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L^p REGULARITY OF THE WEIGHTED BERGMAN PROJECTION ON THE FOCK–BARGMANN–HARTOGS DOMAIN

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Abstract

The Fock–Bargmann–Hartogs domain $D_{n,m}(\mu) := \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\}$, where $\mu > 0$, is an unbounded strongly pseudoconvex domain with smooth real-analytic boundary. We compute the weighted Bergman kernel of $D_{n,m}(\mu)$ with respect to the weight $(-\rho)^\alpha$, where $\rho(z, w) := \|w\|^2 - e^{-\mu\|z\|^2}$ and $\alpha > -1$. Then, for $p \in [1, \infty)$, we show that the corresponding weighted Bergman projection $P_{D_{n,m}(\mu),(-\rho)^\alpha}$ is unbounded on $L^p(D_{n,m}(\mu), (-\rho)^\alpha)$, except for the trivial case $p = 2$. This gives an example of an unbounded strongly pseudoconvex domain whose ordinary Bergman projection is L^p irregular when $p \in [1, \infty) \setminus \{2\}$, in contrast to the well-known positive L^p regularity result on a bounded strongly pseudoconvex domain.

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1. Introduction

1.1. Setup and problems. Let Ω be a domain in $\mathbb{C}^n$ and $\eta(z)$ a nonnegative measurable function on Ω. For $p \in [1, +\infty)$, let $L^p(\Omega, \eta)$ denote the set of all complex measurable functions f with

$$\left(\int_\Omega |f(z)|^p \eta(z)\, dV(z) \right)^{1/p} < +\infty,$$

where $dV(z)$ is the ordinary Lebesgue measure on Ω. We call $\eta(z)$ a weight on Ω and $L^p(\Omega, \eta)$ the weighted L^p space of Ω. The norm on $L^p(\Omega, \eta)$ is defined by

$$\|f\|_{p,\eta} = \left(\int_\Omega |f(z)|^p \eta(z)\, dV(z) \right)^{1/p}.$$

For $p = 2$, $L^2(\Omega, \eta)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_\eta = \int_\Omega f(z)\overline{g(z)}\eta(z)\, dV(z).$$

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The weighted Bergman space of Ω with weight η is defined by

$$A^p(\Omega, \eta) := O(\Omega) \cap L^p(\Omega, \eta),$$

where $O(\Omega)$ is the space of all holomorphic functions on Ω. Thus, $A^2(\Omega, \eta)$ is a subspace of holomorphic functions in $L^2(\Omega, \eta)$. From [18], if η is continuous and never vanishes inside Ω, then $A^p(\Omega, \eta)$ is a closed subspace of $L^2(\Omega, \eta)$ and there is an orthogonal projection, called the weighted Bergman projection,

$$P_{\Omega,\eta} : L^2(\Omega, \eta) \to A^2(\Omega, \eta).$$

This projection is an integral operator with the weighted Bergman kernel, $K_{\Omega,\eta}(z, w)$, that is,

$$P_{\Omega,\eta} f(z) := \int_\Omega K_{\Omega,\eta}(z, w) f(w) \eta(w)\, dV(w).$$

When $\eta(z) \equiv 1$, the weighted Bergman kernel $K_{\Omega,\eta}$ and the weighted Bergman projection $P_{\Omega,\eta}$ degenerate to the ordinary Bergman kernel K_Ω and the ordinary Bergman projection P_Ω, respectively.

For an arbitrary domain $\Omega \subset \mathbb{C}^n$ and a continuous positive weight η on Ω, the corresponding weighted Bergman projection $P_{\Omega,\eta}$ is originally defined on $L^2(\Omega, \eta)$, mapping onto the weighted Bergman space $A^2(\Omega, \eta)$. The weighted Bergman projection $P_{\Omega,\eta}$ on $L^p(\Omega, \eta)$ means $P_{\Omega,\eta}$ on the subspace $L^p(\Omega, \eta) \cap L^2(\Omega, \eta)$ of $L^p(\Omega, \eta)$. Therefore, for any $p \in [1, \infty)$, when we say that the weighted Bergman projection $P_{\Omega,\eta}$ is bounded on $L^p(\Omega, \eta)$, we mean that the weighted Bergman projection $P_{\Omega,\eta}$ mapping $L^p(\Omega, \eta) \cap L^2(\Omega, \eta)$ onto $A^p(\Omega, \eta) \cap L^2(\Omega, \eta)$ is bounded.

