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Nagoya Math. J.
Vol. 106 (1987), 101-111

$\mathfrak{m}$-FULL IDEALS

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Introduction

An ideal $\mathfrak{a}$ of a local ring $(R, \mathfrak{m})$ is called $\mathfrak{m}$-full if $\mathfrak{a}\mathfrak{m}: y = \mathfrak{a}$ for some y in a certain faithfully flat extension of R. The definition is due to Rees (unpublished) and he had obtained some elementary results (also unpublished). The present paper concerns some basic properties of $\mathfrak{m}$-full ideals. One result is the characterization of $\mathfrak{m}$-fullness in terms of the minimal number of generators of ideal, generalizing his result in a low dimensional case (Theorem 2, § 2).

Meanwhile Professor Rees asked me for which ideals is it true that $\mu(\mathfrak{a}) \geqq \mu(\mathfrak{b})$ for all $\mathfrak{b}$ containing $\mathfrak{a}$. Surprisingly enough it turns out that $\mathfrak{m}$-full ideals do have this property (Theorem 3, § 2). To prove this we introduce, in Section 1, three numerical characters Φ, Ψ, $\bar{\mu}$, of $\mathfrak{m}$-primary ideals. Φ and $\bar{\mu}$ are, respectively, the colength and the minimal number of generators of an ideal modulo a general element, and Ψ is the maximum of $\mu(\mathfrak{b})$, where $\mathfrak{b}$ runs over all the ideals containing a given ideal (Definitions 1, 2, 3, § 1).

Theorem 1 in Section 2 shows that these are related by an inequality, from which it immediately follows that an ideal $\mathfrak{a}$ has the property mentioned above if $\mu(\mathfrak{a}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$. And Theorem 2 shows that this is precisely equivalent to the $\mathfrak{m}$-fullness of the ideal.

The purpose of Section 3 is to show that the converse of Theorem 3 holds in a 2-dimensional regular local ring, thanks to the equality $\Psi(\mathfrak{a}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$ for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$. Also we would like to call attention to the fact that Theorem 1 has grown out of the attempt to generalize Lemma 2 which is easily proved homologically. (See Remark 3, § 3).

In Section 4 we present a theorem of Rees which says that any integrally closed ideal is $\mathfrak{m}$-full, and also we prove an interesting formula

Received October 21, 1985.
Revised April 10, 1986.

https://doi.org/10.1017/S0027763000000908 Published online by Cambridge University Press

for the minimal number of generators of certain monomial ideals in polynomial rings.

In Appendix we reproduce another theorem of Rees which is basic in this paper and is used frequently, sometimes as taken for granted. (cf. Remark 2 (i), § 1).

I wish to express my deepest thanks to Professor D. Rees for his new ideas and new problems and for many discussions throughout his stay in Nagoya University early in the year 1983.

Special thanks are also due to Professor H. Yamada who ingeniously pointed out that a theorem of Mordell falls in with the situation of Section 4. Finally I would like to thank Professor H. Matsumura for influential advices and encouragement.

In what follows l stands for length, and μ and τ are respectively the minimal number of generators and the type of an ideal, i.e., $\mu(\mathfrak{a}) = l(\mathfrak{a}/\mathfrak{a}\mathfrak{m})$ and $\tau(\mathfrak{a}) = l(\mathfrak{a}:\mathfrak{m}/\mathfrak{a})$.

§ 1. Definitions and Notation

DEFINITION 1. Let $(R, \mathfrak{m})$ be a local ring, and put $R' = R(X_1, \cdots, X_d)$, the ring of quotients of the polynomial ring by the multiplicative set consisting of polynomials having a unit as a coefficient. Suppose $\mathfrak{m} = (m_1, \cdots, m_d)$ and put $Y = X_1m_1 + \cdots + X_dm_d$. We define Φ, an integer attached to each $\mathfrak{m}$-primary ideal of R, by $\Phi(\mathfrak{a}) = l_{R'}(R'/\mathfrak{a}R' + YR')$.

DEFINITION 2. With the same notation as above, we define $\bar{\mu}$ for $\mathfrak{m}$-primary ideals by $\bar{\mu}(\mathfrak{a}) = \mu_{R'/YR'}(\mathfrak{a}R' + YR'/YR')$.

DEFINITION 3. Let R be a local ring and $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal. We define Ψ by $\Psi(\mathfrak{a}) = \mathrm{Max}\,\{\mu(\mathfrak{b}) \mid \mathfrak{b} \supset \mathfrak{a}\}$.

