FREE PLANES AND COLLINEATIONS

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1. Introduction. Our aim in this paper is to consolidate and extend some of the notions in (**1**; **2**; **5**; **6**) concerning free planes in order to facilitate the study of their collineation groups. An upper bound m_n for the orders of the finite subroups of G_n will be established, where G_n is the collineation group of the free plane F_n of rank $n + 4$. In the process, a result of (**6**) will be generalized. Indeed, m_n will be shown to be the best upper bound for all $n \neq 5$.

In (**2**), a confined configuration is defined. In (**5**), this concept is generalized to a confined configuration over A. We shall define a confinement of A which will differ slightly from each of these concepts. After a brief exposition of the set-theoretic concepts of closure function and intersection class, we shall define two closure functions on an arbitrary incidence structure, one of which nearly coincides with the "skeleton" discussed in (**5**). These closure functions are similar to topological closure operators and will prove useful in obtaining results concerning the finite subgroups of G_n.

2. Basic and free subsets of incidence structures. By an *incidence structure* we mean a quadruple (E, P, L, I) of sets, where P is the set of points, L the set of lines, $E = P \cup L$, and I is a symmetric incidence relation between P and L, i.e., $I \subseteq (P \times L) \cup (L \times P)$. Two distinct points are never incident to two distinct lines. For finite E, $R(E) = 2|E| - (|I|/2)$ is the *rank* of E.

By a collineation of E is meant a permutation α of E which fixes P and L as sets and preserves incidence. The set of collineations of E is a group $G(E)$ called the collineation group of E.

A subset of P is *collinear* provided all of its members are incident to a common line. Likewise, a subset of L is *concurrent* provided its members are all incident to a common point. A *plane* is an incidence structure in which each pair of points is collinear and each pair of lines is concurrent. A plane is *non-degenerate* whenever it contains four points, no three of which are collinear, otherwise, a plane is *degenerate.*

Suppose that E is an incidence structure. Let $E_0 = E$. For each pair of points of E_0 which is not collinear, we add a new line incident to each member of that pair. For a pair of lines which is not concurrent, we add a new point incident to each member of that pair. Let E_1 be the resulting incidence

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structure. For $j \geqq 1$, we add new points and lines to E_j in the same fashion to obtain E_{j+1}. Let $F(E) = \bigcup\{E_j: 0 \leqq j \leqq \infty\}$, and call $F(E)$ the *free completion* of E. $F(E)$ is a plane and is uniquely determined up to isomorphism by E. We call an arbitrary incidence structure E non-degenerate or degenerate depending upon whether $F(E)$ is non-degenerate or degenerate.

For $A \subseteq E$, we associate with A the incidence structure $(A, A \cap P, A \cap L, (A \times A) \cap I)$. For $x \in A$, the *A-degree* of x is the number of elements of A to which x is incident, and we denote this number by $d(x, A)$. We say that A is *complete* in E whenever $x \in E - A$ implies that x is incident to at most one member of A. Let $A_0 = A$ and A_1 denote the union of A with those elements of E which are incident to two or more members of A. Let A_{k+1} denote the union of A_k with the members of E which are incident to at least two members of A_k. Let

$$[A]_E = \bigcup\{A_k: 0 \leqq k \leqq \infty\}.$$

Now $[A]_E$ is the smallest subset of E which contains A and is complete in E. $[A]_E$ is called the E-completion of A. Then set function $[\ ]_{(\ )}$ is monotonic in both arguments. If E is a plane, then $[A]_E$ is a plane, perhaps degenerate, for all $A \subseteq E$.

Suppose that $x \in E$. If $x \in [A]_E$, let

$$s_E(x, A) = \min\{k: x \in A_k\}.$$

If $x \notin [A]_E$, let $s_E(x, A) = \infty$. We call $s_E(x, A)$ the $[A]_E$-step of x.

Let $\mathrm{Fr}(A)$ denote the set of $x \in A$ such that $d(x, A) \leqq 2$. We call $\mathrm{Fr}(A)$ the *fringe* of A. If $\mathfrak{A}$ is a class of incidence structures, then

$$\mathrm{Fr}(\bigcup \mathfrak{A}) \subseteq \bigcup\{\mathrm{Fr}(A): A \in \mathfrak{A}\}.$$

By a *confinement* of A we mean a finite incidence structure K such that $\mathrm{Fr}(K) \subseteq A$. K is a *non-trivial* confinement of A provided $K \not\subseteq A$. A *confined configuration* is a non-trivial confinement of $\emptyset$ (see **2**). In (**5**), a *confined configuration over* A is defined to be an incidence structure K, not necessarily finite, such that $\mathrm{Fr}(K) \subseteq A$. Hence, a confinement of A is simply a finite confined configuration over A.