Fixing a domain Ω and a positive continuous weight η on Ω, we define the operator norm of $P_{\Omega,\eta}$ by

$$\|P_{\Omega,\eta}\|_{p,\eta} := \sup\left\{ \frac{\|P_{\Omega,\eta} f\|_{p,\eta}}{\|f\|_{p,\eta}} : f \in L^p(\Omega, \eta) \cap L^2(\Omega, \eta), f \neq 0 \right\}.$$

It is easy to see that $\|P_{\Omega,\eta}\|_{2,\eta} = 1$ in the case of $p = 2$. A natural and interesting question is to determine the range of $p \in (1, +\infty)$ such that the weighted Bergman projection $P_{\Omega,\eta}$ is bounded on $L^p(\Omega, \eta)$, except for the trivial case $p = 2$. This is the so-called L^p regularity problem.

1.2. Background. The L^p regularity of the (weighted) Bergman projection is of fundamental importance. Even though two domains are biholomorphic equivalent, the corresponding L^p behaviour of the Bergman projection on these two domains may be quite different. There are many papers considering this problem in different settings. One of the most common is a bounded domain with various boundary conditions. For example, positive L^p regularity results have been obtained on the following domains for all $p \in (1, +\infty)$ in the unweighted version.

* Ω is a bounded strongly pseudoconvex domain (see Lanzani and Stein [13] and Phong and Stein [19]).

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- $*$ Ω is a bounded, smooth and pseudoconvex domain of finite type in $\mathbb{C}^2$ (see McNeal [15]).
- $*$ Ω is a bounded, smooth and convex domain of finite type in $\mathbb{C}^n$ (see McNeal [16] and McNeal and Stein [17]).

We refer to Charpentier and Dupain [5] and Huo [9] for positive results on other bounded domains. There are examples of smoothly bounded pseudoconvex domains where the L^p boundedness does not hold on the full interval $(1, +\infty)$ (see Barrett and Şahutoğlu [1]). If the domain Ω has a serious boundary singularity, in general, there will be a restricted range of p for the L^p boundedness of P_Ω (see Chakrabarti and Zeytuncu [4] and Edholm and McNeal [8]). There are also results giving the L^p boundedness of weighted Bergman projections with different types of weights on bounded domains.

- $*$ Consider the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ $(n \geq 1)$. Introduce $\omega = (-\rho)^\alpha$ $(\alpha > -1)$, where $\rho(z) = \|z\|^2 - 1$ is the defining function of $\mathbb{B}^n$. The weighted Bergman projection $P_{\mathbb{B}^n,\omega}$ is bounded from $L^p(\mathbb{B}^n, \omega)$ to $A^p(\mathbb{B}^n, \omega)$ for any $p \in (1, +\infty)$. This result implies that the L^p boundedness is independent of the parameter α (see Rudin [20, Section 7.1]).
- $*$ Consider the Hartogs triangle $\mathbb{H}$ in $\mathbb{C}^2$. Let $\omega(z) := |z_2|^s$ $(s \in \mathbb{R})$, $z = (z_1, z_2) \in \mathbb{H}$. Then the range of p for the L^p boundedness of $P_{\mathbb{H},\omega}$ is related to the power s (see Chen [6]).

For more results on other bounded domains with exponential weights, we refer to Čučković and Zeytuncu [7] and Zeytuncu [23].

There are very few results for the L^p regularity problem on unbounded domains. Krantz and Peloso [12] determined the L^p-mapping properties of the Bergman projection on unbounded, nonsmooth worm domains, facilitated by the fact that the boundaries of these domains are Levi flat. Janson *et al.* [10] determined the L^p-mapping properties of the Bergman projection on L^p space on $\mathbb{C}^n$ with respect to Gaussian weights $\eta_\alpha(z) = e^{-\alpha\|z\|^2}$. Bommier-Hato *et al.* [3] gave criteria for boundedness of the associated Bergman-type projections on L^p space on $\mathbb{C}^n$ with respect to generalised Gaussian weights $e^{-\alpha\|z\|^{2m}}$, where $m > 0$.

In this paper, we focus on the L^p regularity problem on the Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ in $\mathbb{C}^{n+m}$. The behaviour of the (weighted) Bergman projection on $D_{n,m}(\mu)$ serves as a model for the unbounded case.

1.3. The Fock–Bargmann–Hartogs domain. For a given positive real number μ, the Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is a Hartogs domain over $\mathbb{C}^n$ defined by

$$D_{n,m}(\mu) := \{(z, w) \in \mathbb{C}^{n+m} : \|w\|^2 < e^{-\mu\|z\|^2}\},$$

where $\|\cdot\|$ is the standard Hermitian norm. The domain is an unbounded, inhomogeneous strongly pseudoconvex domain in $\mathbb{C}^{n+m}$ with smooth real-analytic boundary. Since each $D_{n,m}(\mu)$ contains $\{(z, 0) \in \mathbb{C}^n \times \mathbb{C}^m\} \cong \mathbb{C}^n$, this domain is not hyperbolic in the sense of Kobayashi. Therefore, it cannot be biholomorphic to any

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bounded domain in $\mathbb{C}^{n+m}$. For more information on the Fock–Bargmann–Hartogs domain, see Bi *et al.* [2], Kim *et al.* [11], Tu and Wang [21] and Yamamori [22].

The Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is defined as a domain in $\mathbb{C}^{m+n}$ with the fibre over $\mathbb{C}^n$ being an m-dimensional ball. Under suitable conditions, we can relate the weighted Bergman kernel of $D_{n,m}(\mu)$ to the weighted Bergman kernel of the base space $\mathbb{C}^n$ and deduce the L^p regular behaviour of the corresponding weighted Bergman projections.

1.4. Main results. Let $\rho(z, w) = \|w\|^2 - e^{-\mu\|z\|^2}$, $(z, w) \in D_{n,m}(\mu)$. For $-1 < \alpha < \infty$, the weighed Bergman space $A^2(D_{n,m}(\mu), (-\rho)^\alpha)$ is defined by

$$A^2(D_{n,m}(\mu), (-\rho)^\alpha) := \Big\{ f \in \mathcal{O}(D_{n,m}(\mu)) : \int_{D_{n,m}(\mu)} |f|^2 (-\rho)^\alpha \, dV < \infty \Big\}.$$

The Bergman kernel of $A^2(D_{n,m}(\mu), (-\rho)^\alpha)$ is denoted by $K_{D_{n,m}(\mu),(-\rho)^\alpha}$. By applying a theorem of Ligocka [14], Yamamori [22] gave an explicit expression of the Bergman kernel of $A^2(D_{n,m}(\mu))$. Following the method in Bi *et al.* [2], we give a formula for the weighted Bergman kernel $K_{D_{n,m}(\mu),(-\rho)^\alpha}$.

THEOREM 1.1. *Let $D_{n,m}(\mu)$ be the Fock–Bargmann–Hartogs domain with the defining function $\rho(z, w) = \|w\|^2 - e^{-\mu\|z\|^2}$, $(z, w) \in D_{n,m}(\mu)$. Then, for $\alpha > -1$, the Bergman kernel of the weighted Hilbert space $A^2(D_{n,m}(\mu), (-\rho)^\alpha)$ can be expressed as*

$$K_{D_{n,m}(\mu),(-\rho)^\alpha}((x, y), (s, t)) = \frac{\mu^n}{\pi^{n+m}} \sum_{k \in \mathbb{N}} \frac{\Gamma(\alpha + m + k + 1)(\alpha + m + k)^n}{\Gamma(\alpha + 1)\Gamma(k + 1)} e^{\mu(\alpha+m+k)\langle x,s \rangle} \langle y, t \rangle^k.$$

As an application of Theorem 1.1, we derive a key relation between the weighted Bergman kernel $K_{D_{n,m}(\mu),(-\rho)^\alpha}$ of $D_{n,m}(\mu)$ and the weighted Bergman kernel $K_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}$ of the base space $\mathbb{C}^n$ (see Lemma 2.4). We use this key relation to study the L^p regularity properties of the weighted Bergman projection $P_{D_{n,m}(\mu),(-\rho)^\alpha}$ on $D_{n,m}(\mu)$ and obtain the following result.

THEOREM 1.2. *Let $P_{D_{n,m}(\mu),(-\rho)^\alpha}$ be the weighted Bergman projection on $D_{n,m}(\mu)$ with the weight $(-\rho)^\alpha$, where $\rho(z, w) = \|w\|^2 - e^{-\mu\|z\|^2}$, $(z, w) \in D_{n,m}(\mu)$ and $\alpha > -1$. Let $1 \le p < \infty$. Then $P_{D_{n,m}(\mu),(-\rho)^\alpha}$ is bounded on $L^p(D_{n,m}(\mu), (-\rho)^\alpha)$ if and only if $p = 2$.*

REMARK 1.3. Setting $\alpha = 0$ in Theorem 1.2, we see that the ordinary Bergman projection $P_{D_{n,m}(\mu)}$ is bounded on $L^p(D_{n,m}(\mu))$ if and only if $p = 2$.

Our proof of Theorem 1.2 employs the technique used by Čučković and Zeytuncu [7] and Zeytuncu [23]. Since $D_{n,m}(\mu)$ is an unbounded domain and it cannot be biholomorphic to any bounded domain in $\mathbb{C}^{n+m}$, the Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ studied here is different from the bounded Hartogs domain Ω_φ in $\mathbb{C}^2$. We give an example of an unbounded strongly pseudoconvex domain whose ordinary Bergman projection is L^p irregular except for the trivial case $p = 2$.