DEFINITION 4. Let $(R, \mathfrak{m})$ be a local ring. (1) Assume first the residue field $R/\mathfrak{m}$ is infinite. Then an ideal $\mathfrak{a}$ of R is said to be $\mathfrak{m}$-full if there exists an element $y \in R$ such that $\mathfrak{a}\mathfrak{m}: y = \mathfrak{a}$. (2) When $R/\mathfrak{m}$ is not necessarily infinite, let R' be a local ring which is faithfully flat over R with infinite residue field and with $\mathfrak{m}R'$ as the maximal ideal. Then an ideal $\mathfrak{a} \subset R$ is called $\mathfrak{m}$-full if $\mathfrak{a}R'$ is "$\mathfrak{m}R'$-full" in the sense of (1).

Remark 1. We will consider mostly $\mathfrak{m}$-primary $\mathfrak{m}$-full ideals. However, $\mathfrak{m}$-full ideals are not necessarily $\mathfrak{m}$-primary. Easy examples are prime ideals. In fact if $\mathfrak{p}$ is a prime ideal, then it is an associated prime of $\mathfrak{m}\mathfrak{p}$, hence there is y such that $\mathfrak{m}\mathfrak{p}: y = \mathfrak{p}$.

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Remark 2. (i) It can be proved that $\Phi(\mathfrak{a}) \leqq l_R(R/\mathfrak{a} + yR)$ for any $y \in \mathfrak{m}$. When the equality holds, we will say that y is a *general element* for $\mathfrak{a}$. It will be proved that if $\mathfrak{a}\mathfrak{m}: y = \mathfrak{a}$ then y is a general element for $\mathfrak{a}$. If the residue field is infinite, a general element exists for any $\mathfrak{m}$-primary ideal or even for any finite number of $\mathfrak{m}$-primary ideals. For details see Appendix.

(ii) It is easy to see that $\Psi(\mathfrak{a}) < \infty$. Indeed we have $\mu(\mathfrak{b}) \leqq \mu(\mathfrak{a}) + l(\mathfrak{b}/\mathfrak{a})$, for any ideal $\mathfrak{b}$ containing $\mathfrak{a}$. Hence $\Phi(\mathfrak{a}) \leqq \mu(\mathfrak{a}) + l(R/\mathfrak{a})$. (cf. Theorem 1.)

§ 2. The inequality $\Psi \leqq \Phi + \bar{\mu}$

THEOREM 1. *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal. Then it holds that $\mu(\mathfrak{a}) \leqq \Psi(\mathfrak{a}) \leqq \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$.*

Proof. Only the second inequality requires proof. We prove it by induction on Φ. As in Definition 1, put $R' = R(X_1, \cdots, X_d)$, $\mathfrak{m} = (m_1, \cdots, m_d)$, $Y = X_1m_1 + \cdots + X_dm_d$, where $d = \mu(\mathfrak{m})$. Let $^{-}$ denote the composition of the natural maps $R \to R' \to R'/YR'$. Assume first $\Phi(\mathfrak{a}) = 1$. This means that $\bar{\mathfrak{a}} = \bar{\mathfrak{m}}$. It follows that $\mathfrak{a}$ contains elements $a_1, a_2, \cdots, a_{d-1} \in R$, such that $(a_1, a_2, \cdots, a_{d-1}, Y)R' = \mathfrak{m}R'$. In particular $\bar{\mu}(\mathfrak{a}) = d - 1$. Now if $\mathfrak{b}$ is any ideal containing $\mathfrak{a}$, then $\mathfrak{b}/(a_1, \cdots, a_{d-1})R$ is a principal ideal because it is an ideal of the ring $R/(a_1, \cdots, a_{d-1})R$ in which the maximal ideal is generated by one element. Hence $\mu(\mathfrak{b}) \leqq d$. Thus $\Psi(\mathfrak{a}) \leqq d$, and we are done with the case $\Phi = 1$. Now assume the inequality to be true for all $\mathfrak{b}$ with $\Phi(\mathfrak{b}) \leqq n - 1$, and assume $\Phi(\mathfrak{a}) = n$. Let $\mathfrak{b}$ be any ideal containing $\mathfrak{a}$. It suffices to show that $\mu(\mathfrak{b}) \leqq \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$. First assume that $\Phi(\mathfrak{b}) < n$. Then by induction hypothesis we have $\mu(\mathfrak{b}) \leqq \Phi(\mathfrak{b}) + \bar{\mu}(\mathfrak{b})$. It is easy to see that $\bar{\mu}(\mathfrak{b}) \leqq l(\bar{\mathfrak{b}}/\bar{\mathfrak{a}}) + \bar{\mu}(\mathfrak{a}) = \Phi(\mathfrak{a}) - \Phi(\mathfrak{b}) + \bar{\mu}(\mathfrak{a})$. Combining these together we have $\mu(\mathfrak{b}) \leqq \Phi(\mathfrak{a}) + \mu(\mathfrak{a})$. Next assume that $\Phi(\mathfrak{b}) = \Phi(\mathfrak{a})$, i.e., $\bar{\mathfrak{b}} = \bar{\mathfrak{a}}$. Let $r = \bar{\mu}(\mathfrak{b})$. Then we may choose a minimal generating set of $\mathfrak{b}R'$ as follows:

$$\mathfrak{b}R' = (z_1, \cdots, z_r, Yx_1, \cdots, Yx_t)R',$$
$$\bar{\mathfrak{b}} = (\bar{z}_1, \cdots, \bar{z}_r)\bar{R},$$

and

$$\mu(\mathfrak{b}) = r + t.$$

(Note $\mu(\mathfrak{b}) = \mu(\mathfrak{b}R')$ since $R \to R'$ is faithfully flat). We want to show that $t \leqq \Phi(\mathfrak{a})$. Obviously $x_i \in \mathfrak{b}R' : Y$ for each i. Denote the images of $x_1, \cdots, x_t$

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in $\mathfrak{b}R' \colon Y/\mathfrak{b}R'$ by $x_1^*, \cdots, x_t^*$. From the exact sequence

$$0 \longrightarrow R'/\mathfrak{b}R' \colon Y \xrightarrow{Y} R'/\mathfrak{b}R' \longrightarrow R'/\mathfrak{b}R' + YR' \longrightarrow 0$$

we have $l_{R'}(\mathfrak{b}R' \colon Y/\mathfrak{b}R') = \Phi(\mathfrak{b}) = n$. Suppose $t > n$. Then $\{x_1^*, \cdots, x_t^*\}$ cannot be a minimal generating set of the ideal they generate in the ring $R'/\mathfrak{b}R'$. Thus there are elements $a_1, \cdots, a_t \in R'$, at least one a unit, such that $a_1x_1^* + \cdots + a_tx_t^* = 0$, i.e., $a_1x_1 + \cdots + a_tx_t \in \mathfrak{b}R'$. Hence $a_1(Yx_1) + \cdots + a_t(Yx_t) \in Y\mathfrak{b}R' \subset \mathfrak{m}\mathfrak{b}R'$. This contradicts the fact that we have chosen Yx_i, $i = 1, \cdots, t$ as a part of a minimal generating set of $\mathfrak{b}R'$. Hence $t \leqq n$, and $\mu(\mathfrak{b}) = t + r \leqq \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$ as wanted. This completes the proof.

COROLLARY. *Let $(A, \mathfrak{m})$ be an Artin local ring. Then for any ideal I of A and for any element y of $\mathfrak{m}$ we have*

$$\mu(I) \leqq l(A/yA) .$$

Proof. Put $R = A$ and $\mathfrak{a} = 0$ in the theorem.

Remark 1. Theorem 1 and its corollary are in fact equivalent. To derive Theorem 1 from Corollary observe that if we put $A = R/\mathfrak{a}\mathfrak{m}$, then (1) $\Phi_A(0) = \Phi_R(\mathfrak{a}\mathfrak{m})$ and (2) $\Psi_A(0) = \Psi_R(\mathfrak{a})$. In fact (1) is obvious. For (2) it suffices to prove that $\Psi_A(0) \leqq \Psi_R(\mathfrak{a})$, the other inequality being obvious. By definition there exists an ideal $\mathfrak{b}$ of R such that $\mu(\mathfrak{b} + \mathfrak{a}\mathfrak{m}/\mathfrak{a}\mathfrak{m}) = \Psi_A(0)$. Consider $\mu(\mathfrak{b} + \mathfrak{a})$. This is equal to $l(\mathfrak{b} + \mathfrak{a}/\mathfrak{b}\mathfrak{m} + \mathfrak{a}\mathfrak{m}) \geqq l(\mathfrak{b} + \mathfrak{a}\mathfrak{m}/\mathfrak{b}\mathfrak{m} + \mathfrak{a}\mathfrak{m}) = \mu(\mathfrak{b} + \mathfrak{a}\mathfrak{m}/\mathfrak{a}\mathfrak{m})$. Hence $\Psi_A(0) \leqq \Psi_R(\mathfrak{a})$ as wanted.

One might ask when does the equality $\Psi_A(0) = \Phi_A(0)$ hold. This will be discussed in [4].

THEOREM 2. *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal. Then the following conditions are equivalent.*

(i) *$\mathfrak{a}$ is $\mathfrak{m}$-full.*

(ii) *$\tau(\mathfrak{a}) = \Phi(\mathfrak{a})$ and $\mu(\mathfrak{a}) = \tau(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$.*

(iii) *$\mu(\mathfrak{a}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$.*

(iv) *$\mu(\mathfrak{a}) = \Phi(\mathfrak{a}\mathfrak{m})$.*

For proof we need an easy

LEMMA 1. *If $\mathfrak{a}\mathfrak{m} \colon y = \mathfrak{a}$ then $\mathfrak{a} \colon \mathfrak{m} = \mathfrak{a} \colon y$.*

Proof. Obviously $\mathfrak{a} \colon \mathfrak{m} \subset \mathfrak{a} \colon y$, while $\mathfrak{a} \colon \mathfrak{m} = (\mathfrak{a}\mathfrak{m} \colon y) \colon \mathfrak{m} = (\mathfrak{a}\mathfrak{m} \colon \mathfrak{m}) \colon y \supset \mathfrak{a} \colon y$.