We call a confinement K of A a *B-confinement* of A whenever $K \subseteq B$. Suppose that K is a B-confinement of A, $A \subseteq A_1$, and $B \subseteq B_1$. Then K is clearly a B_1-confinement of A_1. If $A \subseteq B$ and there exists no non-trivial B-confinement of A, then we say that A *is basic* in B. We can immediately prove that the relation "basic in" is transitive. For, suppose that A is basic in B, B is basic in C, and K is any C-confinement of A. Then K is a C-confinement of B. Since B is basic in C, we have that $K \subseteq B$. Thus, K is a B-confinement of A. It follows that $K \subseteq A$ since A is basic in B. We see that there exists no non-trivial C-confinement of A; thus, A is basic in C. It is also clear that if A is basic in B, then A is basic in D whenever $A \subseteq D \subseteq B$.

We say that A is *free in B* provided the A_m-degree of x is equal to 2 whenever $x \in [A]_B$ and $m = s_B(x, A) \geqq 1$. Our first theorem will establish an equivalent condition that A be free in B. A later theorem will establish the transitivity of the relation "free in". The following constructive lemma will prove helpful.

LEMMA 2.1. *Suppose that $A \subseteq B$ and $s_B(x, A) = m < \infty$. Then there exists an $[A]_B$-confinement K of $A \cup \{x\}$ satisfying*

$$x \in K \subseteq A_m \tag{2.1}$$

and

$$[A \cap K]_K = K. \tag{2.2}$$

If $m \geqq 1$, then K also satisfies

$$d(x, K) \geqq 2 \tag{2.3}$$

and

$$K - A_{m-1} = \{x\}. \tag{2.4}$$

Proof. K is simply an appropriately chosen finite subset of the socle of x over A (see **1** and **4**).

THEOREM 2.1. *A is free in B if and only if A is basic in $[A]_B$.*

Proof. Suppose first that A is free in B. We must show that there exists no non-trivial $[A]_B$-confinement of A. Suppose that K is an $[A]_B$-confinement of A. Since K is finite, we may let

$$m = \max\{s_B(y, A) \colon y \in K\}.$$

Suppose that $m \geqq 1$. Let x be an element of K of $[A]_B$-step equal to m. Since A is free in B and $K \subseteq A_m$, it follows that $d(x, K) \leqq 2$. Thus $x \in \mathrm{Fr}(K)$. This is a contradiction since $x \notin A$, but $\mathrm{Fr}(K) \subseteq A$. It follows that $m = 0$ and $K \subseteq A$. Thus, K is a trivial confinement of A; therefore, A is basic in $[A]_B$.

Suppose, conversely, that A is basic in $[A]_B$. If A is not free in B, then there exists $x \in A_m$ such that $d(x, A_m) \geqq 3$, where $m = s_B(x, A)$. Choosing $z_1, z_2, z_3 \in A_m$, $z_i\ I\ x$, we let K_i be the $[A]_B$-confinement of $A \cup \{z_i\}$ guaranteed by Lemma 2.1. Then $K_1 \cup K_2 \cup K_3 \cup \{x\} = K$ is a non-trivial $[A]_B$-confinement of A, a contradiction. Hence A is free in B.

We need another lemma to establish the transitivity of "free in".

LEMMA 2.2. *If A is basic in B and B is free in C, then A is free in C.*

Proof. Since B is basic in $[B]_C$, it follows from the transitivity of "basic in" that A is basic in $[B]_C$. A is basic in $[A]_C$ since $A \subseteq [A]_C \subseteq [B]_C$. Hence A is free in C.

THEOREM 2.2. *If A is free in B, and B is free in C, then A is free in C.*

Proof. Since $[A]_B$ is complete in B and B is free in C, we have that $[A]_B$ is free in C (see **8**). Now A is basic in $[A]_B$ and $[A]_B$ is free in C; thus, by Lemma 2.2, A is free in C.

3. Closure functions and intersection classes. We now present the definitions and basic properties of two set-theoretic concepts which will be applied to incidence structures in § 4. Let $\mathfrak{P}(S)$ denote the family of subsets of the set S. Following (**3**), we define a *closure function* on S to be a function h: $\mathfrak{P}(S) \rightarrow \mathfrak{P}(S)$ satisfying

$$A \subseteq S \Rightarrow A \subseteq h(A), \tag{3.1}$$

$$A \subseteq B \Rightarrow h(A) \subseteq h(B), \tag{3.2}$$

and

$$h^2 = h. \tag{3.3}$$

Note that a closure function h lacks only the property

$$h(A \cup B) = h(A) \cup h(B)$$

of being a closure operator in the usual topological sense. If h is a closure function on S, we let

$$\mathfrak{F}(h) = \{A \subseteq S: h(A) = A\},$$

and we call $\mathfrak{F}(h)$ the *class of fixed sets* of h.

