0, +\infty)$ be a solution of (**) which is non-oscillatory. Then, since for every solution $x(t)$ of (**), $-x(t)$ is also a solution, we may (and do) assume that $x(t) > 0$, $t \in [t_1, +\infty)$, $t_1 \geqq t_0$. It can be easily seen that $x(t)$ must be concave and strictly increasing on $[t_1, +\infty)$ ($x''(t) < 0$), while its derivative has to be positive and strictly decreasing on the same interval. Thus, if $\lim_{t\to+\infty} x(t) = \lambda$ ($0 < \lambda < +\infty$), then since $\lim_{t\to+\infty} x'(t) = 0$, given a positive number $\varepsilon < \min\{g(\lambda), h(0)\}$, there exists a $t_2 \geqq t_1$ such that

$$g(\lambda)-\varepsilon < g(x(t)) < g(\lambda)+\varepsilon, \quad h(0)-\varepsilon < h(x'(t)) < h(0)+\varepsilon \tag{4}$$

for every $t \geqq t_2$. From (**) by use of (4) we get

$$\begin{aligned} x''(t) &= -p(t)g(x(t))h(x'(t)) \\ &< -L[g(\lambda)-\varepsilon][h(0)-\varepsilon] \\ &= -L^* < 0 \end{aligned} \tag{5}$$

where $p(t) \geqq L$ for $t \geqq t_2$. Obviously (5) implies $x(t) \to -\infty$ as $t \to +\infty$, a contradiction. Thus, every solution of (**) is oscillatory.

To show the decrease of the amplitudes, let $x(t)$ be any solution of (**) with $x'(a) = x'(c) = 0$ and $x(b) = 0$ where $t_0 \leqq a < b < c$. Then after a simple manipulation we obtain from (**):

$$\begin{aligned} H(y(b))/p(b) &= -\int_a^b H(y(t))[p'(t)/p^2(t)]dt + \int_0^{x(a)} g(u)du, \\ -H(y(b))/p(b) &= -\int_b^c H(y(t))[p'(t)/p^2(t)]dt - \int_0^{x(c)} g(u)du \end{aligned} \tag{6}$$

from which, by adding the corresponding sides we get

$$\int_a^c H(y(t))[p'(t)/p^2(t)]dt = \int_{x(c)}^{x(a)} g(u)du. \tag{7}$$

Since g is an odd function, (7) implies that $|x(c)| \leqq |x(a)|$, which proves the decrease of the amplitudes. Now we are ready to show that all nontrivial solutions of (**) tend to 0 as $t \to +\infty$. In fact, let $x = x(t)$ be a solution of (**); then by differentiation of the function

$$W = W(t) = 2G(x)[2P^{1-\alpha}/(\alpha-1)+P'']-2g(x)yP'+4H(y)P/(\alpha-1) \tag{8}$$

where $y = y(t) = x'(t)$, we find

$$\begin{aligned} W' = W'(t) &= 2G(x)P'''(t) \\ &\quad +2[2H(y)/(\alpha-1)+p(t)(g^2(x)h(y)-2G(x))-g'(x)y^2]P' \\ &\leqq 2G(x)P'''(t) \end{aligned} \tag{9}$$

which, by integration from t_1 to t ($t \geqq t_1$) yields

$$\begin{aligned} W(t) &\leqq W(t_1)+2\int_{t_1}^{t} G(x(s))|P'''(s)|ds \\ &\leqq W(t_1)+2[V(t_1)/p(t_1)]\int_{t_1}^{+\infty} P'''(s)|ds = K \qquad \text{(say)}. \end{aligned} \tag{10}$$

Now, following Chang's proof, since P'' is bounded as $t \to +\infty$, given any $\varepsilon > 0$, let $T \geqq t_1$ ($T = T(\varepsilon)$) be such that

$$K/\varepsilon < 2[P(T)]^{1-\alpha}/(\alpha-1)+P''(T),\ x'(T) = 0. \tag{11}$$

Then, finally, $G(x(t)) < \varepsilon$ for every $t \geqq T$. This implies that

$\lim_{t\to+\infty} G(x(t)) = 0$. Suppose now that there exists a sequence $\{t_n\}$ such that $t_n \geqq t_1$, $\lim_{n\to+\infty} t_n = +\infty$, and $\lim_{n\to+\infty} x(t_n) \neq 0$. Then $\lim_{n\to+\infty} G(x(t_n)) > 0$, a contradiction. Thus, $\lim_{t\to+\infty} x(t) = 0$ and the theorem is proved.

REMARK 1. From (9) it turns out that we can replace the integral condition on P''' by the condition $P'''(t) \leqq 0$ for all large t. In fact, this implies that $P''(t) \geqq 0$ for all large t (otherwise we would have $\lim_{t\to+\infty} P(t) = -\infty$) so that $P''(t)$ is bounded on some interval $[c, +\infty)$.

REMARK 2. The condition (S) in (iii) of Theorem 2, is quite artificial and can be replaced by the following one:

$$\text{(S')} \qquad g^2(x)h(y) \geqq 2G(x) \quad \text{for all} \quad (x, y) \in \mathbf{R}^2,$$

$y(t) = x'(t)$ is bounded for all solutions $x(t)$ of (**), and

$$\int_{t_0}^{+\infty} |P'(t)|dt < +\infty.$$

In fact, if we take into account (S'), then from the first of (9) we obtain

$$W'(t) \leqq 2G(x(t))|P'''(t)| + |g'(x(t))|y^2(t)|P'(t)|$$

and

$$W(t) \leqq K + \int_{t_0}^{+\infty} |g'(x(t))|y^2(t)|P'(t)|dt < +\infty.$$

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