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Proof of Theorem 2. Let $^{-}$ denote the natural map $R \to R'/YR'$ as before. Then, by definition, $\Phi(\mathfrak{am}) = l(\bar{R}/\overline{\mathfrak{am}}) = l(\bar{R}/\bar{\mathfrak{a}}) + l(\bar{\mathfrak{a}}/\overline{\mathfrak{am}}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$. Hence (iii) and (vi) are equivalent. We prove (i) implies (ii). Since μ, τ, $\bar{\mu}$, and Φ are all stable under a faithfully flat extenstion, we may assume $R/\mathfrak{m}$ is infinite. Choose y such that $\mathfrak{am}: y = \mathfrak{a}$ and consider the exact sequences

$$(*) \qquad 0 \longrightarrow R/\mathfrak{a}: y \xrightarrow{y} R/\mathfrak{a} \longrightarrow R/\mathfrak{a} + yR \longrightarrow 0$$

and

$$(**) \qquad 0 \longrightarrow R/\mathfrak{am}: y \xrightarrow{y} R/\mathfrak{am} \longrightarrow R/\mathfrak{am} + yR \longrightarrow 0\,.$$

From (*) and from Lemma 1, we have $l(R/\mathfrak{a} + yR) = l(\mathfrak{a}: y/\mathfrak{a}) = l(\mathfrak{a}: \mathfrak{m}/\mathfrak{a}) = \tau(\mathfrak{a})$. From the similar exact sequence as (*) with $\mathfrak{a}R'$ and Y in place of $\mathfrak{a}$ and y, we have $\Phi(\mathfrak{a}) = l_{R'}(R'/\mathfrak{a}R' + YR') \geqq \tau(\mathfrak{a}R')$. Since $R \to R'$ is faithfully flat, $\tau(\mathfrak{a}) = \tau(\mathfrak{a}R')$, and from Remark 2 (i) it follows that $\tau(\mathfrak{a}) = \Phi(\mathfrak{a})$. This is the first equality of (ii). From (**) we have $l(R/\mathfrak{am} + yR) = l(\mathfrak{am}: y/\mathfrak{am}) = l(\mathfrak{a}/\mathfrak{am}) = \mu(\mathfrak{a})$. Exactly the same argument as for $\Phi(\mathfrak{a}) = \tau(\mathfrak{a})$ shows that $\Phi(\mathfrak{am}) = \mu(\mathfrak{a})$, proving the second equality. The implication (ii) $\Rightarrow$ (iii) is trivial. We have already shown (iii) $\Leftrightarrow$ (iv). It remains to prove (iv) $\Rightarrow$ (i). Note that we may assume $R/\mathfrak{m}$ is infinite. Then we can choose a general element y for $\mathfrak{am}$. By definition $\Phi(\mathfrak{am}) = l(R/\mathfrak{am} + yR)$. From the short exact sequence (**) above, it follows that $l(\mathfrak{am}: y/\mathfrak{am}) = \Phi(\mathfrak{am})$. But $\mu(\mathfrak{a}) = l(\mathfrak{a}/\mathfrak{am})$. Thus (iv) implies $\mathfrak{am}: y = \mathfrak{a}$, i.e., $\mathfrak{a}$ is $\mathfrak{m}$-full.

We will say that an ($\mathfrak{m}$-primary) ideal $\mathfrak{a}$ has the Rees property if $\mu(\mathfrak{a}) \geqq \mu(\mathfrak{b})$ for any ideal $\mathfrak{b}$ containing $\mathfrak{a}$.

THEOREM 3. *Let $(R, \mathfrak{m})$ be a local ring. Then an $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal has the Rees property.*

Proof. By Theorem 2, we have $\mu(\mathfrak{a}) = \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a})$. By Theorem 1, we have $\mu(\mathfrak{a}) = \Psi(\mathfrak{a})$. This is the Rees property.

§ 3. The $\mathfrak{m}$-full ideals in a two dimensional regular local ring

THEOREM 4. *Let $(R, \mathfrak{m})$ be a two dimensional regular local ring. Suppose that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal and n is the integer such that $\mathfrak{a} \subset \mathfrak{m}^n$ and $\mathfrak{a} \not\subset \mathfrak{m}^{n+1}$. Then the following conditions are equivalent.*

(i) *$\mathfrak{a}$ is $\mathfrak{m}$-full.*

(ii) $\mu(\mathfrak{a}) = n + 1$.