Suppose that $\mathfrak{C} \subseteq \mathfrak{P}(S)$. $\mathfrak{C}$ is called an *intersection class* on S provided

$$S \in \mathfrak{C} \tag{3.4}$$

and

$$\mathfrak{A} \subseteq \mathfrak{C} \Rightarrow \bigcap \mathfrak{A} \in \mathfrak{C}. \tag{3.5}$$

If $\mathfrak{C}$ is an intersection class on S, we associate with $\mathfrak{C}$ the set function $h(\mathfrak{C})$: $\mathfrak{P}(S) \rightarrow \mathfrak{P}(S)$ as follows:

$$h(\mathfrak{C}): A \rightarrow \bigcap \{X: A \subseteq X \in \mathfrak{C}\}$$

for $A \subseteq S$. Here, h is acting as a function with the family of intersection classes as domain. The closure function is not h, but $h(\mathfrak{C})$.

The following theorems connect the notions of closure function and intersection class in a natural way.

THEOREM 3.1. *If h is a closure function on S, then the class of fixed sets $\mathfrak{F}(h)$ is an intersection class on S. Moreover, $h(\mathfrak{F}(h)) = h$.*

Proof. Since h satisfies (3.1), we have that $h(S) = S$. Thus $S \in \mathfrak{F}(h)$. Now suppose that $\mathfrak{A} \subseteq \mathfrak{F}(h)$. For each $A \in \mathfrak{A}$, we have that $\bigcap\mathfrak{A} \subseteq A$. Thus, $h(\bigcap\mathfrak{A}) \subseteq h(A) = A$. It follows that $h(\bigcap\mathfrak{A}) \subseteq \bigcap\mathfrak{A}$. But $\bigcap\mathfrak{A} \subseteq h(\bigcap\mathfrak{A})$ from

(3.1). Hence, $h(\bigcap\mathfrak{A}) = \bigcap\mathfrak{A}$, and therefore, $\bigcap\mathfrak{A} \in \mathfrak{F}(h)$. $\mathfrak{F}(h)$ satisfies (3.4) and (3.5); therefore, $\mathfrak{F}(h)$ is an intersection class on S. It remains to show that $h(\mathfrak{F}(h)) = h$. Suppose that $A \subseteq S$. Let

$$\mathfrak{D} = \{X: A \subseteq X \in \mathfrak{F}(h)\}.$$

Let

$$g = h(\mathfrak{F}(h)).$$

By definition, $g(A) = \bigcap \mathfrak{D}$. Since $h(A) \in \mathfrak{D}$, it follows that $g(A) \subseteq h(A)$. On the other hand, if $X \in \mathfrak{D}$, then $h(A) \subseteq h(X) = X$. Hence, $h(A) \subseteq X$, and therefore, $h(A) \subseteq g(A)$. Thus, $h = g$ and the theorem is proved.

THEOREM 3.2. *If $\mathfrak{F}$ is an intersection class on S, then $h(\mathfrak{F})$ is a closure function on S, and $\mathfrak{F}(h(\mathfrak{F})) = \mathfrak{F}$.*

The proof is in the same spirit as that of Theorem 3.1.

We have established a natural 1-1 correspondence $h \to \mathfrak{F}(h)$ between the closure functions and the intersection classes on a given set S.

4. The basic and free closure functions. We shall show that both the class of basic subsets of an incidence structure E and the class of free subsets are intersection classes on E. Applying the results of §3 we may define corresponding closure functions on E, the free closure function being a slight generalization of the "skeleton" discussed in (**4**).

THEOREM 4.1. *Suppose that E is an incidence structure. The class of basic subsets of E and the class of free subsets of E are intersection classes. Moreover, the basic subsets of E form a subclass of the free subsets.*

Proof. Let $\mathfrak{B}$ be the class of basic subsets and $\mathfrak{C}$ the class of free subsets. Clearly, $E \in \mathfrak{B}$ and $E \in \mathfrak{C}$. Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$. Now suppose that K is an E-confinement of $\bigcap\mathfrak{A}$. Then K is an E-confinement of A for each $A \in \mathfrak{A}$. Since $\mathfrak{A} \subseteq \mathfrak{B}$, it follows that $K \subseteq A$ for every $A \in \mathfrak{A}$. Hence $K \subseteq \bigcap \mathfrak{A}$. Thus, $\bigcap\mathfrak{A} \in \mathfrak{B}$ and $\mathfrak{B}$ is an intersection class. Likewise, suppose that $\mathfrak{A} \subseteq \mathfrak{C}$ and let $V = \bigcap\mathfrak{A}$. Let K be a $[V]_E$-confinement of V. Then for each $A \in \mathfrak{A}$, K is an $[A]_E$-confinement of A, since $V \subseteq A$. It follows that $K \subseteq A$ for every $A \in \mathfrak{A}$; therefore $K \subseteq V$. Thus, $V \in \mathfrak{C}$ and $\mathfrak{C}$ is an intersection class. For $B \in \mathfrak{B}$, B is basic in E. Since $[B]_E \subseteq E$, we have B basic in $[B]_E$. Thus, B is free in E. It follows that $\mathfrak{B} \subseteq \mathfrak{C}$.