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2. Preliminaries

LEMMA 2.1. *For $\alpha > -1$, the following multiple integral exists:*

$$\int_0^1 dx_m \cdots \int_0^{1-\sum_{i=2}^m x_i} \Big(1 - \sum_{i=1}^m x_i\Big)^\alpha \prod_{i=1}^m x_i^{q_i}\, dx_1 = \frac{\prod_{i=1}^m \Gamma(q_i+1)\Gamma(\alpha+1)}{\Gamma(\alpha + \sum_{i=1}^m q_i + m + 1)},$$

where $q = (q_1, \ldots, q_m) \in (\mathbb{R}_+)^m$. Here $\mathbb{R}_+$ denotes the set of positive real numbers.

PROOF. By computation,

$$\begin{aligned}
&\int_0^1 dx_m \cdots \int_0^{1-\sum_{i=2}^m x_i} \Big(1 - \sum_{i=1}^m x_i\Big)^\alpha \prod_{i=1}^m x_i^{q_i}\, dx_1 \\
&= \int_0^1 dx_m \cdots \int_0^{1-\sum_{i=3}^m x_i} \prod_{i=2}^m x_i^{q_i} \Big(\int_0^{1-\sum_{i=2}^m x_i} \Big(1 - \sum_{i=2}^m x_i - x_1\Big)^\alpha x_1^{q_1}\, dx_1 \Big) dx_2 \\
&= B(q_1+1, \alpha+1) \int_0^1 dx_m \cdots \int_0^{1-\sum_{i=3}^m x_i} \prod_{i=2}^m x_i^{q_i} \Big(1 - \sum_{i=2}^m x_i\Big)^{\alpha+q_1+1} dx_2 \\
&= B(q_1+1, \alpha+1) B(q_2+1, \alpha+q_1+2) \cdots B\Big(q_m+1, \alpha + \sum_{i=1}^{m-1} q_i + m - 1\Big) \\
&= \frac{\prod_{i=1}^m \Gamma(q_i+1)\Gamma(\alpha+1)}{\Gamma(\alpha + \sum_{i=1}^m q_i + m + 1)}.
\end{aligned}$$

□

LEMMA 2.2. *For any $p \in \mathbb{N}^n$, $q \in \mathbb{N}^m$ and $\alpha > -1$,*

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = \pi^{n+m} \frac{\prod_{i=1}^n \Gamma(p_i+1) \prod_{i=1}^m \Gamma(q_i+1)\Gamma(\alpha+1)}{\Gamma(\alpha+m+1+|q|)[\mu(\alpha+m+|q|)]^{|p|+n}},$$

where w^q, $|q|$, $\|z^p w^q\|^2_{2,(-\rho)^\alpha}$ are given by

$$w^q = \prod_{i=1}^m w_i^{q_i}, \quad |q| = \sum_{i=1}^m q_i, \quad \|z^p w^q\|^2_{2,(-\rho)^\alpha} = \int_{D_{n,m}(\mu)} |z^p w^q|^2 (e^{-\mu\|z\|^2} - \|w\|^2)^\alpha\, dV(z,w)$$

for $w = (w_1, \ldots, w_m)$ and $q = (q_1, \ldots, q_m)$.

PROOF. From the definition,

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = \int_{D_{n,m}(\mu)} |z^p w^q|^2 (e^{-\mu\|z\|^2} - \|w\|^2)^\alpha\, dV(z,w).$$

By setting $z_j = r_j e^{i\theta_j}$ $(1 \le j \le n)$, $w_l = k_l e^{i\theta_l}$ $(1 \le l \le m)$,

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = (2\pi)^{n+m} \int_{\substack{\|k\|^2 < e^{-\mu\|r\|^2} \\ k \ge 0, r \ge 0}} r^{2p+1} k^{2q+1} \Big(e^{-\mu \sum_{j=1}^n r_j^2} - \sum_{l=1}^m k_l^2\Big)^\alpha dr\, dk,$$

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where $r = (r_1, \ldots, r_n)$, $k = (k_1, \ldots, k_m)$. Now, by setting $s_i = r_i^2$ $(1 \le i \le n)$ and $t_j = k_j^2$ $(1 \le j \le m)$,

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = \pi^{n+m} \int_{\substack{\sum_{j=1}^m t_j < e^{-\mu \sum_{i=1}^n s_i} \\ t_j \ge 0, s_i \ge 0}} s^p t^q \Big(e^{-\mu \sum_{i=1}^n s_i} - \sum_{j=1}^m t_j \Big)^\alpha \, ds\, dt.$$