(iii) *$\mathfrak{a}$ has the Rees property.*

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Proof. First we prove that $\Phi(\mathfrak{a}) = n$. As before we may assume $R/\mathfrak{m}$ is infinite. Let $y \in \mathfrak{m}$ be any element. Then we have $\mathfrak{m} + yR \supsetneqq \mathfrak{m}^2 + yR \supsetneqq \cdots \supsetneqq \mathfrak{m}^n + yR \supset \mathfrak{a} + yR$, and $l(R/\mathfrak{a} + yR) \geqq n$. Hence in order to prove $\Phi(\mathfrak{a}) = n$ it suffices to show the existence of y such that $l(R/\mathfrak{a} + yR) = n$. The condition $\mathfrak{a} \subset \mathfrak{m}^n$ and $\mathfrak{a} \not\subset \mathfrak{m}^{n+1}$ means that $\mathfrak{a}$ contains an element a such that $a \in \mathfrak{m}^n$ and $a \notin \mathfrak{m}^{n+1}$. Let $\mathfrak{m} = (t_1, t_2)R$. Then a may be written as a homogeneous from in t_1 and t_2 of degree n with coefficients in R, and not all of them in $\mathfrak{m}$. Since $R/\mathfrak{m}$ is infinite, we may assume, after a suitable linear transformation of t_1 and t_2, that a is of the form

$$a = (\text{unit})t_1^n + c_1 t_1^{n-1} t_2 + \cdots .$$

Let $y = t_2$. Then R/yR is a DVR with $(\bar{t}_1)$ as the maximal ideal. Clearly $\mathfrak{a} \equiv t_1^n R \bmod yR$, and we have $l(R/\mathfrak{a} + yR) = n$, as wanted. We note also that $\bar{\mu}(\mathfrak{a}) = 1$, which is clear because $\mathfrak{a} \bmod yR$ is an ideal of a DVR.

Now the equivalence of (i) and (ii) follows immediately from Theorem 2. The implication (i) $\Rightarrow$ (iii) was proved in Theorem 3. Assume (iii). Then $\mu(\mathfrak{a}) \geqq \mu(\mathfrak{m}^n) = n + 1$. On the other hand, $\mu(\mathfrak{a}) \leqq \Phi(\mathfrak{a}) + \bar{\mu}(\mathfrak{a}) \leqq n + 1$, Thus (ii) follows.

Remark 3. The implication (i) $\Rightarrow$ (ii) and (iii) in the theorem is due to Rees. He proved it using Lemma 2 below. Lemma 2 itself was originally proved by Burch homologically ([1] Corollary 2). (We outline her proof below). On the other hand the same lemma is proved, as was done in the proof of the Theorem 4 above, as a consequence of Theorem 1. It gives us a non-homological proof. Thus we may consider Theorem 1 as a broad generalization of Lemma 2.

LEMMA 2 (Burch). *Let $(R, \mathfrak{m})$ be a regular local ring of dimension* 2. *Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R such that $\mathfrak{a} \subset \mathfrak{m}^n$ and $\mathfrak{a} \not\subset \mathfrak{m}^{n+1}$. Then we have $\mu(\mathfrak{a}) \leqq n + 1$.*

Proof. Write a minimal free resolution of $\mathfrak{a}$:

$$0 \longrightarrow R^{\mu-1} \xrightarrow{M} R^{\mu} \longrightarrow \mathfrak{a}\,.$$

Then $\mathfrak{a}$ is generated by the $(\mu - 1) \times (\mu - 1)$-minor determinants of the matix M. Hence $\mathfrak{a} \subset \mathfrak{m}^{\mu-1}$. Hence $\mu - 1 \leqq n$.

§ 4. Application

THEOREM 5 (Rees). *Let $(R, \mathfrak{m})$ be an integrally closed integral domain. Then any integrally closed ideal $\mathfrak{a}$ (not necessarily $\mathfrak{m}$-primary) is $\mathfrak{m}$-full.*

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Proof. Let $(\mathfrak{a}\mathfrak{m})'$ be the integral closure of $\mathfrak{a}\mathfrak{m}$. It suffices to find an element z (in a suitable faithfully flat extension of R) such that $(\mathfrak{a}\mathfrak{m})' : z = \mathfrak{a}$, because, then $\mathfrak{a} \subset \mathfrak{a}\mathfrak{m} : z \subset (\mathfrak{a}\mathfrak{m})' : z = \mathfrak{a}$. Recall that an integrally closed ideal $\mathfrak{a}$ is a finite intersection of valuation ideals. I.e., there are a finite set of discrete valuations v_i of rank 1, non-negative on R, and positive integers e_i such that for $x \in R$,

$$x \in \mathfrak{a} \Longleftrightarrow v_i(x) \geqq e_i \qquad \text{for all } i. \text{ (See [2] p. 353 Theorem 3.)}$$