For an incidence structure E we let b_E and f_E be the closure functions on E associated with the intersection classes of basic and free subsets of E, respectively. That is, $b_E = h(\mathfrak{B})$ and $f_E = h(\mathfrak{C})$ in the notation of §3. If $B \subseteq E$, then the functions b_B and f_B are defined in the same way for the incidence structure B. If $A \subseteq B$, we call $b_B(A)$ the *B-basic closure* of A and $f_B(A)$ the *B-free* *closure* of A. It should be noted that $[\ ]_B$ is also a closure function on B. The following relations among the three functions f, b, and $[\ ]$ are not difficult to prove:

(4.1) The $[A]_B$-free closure of A is $f_B(A)$;

(4.2) The $[A]_B$-basic closure of A is $f_B(A)$;

(4.3) The B-completion of $f_B(A)$ is $[A]_B$.

Suppose that $A \subseteq B$. By the total *B-confinement* of A we mean the union of all of the B-confinements of A, and we denote this by $K_B(A)$. Suppose that $x \in K_B(A)$, but $x \notin b_B(A)$. There exists a B-confinement K of A which contains x. Thus, K is a B-confinement of $b_B(A)$, but $K \not\subseteq b_B(A)$. This is a contradiction since $b_B(A)$ is basic in B. We conclude that $x \in b_B(A)$, and $K_B(A) \subseteq b_B(A)$.

THEOREM 4.2. *If* $A \subseteq B$, *then* $b_B(A) = K_B(A)$.

Proof. It suffices to show that $b_B(A) \subseteq K_B(A)$. For this we need only show that $A \subseteq K_B(A)$ and $K_B(A)$ is basic in B. Suppose that $a \in A$. Then $\{a\}$ is a B-confinement of A; therefore, $a \in K_B(A)$. Thus, $A \subseteq K_B(A)$. Now suppose that K is a B-confinement of $K_B(A)$. For $x \in \mathrm{Fr}(K) \subseteq K_B(A)$ there exists a B-confinement K_x of A such that $x \in K_x \subseteq K_B(A)$, by definition of $K_B(A)$. Let

$$L = K \cup \bigcup\{K_x : x \in \mathrm{Fr}(K)\}.$$

We have that

$$\mathrm{Fr}(L) \subseteq \mathrm{Fr}(K) \cup \bigcup\{\mathrm{Fr}(K_x) : x \in \mathrm{Fr}(K)\}.$$

Since $\mathrm{Fr}(K_x) \subseteq A$, we have that $\mathrm{Fr}(L) \subseteq \mathrm{Fr}(K) \cup A$. For $x \in \mathrm{Fr}(K)$, either $x \in A$ or $d(x, L) \geqq d(x, K_x) \geqq 3$. Thus, $\mathrm{Fr}(L) \subseteq A$, and L is a B-confinement of A. Hence, $L \subseteq K_B(A)$. Since $K \subseteq L$, it follows that $K \subseteq K_B(A)$, and $K_B(A)$ is basic in B.

5. Finite subgroups of G_n. We shall show that a finite subgroup H of G_n fixes, setwise, some finite free generator B of F_n. The representation of H in $G(A)$ defined by $\alpha \rightarrow \alpha | A$, $\alpha \in H$, is 1-1 since A generates F_n. Thus, H may be considered as a subgroup of $G(A)$. A lemma will enable us to restrict the discussion to a subclass of these generators. Ultimately, an extensive case analysis of a family of finite incidence structures is necessary in order to establish an upper bound for the orders of the finite subgroups.

E is called a *free extension* of A provided A is free in E, and A generates E. If E is a free extension of A, we call A a *free generator* of E.

Let P_n be the incidence structure consisting of the n points $x_1, x_2, \ldots, x_n$ and the line y with x_j incident to y for $1 \leqq j \leqq n - 2$. For $n \geqq 4$, we let $F_n = F(P_n)$ and call F_n the *free plane of rank* $n + 4$. Then $G_n = G(F_n)$.