Let $\widetilde{t_j} = e^{\mu \sum_{i=1}^n s_i} t_j$. Then

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = \pi^{n+m} \int_{(\mathbb{R}_+)^n} s^p e^{-\mu(\alpha+m+|q|) \sum_{i=1}^n s_i} \, ds \int_{\substack{\sum_{j=1}^m \widetilde{t_j} < 1 \\ \widetilde{t_j} \ge 0}} \Big(1 - \sum_{j=1}^m \widetilde{t_j}\Big)^\alpha \widetilde{t}^q \, d\widetilde{t}.$$

Since $\alpha > -1$, by Lemma 2.1,

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = \pi^{n+m} \frac{\prod_{i=1}^m \Gamma(q_i + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha + m + 1 + \sum_{i=1}^m q_i)} \int_{(\mathbb{R}_+)^n} s^p e^{-\mu(\alpha+m+|q|) \sum_{i=1}^n s_i} \, ds.$$

Since

$$\int_0^\infty s_i^{p_i} e^{-\mu(\alpha+m+|q|)s_i} \, ds_i = [\mu(\alpha + m + |q|)]^{-p_i-1} \Gamma(p_i + 1),$$

we obtain

$$\|z^p w^q\|^2_{2,(-\rho)^\alpha} = \pi^{n+m} \frac{\prod_{i=1}^n \Gamma(p_i + 1) \prod_{i=1}^m \Gamma(q_i + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha + m + 1 + |q|)[\mu(\alpha + m + |q|)]^{|p|+n}}. \qquad \square$$

Let $A^p(\mathbb{C}^n, \eta_\alpha)$ be the space of all entire functions f on $\mathbb{C}^n, n \ge 1$, such that $|f|^p$ is integrable with respect to the Gaussian

$$\eta_\alpha(z) = e^{-\alpha \|z\|^2},$$

where $\alpha > 0, 1 \le p < \infty$. Equipped with the norm inherited from $L^p_\alpha(\mathbb{C}^n, \eta_\alpha)$, the spaces $A^p(\mathbb{C}^n, \eta_\alpha)$ become Banach spaces. In particular, $A^2(\mathbb{C}^n, \eta_\alpha)$ is the Segal–Bargmann–Fock space of quantum mechanics with parameter α. The function

$$K_{\mathbb{C}^n,\eta_\alpha}(x, y) = \Big(\frac{\alpha}{\pi}\Big)^n e^{\alpha\langle x,y\rangle}, \quad x, y \in \mathbb{C}^n, \tag{2.1}$$

is the Bergman kernel for $A^2(\mathbb{C}^n, \eta_\alpha)$ (see [3] for further details).

The integral operator defined by

$$P_{\mathbb{C}^n,\eta_\alpha} f(x) = \int_{\mathbb{C}^n} f(y) K_{\mathbb{C}^n,\eta_\alpha}(x, y) \eta_\alpha(y) \, dV(y), \quad x \in \mathbb{C}^n,$$

is the orthogonal projection in $L^2(\mathbb{C}^n, \eta_\alpha)$ onto $A^2(\mathbb{C}^n, \eta_\alpha)$. The operator $P_{\mathbb{C}^n,\eta_\alpha}$ is bounded on $L^2(\mathbb{C}^n, \eta_\alpha)$, but this turns out to be no longer the case for $L^p(\mathbb{C}^n, \eta_\alpha)$ with $p \ne 2$. Janson *et al.* proved the following assertions.

THEOREM 2.3 [10]. *Suppose that $\alpha \in \mathbb{R}, \beta > 0, 1 \le p < \infty$ satisfy $\beta p > \alpha$. Then $P_{\mathbb{C}^n,\eta_\beta}$ is bounded from $L^p(\mathbb{C}^n, \eta_\alpha)$ into $L^p(\mathbb{C}^n, \eta_\gamma)$, where $1/\gamma = 4(\beta p - \alpha)/p^2\beta^2$. In particular, $P_{\mathbb{C}^n,\eta_\alpha}$ is bounded on $L^p(\mathbb{C}^n, \eta_\alpha)$ if and only if $p = 2$.*

Comparing the expression of the Bergman kernel in Theorem 1.1 and (2.1) gives the following lemma.