It is harmless to add some additional v_j (non-negative on R) with $e_j = v_j(\mathfrak{a}) := \mathrm{Min}_{a \in \mathfrak{a}} v_j(a)$. Therefore we can choose v_i $(1 \leqq i \leqq n)$ which work both for $\mathfrak{a}$ and for $(\mathfrak{a}\mathfrak{m})'$, i.e. such that

$$\mathfrak{a} = \{z \in R \mid v_i(z) \geqq v_i(\mathfrak{a}) \text{ for all } i\},$$
$$(\mathfrak{a}\mathfrak{m})' = \{z \in R \mid v_i(z) \geqq v_i((\mathfrak{a}\mathfrak{m})') \text{ for all } i\}.$$

Note that $v_i((\mathfrak{a}\mathfrak{m})') = v_i(\mathfrak{a}\mathfrak{m}) = v_i(\mathfrak{a}) + v_i(\mathfrak{m})$. If $R/\mathfrak{m}$ is finite, we replace R by $R(X)$ (as in Definition 1), and we may assume $R/\mathfrak{m}$ is infinite. Then we can find an element $z \in \mathfrak{m}$ such that $v_i(z) = v_i(\mathfrak{m})$ for all i. Hence $(\mathfrak{a}\mathfrak{m})' : z = \mathfrak{a}$ as desired.

THEOREM 6. *Let $R = k[X_1, \cdots, X_n]$ be the polynomial ring over any field k. Let $a_1 \leqq a_2 \leqq \cdots \leqq a_n$ be fixed positive integers and let $\mathfrak{a}$ be the ideal of R spanned by the set of monomials $\{X_1^{p_1} \cdots X_n^{p_n} \mid p_1/a_1 + p_2/a_2 + \cdots + p_n/a_n \geqq 1\}$. We define $N_i(a_1, \cdots, a_i)$ to be the number of non-negative integer solutions $(p_1, \cdots, p_i)$ in Z^i of the inequalities $0 \leqq p_1/a_1 + \cdots + p_i/a_i < 1$. Then*

$$\mu(\mathfrak{a}) = N_{n-1}(a_1, \cdots, a_{n-1}) + N_{n-2}(a_1, \cdots, a_{n-2}) + \cdots + N_1(a_1) + N_0,$$

with the convention $N_0 = 1$.

Proof. Put $\mathfrak{m} = (X_1, \cdots, X_n)$, $R' = R_{\mathfrak{m}}$, and $\mathfrak{a}' = \mathfrak{a}R_{\mathfrak{m}}$. One sees easily that $\mathfrak{a}\mathfrak{m} : X_n = \mathfrak{a}$. Hence $\mathfrak{a}'$ is $\mathfrak{m}$-full in the local ring R'. By Theorem 2 we have $\mu(\mathfrak{a}') = \Phi(\mathfrak{a}') + \bar{\mu}(\mathfrak{a}')$. Since $\mathfrak{a}$ is a graded ideal of R, $\mu_R(\mathfrak{a}) = \mu_{R'}(\mathfrak{a}')$. Now we calculate $\Phi(\mathfrak{a}')$ and $\bar{\mu}(\mathfrak{a}')$. Since $\mathfrak{a}$ is generated by monomials, $z = X_1 + \cdots + X_n$ is a general element for $\mathfrak{a}$. Put $\bar{R} = R/zR$, and consider $\bar{\mathfrak{a}} = \mathfrak{a}\bar{R}$. If we eliminate X_n, we see that $\bar{\mathfrak{a}}$ is the ideal of $k[X_1, \cdots, X_{n-1}]$ spanned by the monomials $\{X_1^{p_1} \cdots X_{n-1}^{p_{n-1}} \mid p_1/a_1 + \cdots + p_{n-1}/a_{n-1} \geqq 1\}$. Hence by induction on n, $\bar{\mu}(\mathfrak{a}) = N_{n-2} + \cdots + N_0$. For $\Phi(\mathfrak{a}')$ we count the number of the monomials of $k[X_1, \cdots, X_{n-1}]$ which are not contained in $\bar{\mathfrak{a}}$. This is precisely equal to $N_{n-1}(a_1, a_2, \cdots, a_{n-1})$. Q.E.D.

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Remark 4. It is easy to see that $N_1(a) = a$ and $N_2(a, b) = \frac{1}{2}\{(a+1)\cdot(b+1) - \operatorname{GCD}(a, b) - 1\}$. Hence in the case $n = 3$, we have

$$\mu(\mathfrak{a}) = N_2(a_1, a_2) + N_1(a_1) + N_0 = \tfrac{1}{2}\{(a_2+1)(a_1+3) - \operatorname{GCD}(a_1, a_2) - 1\}.$$

As to $N_3(a, b, c)$ there is a theorem due to Mordell describing this number using certain number theoretic function (Dedekind sums) provided that a, b, c are pairwise coprime. (See [3] p. 39 Theorem 5.) For higher dimensional case no results of this kind seem to be available.