An incidence structure E is open provided $\emptyset$ is basic in E. If A is a free generator of E and A is open, then E is open. For, suppose that K is an E-confinement of $\emptyset$. Then K is an E-confinement of A. But A is basic in $[A]_E$ and $[A]_E = E$. Thus, $K \subseteq A$. Now K is an A-confinement of $\emptyset$. Since A is open, $K \subseteq \emptyset$. Thus, $\emptyset$ is basic in E, and therefore E is open.

The finite free generators of F_n are precisely those finite subsets $A \subseteq F_n$ such that $R(A) = n + 4$ and A generates F_n (see **2** and **8**). If A is any finite, open, non-degenerate incidence structure of rank $n + 4$, then $F(A)$ is isomorphic to F_n (see **2**).

Henceforth, we shall simply write s, $[\ ]$, f, and b in place of the functions s_{F_n}, $[\ ]_{F_n}$, f_{F_n}, and b_{F_n}. The following lemma gives us some fundamental information regarding the basic and free closure functions on F_n.

LEMMA 5.1. *If A is a finite subset of F_n, then $b(A)$ and $f(A)$ are finite.*

Proof. Let P be a finite free generator of F_n. Since A is finite and $A \subseteq [P]$, we may let

$$r = \max\{s(a, P) \colon a \in A\}.$$

Since A_r is a free generator of F_n, A_r is basic in F_n. Since $A \subseteq A_r$, we have that $b(A) \subseteq A_r$. But A_r is finite since P is finite. Thus, $b(A)$ is finite. Since $f(A) \subseteq b(A)$, $f(A)$ is also finite.

The next lemma establishes the commutativity between a collineation and the basic and free closure functions.

LEMMA 5.2. *If $\alpha \in G_n$ and $A \subseteq F_n$, then $b(A)\alpha = b(A\alpha)$ and $f(A)\alpha = f(A\alpha)$.*

Proof. Suppose that $x \in b(A)\alpha$. Then $x = y\alpha$ for some $y \in b(A)$. Thus, there exists an F_n-confinement K of A such that $y \in K$, by Theorem 4.2. Hence, $y\alpha \in K\alpha$, thus $x \in K\alpha$. But $K\alpha$ is an F_n-confinement of $A\alpha$; thus, $x \in b(A\alpha)$. Hence $b(A)\alpha \subseteq b(A\alpha)$. Conversely, suppose that $x \in b(A\alpha)$. There exists an F_n-confinement K of $A\alpha$ such that $x \in K$. Thus, $x\alpha^{-1} \in K\alpha^{-1}$. But $K\alpha^{-1}$ is an F_n-confinement of $(A\alpha)\alpha^{-1} = A$. Thus, $x\alpha^{-1} \in b(A)$, and therefore, $x \in b(A)\alpha$. It follows that $b(A\alpha) \subseteq b(A)\alpha$, and therefore $b(A)\alpha = b(A\alpha)$. The proof of the analogous result for f is similar.

The following theorem lays the groundwork for our discussion of the finite subgroups of G_n. This theorem is a generalization of a result of Lippi in (**6**) concerning a single finite collineation.

THEOREM 5.1. *If H is a finite subgroup of G_n, then there exists a finite free generator A of F_n such that $AH = A$.*

Proof. Let P be a finite free generator of F_n. Let $A_0 = PH$. We claim that $A = b(A_0)$ satisfies the requirements of the theorem. Clearly, A is finite since P and H are finite. From Lemma 5.1, it follows that A is finite. Since $A_0 \subseteq A$ and A_0 generates F_n, we know that A generates F_n. Thus, from Theorem 2.1, A is a finite free generator of F_n. For $\alpha \in H$, we have that $A\alpha = b(A_0)\alpha$.

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Applying Lemma 5.2 we obtain $b(A_0)\alpha = b(A_0\alpha) = b((PH)\alpha) = b(P(H\alpha)) = b(PH) = b(A_0) = A$. Thus, $A\alpha = A$ for each $\alpha \in H$, and therefore, $AH = A$.

We see that every finite subgroup of G_n occurs as a subgroup of some $G(A)$, where A is a finite, open, non-degenerate incidence structure of rank $n + 4$. We now look for a smaller class of incidence structures whose collineation groups represent all finite subgroups of G_n in the same manner.

LEMMA 5.3. *Suppose that A is a finite incidence structure. Let S be the set of points (lines) of A of degree 2. Then $DG(A) = D$, where $D = A - S$.*

Proof. Since collineations preserve degree, $S\alpha = S$ for all $\alpha \in G(A)$. Thus, $D\alpha = D$ for all $\alpha \in G(A)$.