LEMMA 2.4. *Let* $K_{D_{n,m}(\mu),(-\rho)^\alpha}$ *be the Bergman kernel for* $A^2(D_{n,m}(\mu),(-\rho)^\alpha)$ *and* $K_{\mathbb{C}^n,\eta_\alpha}$ *the Bergman kernel for* $A^2(\mathbb{C}^n,\eta_\alpha)$. *Then*

$$K_{D_{n,m}(\mu),(-\rho)^\alpha}((x,0),(s,0)) = \frac{\Gamma(\alpha+m+1)}{\pi^m\Gamma(\alpha+1)}K_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}(x,s), \quad x,s\in\mathbb{C}^n.$$

3. Proofs of the main results

PROOF OF THEOREM 1.1. Since $\{z^pw^q/\|z^pw^q\|_{2,(-\rho)^\alpha}\}$ constitutes an orthonormal basis of $A^2(D_{n,m}(\mu),(-\rho)^\alpha dv)$ and the Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is a Reinhardt domain,

$$K_{D_{n,m}(\mu),(-\rho)^\alpha}((x,y),(s,t)) = \sum_{p\in\mathbb{N}^n,q\in\mathbb{N}^m}\frac{x^py^q\overline{s^pt^q}}{\|x^py^q\|_{2,(-\rho)^\alpha}\|s^pt^q\|_{2,(-\rho)^\alpha}}.$$

By Lemma 2.1,

$$\begin{aligned}
&K_{D_{n,m}(\mu),(-\rho)^\alpha}((x,y),(s,t))\\
&\quad=\sum_{p\in\mathbb{N}^n,q\in\mathbb{N}^m}\frac{\Gamma(\alpha+m+1+|q|)[\mu(\alpha+m+|q|)]^{n+|p|}}{\pi^{n+m}\prod_{i=1}^n\Gamma(p_i+1)\prod_{i=1}^m\Gamma(q_i+1)\Gamma(\alpha+1)}x^py^q\overline{s^pt^q}\\
&\quad=\frac{1}{\pi^{n+m}}\sum_{q\in\mathbb{N}^m}\phi(x,s)\frac{\Gamma(\alpha+m+1+|q|)[\mu(\alpha+m+|q|)]^n}{\prod_{i=1}^m\Gamma(q_i+1)\Gamma(\alpha+1)}y^q\overline{t^q},
\end{aligned}\tag{3.1}$$

where $\phi(x,s)=\sum_{p\in\mathbb{N}^n}([\mu(\alpha+m+|q|)]^{|p|}/\prod_{i=1}^n\Gamma(p_i+1))x^p\overline{s^p}$. By an easy calculation,

$$\sum_{p\in\mathbb{N}^n}\frac{[\mu(\alpha+m+|q|)]^{|p|}}{\prod_{i=1}^n\Gamma(p_i+1)}x^p\overline{s^p}=e^{\mu(\alpha+m+|q|)\langle x,s\rangle}.\tag{3.2}$$

Substituting (3.2) into (3.1),

$$\begin{aligned}
&K_{D_{n,m}(\mu),(-\rho)^\alpha}((x,y),(s,t))\\
&\quad=\frac{1}{\pi^{n+m}}\sum_{q\in\mathbb{N}^m}e^{\mu(\alpha+m+|q|)\langle x,s\rangle}\frac{\Gamma(\alpha+m+1+|q|)[\mu(\alpha+m+|q|)]^n}{\prod_{i=1}^m\Gamma(q_i+1)\Gamma(\alpha+1)}y^q\overline{t^q}\\
&\quad=\frac{\mu^n}{\pi^{n+m}}\sum_{q\in\mathbb{N}^m}\frac{\Gamma(\alpha+m+1+|q|)(\alpha+m+|q|)^n}{\prod_{i=1}^m\Gamma(q_i+1)\Gamma(\alpha+1)}e^{\mu(\alpha+m+|q|)\langle x,s\rangle}y^q\overline{t^q}\\
&\quad=\frac{\mu^n}{\pi^{n+m}}\sum_{k\in\mathbb{N}}\frac{\Gamma(\alpha+m+k+1)(\alpha+m+k)^n}{\Gamma(\alpha+1)\Gamma(k+1)}e^{\mu(\alpha+m+k)\langle x,s\rangle}\langle y,t\rangle^k.
\end{aligned}$$

PROOF OF THEOREM 1.2. For a given $p\in[1,\infty)\setminus\{2\}$, by Theorem 2.3, $P_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}$ is unbounded on $L^p(\mathbb{C}^n,\eta_{\mu(\alpha+m)})$, where $\eta_{\mu(\alpha+m)}=e^{-\mu(\alpha+m)\|z\|^2}$. Therefore, there exists a

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sequence $\{f_n(z)\}$ in $L^p(\mathbb{C}^n, \eta_{\mu(\alpha+m)}) \cap L^2(\mathbb{C}^n, \eta_{\mu(\alpha+m)})$ such that

$$\lim_{n\to\infty} \frac{\|P_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}} f_n\|^p_{p,\eta_{\mu(\alpha+m)}}}{\|f_n\|^p_{p,\eta_{\mu(\alpha+m)}}} = \infty. \tag{3.3}$$