Appendix

THEOREM A. *Let $(R, \mathfrak{m})$ be a local ring with residue field $k = R/\mathfrak{m}$. Let $m_1, \cdots, m_t$ be elements of $\mathfrak{m}$ and let $y = \sum x_i m_i$ with $x_i \in R$. Let $X_1, \cdots, X_t$ be indeterminates over R, let $S = R[X_1, \cdots, X_t]$, and let $Y = \sum X_i m_i$, and let R' denotes S localized at $\mathfrak{m}R[X_1, \cdots, X_t]$. Finally let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. Then $l_{R'}(R'/\mathfrak{a}R' + YR') \leqq l_R(R/\mathfrak{a} + yR)$. Further, if k is infinite, then there exists a non-zero radical ideal $\mathfrak{b}$ of $k[X_1, \cdots, X_t]$ such that the equality holds above if and only if the ideal $(X_1 - \bar{x}_1, \cdots X_t - \bar{x}_t)$, does not contain $\mathfrak{b}$. ($\bar{x}_i$ denotes the image of x_i in k.)*

To prove the first part of this Theorem we may assume that R is Artinian and $\mathfrak{a} = 0$. Since $R' = R(X_1, \cdots, X_t) = R(X_1, \cdots, X_{t-1})(X_t)$, the induction on t reduces the assertion to the following

PROPOSITION B. *Let $(R, \mathfrak{m}, k)$ be an Artin local ring, $m_1, m_2 \in \mathfrak{m}$ and let X be an indeterminate and put $R' = R(X)$. Then $l_{R'}(R'/(Xm_1 + m_2)R') \leqq l_R(R/(xm_1 + m_2)R)$, for any x in R.*

Proof. Set $A = R[X]/(Xm_1 + m_2)R[X]$, $B = R(X)/(Xm_1 + m_2)R(X)$, and consider the natural homomorphism $\phi: A \to B$. Let $B = J_0 \supset J_1 \supset \cdots \supset J_n = (0)$ be a composition series of B, and let $I_i = \phi^{-1}(J_i)$. Then we obtain a chain of ideals of A: $I_0 \supset I_1 \supset \cdots \supset I_n$. Let x be any element of R. Then it holds that $I_i : (X - x) = I_i$ for all i, since $(X - x)$ is a unit in B. Hence, by letting $\bar{I}_i =$ image of I_i under the natural map

$$A \longrightarrow A_{R[X]}R[X]/(X - x)R[X] \simeq R/(xm_1 + m_2)R,$$

we obtain a chain of ideals $\bar{I}_0 \supsetneqq \bar{I}_1 \supsetneqq \cdots \supsetneqq \bar{I}_n$. This proves the proposition.

Remark. It is easy to see that in this proposition $l_{R'}(R'/YR')$ depends only on the ideal $(m_1, m_2, \cdots, m_t)$ and not on a particular choice of elements.

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Proof of the second part of Theorem A. As in the first part, we may assume that R is Artinian and $\mathfrak{a} = 0$. Let $S = R[X_1, \cdots, X_t]$ and $Y = m_1X_1 + \cdots + m_tX_t$. We fix a chain of ideals of S:

$$(*) \qquad S = I_0 \supset I_1 \supset \cdots \supset I_n = YS\,,$$

such that $I_i/I_{i+1} \simeq S/P_i$, $P_i \in \operatorname{Spec}(S)$. Since any prime ideal of S contains $\mathfrak{m}S$, we may think $S/P_i = k[X_1, \cdots, X_t]/\mathfrak{p}_i$ for prime ideals $\mathfrak{p}_i$ of $k[X_1, \cdots, X_t]$. By localization at the prime ideal $\mathfrak{m}S$, the chain (*) becomes a chain of ideals in R'. The isomorphisms

$$S/P_i \otimes_S R' \simeq \begin{cases} k(X_1, \cdots, X_t) & \text{if } \mathfrak{p}_i = 0 \\ 0 & \text{if } \mathfrak{p}_i \neq 0 \end{cases}$$

imply

$$(1) \qquad l_{R'}(R'/YR') = \sharp\{i \mid \mathfrak{p}_i = 0\}\,.$$

Next we consider $l_R(R/yR)$. Let $x_1, \cdots, x_t \in R$, $M = (X_1 - x_1, \cdots, X_t - x_t)S$, and let $f\colon S \longrightarrow S/M \simeq R$ be the natural homomorphism. Applying f to the chain (*) above, we have $R = f(I_0) \supset f(I_1) \supset \cdots \supset f(I_n) = yR$. From the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_i/I_{i+1} & \xrightarrow{h_i} & S/I_{i+1} & \longrightarrow & S/I_i & \longrightarrow & 0 \\ & & \wr\!\wr & & & & & & \\ & & S/P_i & & & & & & \end{array}$$