THEOREM 5.2. *If A is a finite incidence structure, then there exists a finite free generator D of A such that $DG(A) = D$ and D contains no elements of degree 2.*

Proof. Let D_1 be the result of removing from A its points of degree 2. Let D_2 be the result of removing from D_1 its lines of degree 2. We repeat this procedure obtaining a decreasing sequence of incidence structures A, D_1, D_2, $\ldots$. Since A is finite, there exists an integer r such that $D_r = D_{r+1} = \ldots$. Let $D = D_r$. Clearly, D contains no elements of degree 2. Repeated application of Lemma 5.3 proves the remaining condition required of D by the theorem.

Suppose that H is a finite subgroup of G_n. Let A be the finite free generator of F_n guaranteed by Theorem 5.1. Let D be the free generator of A guaranteed by Theorem 5.2. Then D is a finite free generator of F_n; thus, the representation $H \to G(D)$ defined by $\alpha \to \alpha|D$ is 1-1. Thus, we have the group H embedded in the collineation group of a finite free generator of F_n which contains no elements of degree 2. Next we consider the result of removing elements of degree 1.

LEMMA 5.4. *Suppose that A is a finite incidence structure. Let T be the set of points (lines) of A of degree 1. Let $t = |T|$. Then $DG(A) = D$, where $D = B - T$, and $|G(A)| \leqq t!|G(D)|$.*

Proof. As in the proof of Lemma 5.3, $T\alpha = T$ for all $\alpha \in G(A)$ since collineations preserve degree. Thus, $D\alpha = D$ for all $\alpha \in G(A)$. Let $S(T)$ denote the group of all permutations of T. Now consider the representation $G(A) \to S(T) \times G(D)$ defined by $\alpha \to (\alpha|T, \alpha|D)$. Since this representation is 1-1, we have that $|G(A)| \leqq |S(T) \times G(D)| = |S(T)|\,|G(D)| = t!|G(D)|$, and the lemma is proved.

A finite, non-degenerate, open incidence structure A is said to be of *type U* provided: either A consists only of elements of degree 0, or A contains no elements of degree 2, and A becomes degenerate upon removal of its points of degree 1. The following theorem relates the order of a finite subgroup of G_n to that of a collineation group of a type U incidence structure.

THEOREM 5.3. *If H is a finite subgroup of G_n, then there exists an incidence structure A of type U such that $|H| \leqq t!|G(A)|$, where $R(A) = n - t + 4$ and $0 \leqq t \leqq n - 4$.*

Proof. Our proof is based upon alternate applications of Theorem 5.2 and Lemma 5.4 to the incidence structure guaranteed by Theorem 5.1. Let B be the generator of F_n guaranteed by Theorem 5.1. Then $|H| \leqq |G(B)|$. Let A_1 be the subset of B guaranteed by Theorem 5.2. Then $|G(B)| \leqq G(A_1)|$. Applying Lemma 5.4 to A_1 we obtain T_1. Letting t_1 be the associated integer of Lemma 5.4, we have that $R(T_1) = R(A_1) - t_1 = n - t_1 + 4$, and $|G(A_1)| \leqq t!|G(T_1)|$. Now we apply Theorem 5.2 to T_1 to obtain A_2. We then apply Lemma 5.4 to A_2 to obtain T_2. Continuing in this fashion we obtain a sequence A_1, T_1, A_2, T_2, ... $R(A_{k+1}) = R(T_k) = R(A_k) - t_k = n - (t_1 + t_2 + \ldots + t_k) + 4$, and $|H| \leqq |G(A_1)| \leqq t_1!|G(A_2)| \leqq \ldots \leqq t_1!\, t_2! \ldots t_k!\, |G(A_{k+1})|$. This process terminates when A_k contains no elements of degree 1. Suppose that this occurs at $k = s$. We let $C = A_r$, where A_r is the last non-degenerate incidence structure in the sequence, and we let $t = t_1 + \ldots + t_{r-1}$. Then

$$|H| \leqq t_1! \ldots t_{r-1}!|G(A_r)| \leqq t!|G(A_r)|.$$

Likewise, $R(A_r) = n - (t_1 + \ldots + t_{r-1}) + 4 = n - t - 4$. Clearly, $0 \leqq t \leqq n - 4$ since $R(C) \geqq 8$. If $r = s$, then C consists only of elements of degree 0, and we let $A = C$. If not, then C contains no elements of degree 2 and becomes degenerate upon removal either of points of degree 1 or of lines of degree 1. If C becomes degenerate upon removal of its points of degree 1, then we again let $A = C$. If not, then we let A be the dual of C. In any case, A is of type U and satisfies the requirements of the theorem.