Define $F_n(z, w) = f_n(z)$. Then

$$\begin{aligned}\|F_n\|^p_{p,(-\rho)^\alpha} &= \int_{D_{n,m}(\mu)} |F_n(z, w)|^p (e^{-\mu\|z\|^2} - \|w\|^2)^\alpha \, dV(z, w)\\ &= \int_{\mathbb{C}^n} |f_n(z)|^p e^{-\mu\alpha\|z\|^2} \Big(\int_{\|w\|^2<e^{-\mu\|z\|^2}} (1 - e^{\mu\|z\|^2}\|w\|^2)^\alpha dV(w)\Big) dV(z).\end{aligned}$$

Let σ be the rotation-invariant positive Borel measure on $\partial\mathbb{B}^m$, the surface of the unit ball of complex dimension m, with $\sigma(\partial\mathbb{B}^m) = 1$, and let $w = r\zeta,\ \zeta \in \partial\mathbb{B}^m$. Then

$$\begin{aligned}\|F_n\|^p_{p,(-\rho)^\alpha} &= \int_{\mathbb{C}^n} |f_n(z)|^p e^{-\mu\alpha\|z\|^2} \int_0^{e^{-\mu\|z\|^2/2}} 2mV(\mathbb{B}^m) r^{2m-1}(1 - e^{\mu\|z\|^2} r^2)^\alpha \, dr \, dV(z)\\ &= mB(m, \alpha + 1)V(\mathbb{B}^m) \int_{\mathbb{C}^n} |f_n(z)|^p e^{-\mu(\alpha+m)\|z\|^2} \, dV(z)\\ &= mB(m, \alpha + 1)V(\mathbb{B}^m)\|f_n\|^p_{p,\eta_{\mu(\alpha+m)}},\end{aligned} \tag{3.4}$$

where $B(m, \alpha + 1)$ is the beta function. Therefore, $F_n(z, w) \in L^p(D_{n,m}(\mu), (-\rho)^\alpha)$ for any n. Next,

$$\begin{aligned}&P_{D_{n,m}(\mu),(-\rho)^\alpha} F_n(z, 0)\\ &\quad= \int_{D_{n,m}(\mu)} K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, 0), (s, t))F_n(s, t)(-\rho(s, t))^\alpha \, dV(s, t)\\ &\quad= \int_{\mathbb{C}^n} \int_{\|t\|^2<e^{-\frac{\mu}{2}\|s\|^2}} K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, 0), (s, t)) f_n(s)(-\rho(s, t))^\alpha \, dV(t) \, dV(s)\\ &\quad= \int_{\mathbb{C}^n} \Big(\int_0^{e^{-\mu\|s\|^2/2}} 2mV(\mathbb{B}^m) r^{2m-1} \, dr\\ &\qquad\quad \times \int_{\partial\mathbb{B}^m} K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, 0), (s, r\zeta)) f_n(s)(e^{-\mu\|s\|^2} - r^2)^\alpha d\sigma(\zeta)\Big) dV(s)\\ &\quad= \int_{\mathbb{C}^n} \Big(2mV(\mathbb{B}^m) f_n(s) \int_0^{e^{-\mu\|s\|^2/2}} r^{2m-1}(e^{-\mu\|s\|^2} - r^2)^\alpha \, dr\\ &\qquad\quad \times \int_{\partial\mathbb{B}^m} K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, 0), (s, r\zeta)) d\sigma(\zeta)\Big) dV(s).\end{aligned} \tag{3.5}$$

Since $K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, w), (s, t))$ is antiholomorphic in t, by the mean value property,

$$\int_{\partial\mathbb{B}^m} K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, 0), (s, r\zeta)) \, d\sigma(\zeta) = V(\partial\mathbb{B}^m) K_{D_{n,m}(\mu),(-\rho)^\alpha}((z, 0), (s, 0)), \tag{3.6}$$

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where $V(\partial\mathbb{B}^m)$ is the volume of $\partial\mathbb{B}^m$. Putting (3.6) into (3.5),

$$\begin{aligned} P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,0) &= \int_{\mathbb{C}^n}\Big(2mV(\partial\mathbb{B}^m)V(\mathbb{B}^m)K_{D_{n,m}(\mu),(-\rho)^\alpha}((z,0),(s,0))f_n(s) \\ &\qquad\times\int_0^{e^{-\frac{\mu}{2}\|s\|^2}} r^{2m-1}(e^{-\mu\|s\|^2}-r^2)^\alpha dr\Big)dV(s) \\ &= mV(\partial\mathbb{B}^m)V(\mathbb{B}^m)B(m,\alpha+1) \\ &\qquad\times\int_{\mathbb{C}^n} K_{D_{n,m}(\mu),(-\rho)^\alpha}((z,0),(s,0))f_n(s)e^{-\mu(\alpha+m)\|s\|^2}\,dV(s). \end{aligned} \tag{3.7}$$