it follows that

$$f(I_i)/f(I_{i+1}) \simeq \begin{cases} k & \text{if } S/P_i \otimes_S S/M \neq 0 \quad \text{and} \quad h_i \otimes_S S/M \neq 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Accordingly

$$(2) \qquad l_R(R/yR) = \sharp\left\{ i \,\middle|\, \begin{array}{l} \text{(a)} \ \ \mathfrak{p}_i \subset (X_1 - \bar{x}_1, \cdots, X_t - \bar{x}_t)k[X_1, \cdots, X_t]\,, \\ \text{(b)} \ \ h_i \otimes S/M \neq 0\,. \end{array} \right\}$$

where $\bar{x}_i$ is the image of x_i in $R/\mathfrak{m} = k$. Let $\mathfrak{b} = \cap_{\mathfrak{p}_i \neq 0}\mathfrak{p}_i$. Suppose a set of elements $x_1, \cdots, x_t \in R$ is such that $(X_1 - \bar{x}_1, \cdots, X_t - \bar{x}_t) \not\supset \mathfrak{b}$. Then only the null ideal can satisfy the condition (a) of the right hand side of (2). Hence the set (1) contains the set (2). I.e., $l_{R'}(R'/YR') \geqq l_R(R/yR)$. The other inequality was proved in Proposition B. Q.E.D.

Following is a generalization of Theorem A.

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THEOREM C. *Let* $(R, \mathfrak{m})$ *be a local ring and* $\mathfrak{a}$ *an arbitrary ideal. Let* R' *and* Y *be as in Theorem* A. *Then*

$$l_{R'}(\mathrm{Ker}\,[Y\colon R'/\mathfrak{a}R' \longrightarrow R'/\mathfrak{a}R']) \leqq l_R(\mathrm{Ker}\,[y\colon R/\mathfrak{a}R \longrightarrow R/\mathfrak{a}R])\,.$$

(*Note that length can be* ∞.)

Proof. We may assume $\mathfrak{a} = 0$. If R is Artinian $l(0\colon y) = l(R/yR)$ as is easily seen from the exact sequence

$$0 \longrightarrow 0\colon y \longrightarrow R \xrightarrow{\cdot y} R \longrightarrow R/yR \longrightarrow 0\,,$$

and this case is contained in Theorem A.

First we assume that the left hand side of the inequality is finite. In this case we have to prove the inequality only when the right hand side is finite. Consider $H = H^0_{\mathfrak{m}}(R)$ and $H' = H^0_{\mathfrak{m}R'}(R')$. We note that $H' = H \otimes_R R'$. Suppose $0\colon y$ is a module of finite length. Then, with the identification $H = \cup_n (0\colon \mathfrak{m}^n)$, it is contained in H, hence we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & 0\colon y & \longrightarrow & R & \xrightarrow{\cdot y} & R & & & \\ & \| & & \cup & & \cup & & & \\ 0 \longrightarrow & 0\colon_H y & \longrightarrow & H & \xrightarrow{\cdot y} & H & \longrightarrow & H/yH & \longrightarrow 0\,. \end{array}$$

Since H is a module of finite length, $l(\mathrm{Ker}\,[y\colon R \to R]) = l(H/yH)$. The same is true for R' and Y and H'. Observe that Theorem A is actually true for H (or for any module of finite length), instead of $R/\mathfrak{a}$ for an $\mathfrak{m}$-primary ideal $\mathfrak{a}$, the proof being valid verbatim. Thus we have settled this case. It remains to prove that if the left hand side is infinite then so is the right hand side. Let $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a shortest primary decomposition of 0 in R. Then we have $0 = \mathfrak{q}_1 R' \cap \cdots \cap \mathfrak{q}_n R'$ as a shortest primary decomposition in R'. The assumption $l(\mathrm{Ker}\,[Y\colon R' \to R']) = \infty$ implies that Y is contained in $\sqrt{\mathfrak{q}_i}R'$ for some i such that $\sqrt{\mathfrak{q}_i}R' \neq \mathfrak{m}R'$. This means that all the elements m_j (which occur in $Y = \sum X_j m_j$) are contained in $\sqrt{\mathfrak{q}_i}$. Therefore $y \in \sqrt{\mathfrak{q}_i}$, and $l(\mathrm{Ker}\,[y\colon R \to R]) = \infty$. Q.E.D.

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https://doi.org/10.1017/S0027763000000908 Published online by Cambridge University Press