Let $\mathfrak{U}$ be the class of type U incidence structures, and let

$$\mathfrak{U}_r = \{A \in \mathfrak{U}: R(A) = r\}.$$

Now let

$$u_r = \max\{|G(A)|: A \in \mathfrak{U}_r\}.$$

Suppose that H is a finite subgroup of G_n. From Theorem 5.3 we have that $|H| \leqq t!\, u_{n-t+4}$ for some integer t, $0 \leqq t \leqq n - 4$. Letting

$$m_n = \max\{t!\, u_{n-t+4}: 0 \leqq t \leqq n - 4\},$$

we see that m_n is an upper bound for the orders of the finite subgroups of G_n. We shall discover that for $n \neq 5$, m_n is in fact the best such upper bound.

After calculating u_r we shall see that $m_5 = u_8$, and $m_n = u_{n+4}$ for $n \neq 5$. Our major task is the calculation of u_r. Let $\mathfrak{Z}$ be the subclass of $\mathfrak{U}$ consisting of those members of $\mathfrak{U}$ which contain only elements of degree 0. Let $\mathfrak{W} = \mathfrak{U} - \mathfrak{Z}$. If $A \in \mathfrak{W}$, then A contains no elements of degree 2, and A^* is degenerate, where A^* is the result of removing from A its points of degree 1. Let $\mathfrak{V}$ be the class of V such that $V = A^*$, for some $A \in \mathfrak{W}$. An incidence structure $V \in \mathfrak{V}$ has the following properties: V is degenerate, V contains neither points of degree 1 nor points of degree 2, and a non-degenerate incidence structure, namely some $A \in \mathfrak{W}$, may be obtained from V by adding some new points each of which is incident to exactly one line of V.

Let z_r and w_r denote the maxima of the orders of the collineation groups of rank r members of $\mathfrak{Z}$ and $\mathfrak{W}$, respectively. Then $u_r = \max\{z_r, w_r\}$.

First, we shall compute z_r. If $A \in \mathfrak{Z}$, then A consists of a points and b lines with no incidences. $R(A) = 2(a + b)$, and we see that

$$z_r = \max\{a!b!: 2(a + b) = r\}.$$

Hence,

$$z_r = \begin{cases} (\frac{1}{2}r)! & \text{for } r \text{ even,} \\ 0 & \text{for } r \text{ odd.} \end{cases}$$

In order to compute w_r we need a survey of the members of the class $\mathfrak{W}$. This, in turn, requires a survey of the class $\mathfrak{V}$. We shall decompose $\mathfrak{V}$ into nine disjoint classes $\mathfrak{V}(1), \ldots, \mathfrak{V}(9)$. This decomposition of $\mathfrak{V}$ induces a decomposition of $\mathfrak{W}$ as follows. Let $\mathfrak{W}(i)$ be the subclass of $\mathfrak{W}$ consisting of those $A \in \mathfrak{W}$ for which $A^* \in \mathfrak{V}(i)$. Now $\mathfrak{W}$ is decomposed into the classes $\mathfrak{W}(1), \ldots, \mathfrak{W}(9)$. Clearly,

$$w_r = \max\{w_r(i): 1 \leqq i \leqq 9\},$$

where $w_r(i)$ is the maximum of the orders of the collineation groups of rank r members of $\mathfrak{W}(i)$.

If $V \in \mathfrak{V}$, then $F(V)$ is a degenerate plane. Hence, all members of $\mathfrak{V}$ may be found among the free generators of degenerate planes. A discussion of degenerate planes will prove helpful.

Let $\mathfrak{D}$ denote the class of degenerate planes. Let $\mathfrak{D}(1) = \{DP\}$, where DP denotes the plane consisting of a single point x. Let $\mathfrak{D}(2) = \{DL\}$, where DL denotes the planes consisting of a single line y. In (5) it has been shown that every other degenerate plane contains a point x and a line y such that every line except perhaps y is incident to x, and every point except perhaps x is incident to y; x and y may or may not be incident to one another. Let $\mathfrak{D}(3)$ be the class of such planes in which x and y are incident, and let $\mathfrak{D}(4)$ be the class of such planes in which x and y are not incident. Let $D_3(a, b)$ be the plane containing the $a + 1$ points $x, x_1, x_2, \ldots, x_a$ and the $b + 1$ lines $y, y_1, y_2, \ldots, y_b$ with x incident to y, x_i incident to y for $1 \leqq i \leqq a$, and x incident to y_j for $1 \leqq j \leqq b$. Then

$$\mathfrak{D}(3) = \{D_3(a, b): a, b \geqq 0\}.$$

Let $D_4(c)$ be the plane containing the $c + 1$ points $x, x_1, x_2, \ldots, x_c$ and the $c + 1$ lines $y, y_1, y_2, \ldots, y_c$ with x incident to y_i, y incident to x_i, and x_i incident to y_i, for $1 \leqq i \leqq c$. Then