Applying Lemma 2.4 to (3.7),

$$\begin{aligned} P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,0) &= c\int_{\mathbb{C}^n} K_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}(z,s)f_n(s)e^{-\mu(\alpha+m)\|s\|^2}\,dV(s) \\ &= cP_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}f_n(z), \end{aligned} \tag{3.8}$$

where $c = mV(\partial\mathbb{B}^m)V(\mathbb{B}^m)B(m,\alpha+1)\Gamma(\alpha+m+1)/\pi^m\Gamma(\alpha+1)$.

Next, we estimate the norm of $P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n$:

$$\begin{aligned} &\|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n\|^p_{p,(-\rho)^\alpha} \\ &\quad= \int_{D_{n,m}(\mu)} |P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,w)|^p(e^{-\mu\|z\|^2}-\|w\|^2)^\alpha\,dV(z,w) \\ &\quad= \int_{\mathbb{C}^n}\int_{\|w\|^2<e^{-\mu\|z\|^2}} |P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,w)|^p(e^{-\mu\|z\|^2}-\|w\|^2)^\alpha\,dV(w)\,dV(z) \\ &\quad= \int_{\mathbb{C}^n}\Big(\int_0^{e^{-\mu/2\|z\|^2}} 2mV(\mathbb{B}^m)r^{2m-1}(e^{-\mu\|z\|^2}-r^2)^\alpha\,dr \\ &\qquad\times\int_{\partial\mathbb{B}^m}|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,r\zeta)|^p d\sigma(\zeta)\Big)dV(z). \end{aligned} \tag{3.9}$$

Since $P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,w)$ is holomorphic in w, by the submean value property,

$$\int_{\partial\mathbb{B}^m}|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,r\zeta)|^p\,d\sigma(\zeta) \geq V(\partial\mathbb{B}^m)|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,0)|^p. \tag{3.10}$$

Substituting (3.10) in (3.9) and invoking (3.8),

$$\begin{aligned} &\|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n\|^p_{p,(-\rho)^\alpha} \\ &\geq \int_{\mathbb{C}^n}\Big(\int_0^{e^{-\mu 2\|z\|^2/2}} 2mV(\mathbb{B}^m)r^{2m-1}(e^{-\mu\|z\|^2}-r^2)^\alpha V(\partial\mathbb{B}^m)|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n(z,0)|^p dr\Big)dV(z) \\ &= mB(m,\alpha+1)V(\partial\mathbb{B}^m)V(\mathbb{B}^m)c^p\int_{\mathbb{C}^n}|P_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}f_n(z)|^p e^{-\mu(\alpha+m)\|z\|^2}\,dV(z) \\ &= mB(m,\alpha+1)V(\partial\mathbb{B}^m)V(\mathbb{B}^m)c^p\|P_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}f_n\|^p_{p,\eta_{\mu(\alpha+m)}}. \end{aligned} \tag{3.11}$$

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By (3.4) and (3.11),

$$\frac{\|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n\|^p_{p,(-\rho)^\alpha}}{\|F_n\|^p_{p,(-\rho)^\alpha}} \geq V(\partial\mathbb{B}^m)c^p\frac{\|P_{\mathbb{C}^n,\eta_{\mu(\alpha+m)}}f_n\|^p_{p,\eta_{\mu(\alpha+m)}}}{\|f_n\|^p_{p,\eta_{\mu(\alpha+m)}}}. \tag{3.12}$$

Thus, by (3.3) and (3.12),

$$\lim_{n\to\infty}\frac{\|P_{D_{n,m}(\mu),(-\rho)^\alpha}F_n\|^p_{p,(-\rho)^\alpha}}{\|F_n\|^p_{p,(-\rho)^\alpha}} = \infty.$$

This means that $P_{D_{n,m}(\mu),(-\rho)^\alpha}$ is unbounded on $L^p(D_{n,m}(\mu),(-\rho)^\alpha)$ for $p \in [1,\infty)\setminus\{2\}$ and it follows that $P_{D_{n,m}(\mu),(-\rho)^\alpha}$ is bounded on $L^p(D_{n,m}(\mu),(-\rho)^\alpha)$ if and only if $p = 2$.

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