$$\mathfrak{D}(4) = \{D_4(c): c \geqq 0\},$$

and therefore

$$\mathfrak{D} = \bigcup\{\mathfrak{D}(i): 1 \leqq i \leqq 4\}.$$

Since $F(V) \in \mathfrak{D}$ for all $V \in \mathfrak{V}$, we may obtain any member of $\mathfrak{V}$ by successively deleting elements of degree 2 from some member of $\mathfrak{D}$. We shall exhaust the class $\mathfrak{V}$ by considering all such possible successions of deletions from members of $\mathfrak{D}$. We recall that a member V of $\mathfrak{V}$ contains neither points of degree 1 nor degree 2, and that a member of $\mathfrak{W}$ may be obtained by adding points incident to a single line of V. In particular, V must contain at least one line.

First consider $D(1)$. No deletions may be made from DP; thus, DP itself is the only candidate for membership in $\mathfrak{V}$. But DP contains no lines; thus, $DP \notin \mathfrak{V}$. Considering $\mathfrak{D}(2)$, we see that although DL contains the line y, no matter how many points are added incident to y, the resulting incidence structure is degenerate. Hence, $DL \notin \mathfrak{V}$.

Now consider $\mathfrak{D}(3)$. No deletions may be made from $D_3(0, 0)$. D_3 $(0, 0)$ contains a point of degree 1; thus, it is not a member of $\mathfrak{V}$. From $D_3(0, 1)$ we may delete the point x obtaining the plane V_1 consisting of the two lines y, y_1. By adding three points incident to y and three points incident to y_1 we obtain a member of $\mathfrak{W}$. Thus, $V_1 \in \mathfrak{V}$ and we let $\mathfrak{V}(1) = \{V_1\}$.

Consider $D_3(0, b)$, $b \geqq 2$. Since the degree of x is $b + 1 \geqq 3$, no deletions may be made. Again we may obtain a member of $\mathfrak{W}$ by adding sufficiently many points to y and y_1; thus, $D_3(0, b) \in \mathfrak{V}$. Let

$$\mathfrak{V}(2) = \{D_3(0, b) \colon b \geqq 2\}.$$

From $D_3(1, 1)$ we may delete either x or y, but not both. If we delete y, then x still has degree 1; thus $D_3(1, 1) - \{y\} \notin \mathfrak{V}$. If we delete x, then x_1 still has degree 1. We conclude that $D_3(1, 1) - \{x\} \notin \mathfrak{V}$.

$D_3(1, 2)$ contains the point x_1 of degree 1; thus, we must delete y. Now x has degree 2 and therefore we must delete x. Let

$$V_3 = D_3(1, 2) - \{x, y\}.$$

By adding sufficiently many points to y in V_3 we obtain a member of $\mathfrak{W}$. Thus, we let $\mathfrak{V}(3) = \{V_3\}$.

Now consider $D_3(1, b)$, $b \geqq 3$. Since x has degree 1 in $D_3(1, b)$, we must delete y. Let

$$V_4(b) = D_3(1, b) - \{y\}.$$

By adding at least two points to y_1 we obtain a member of $\mathfrak{W}$. Thus, $V_4(b) \in \mathfrak{V}$. Let

$$\mathfrak{V}(4) = \{V_4(b) \colon b \geqq 2\}.$$

Now consider $D_3(1, 0)$. The only element which may be deleted is y. But deletion of y leaves no lines. Thus $D_3(1, 0) - \{y\} \notin \mathfrak{V}$. No deletions may be made from $D_3(a, 0)$, $a \geqq 2$. Moreover, $D_3(a, 0)$ contains points of degree 1; thus, $D_3(a, 0) \notin \mathfrak{V}$.

Consider $D_3(2, 1)$. Since x has degree 2, it must be deleted. This leaves y of degree 2 and x, x_1 of degree 1. Hence, we delete y to obtain $V_5 = D_3(2, 1) - \{x, y\}$. By adding three or more points to y_1 we obtain a member of $\mathfrak{W}$. Thus, $V_5 \in \mathfrak{V}$. Let $\mathfrak{V}(5) = \{V_5\}$.

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$D_3(a, 1)$, $a \geqq 3$, contains the points $x_1, \ldots, x_a$ of degree 1, which cannot be deleted. Neither can y be deleted, since $a \geqq 3$. Even if we delete x, y still has degree $a \geqq 3$. We conclude that $D_3(a, 1)$ yields no new members of $\mathfrak{V}$. From $D_3(a, b)$, $a \geqq 2$, $b \geqq 2$, no deletions are possible. Since $D_3(a, b)$ contains points of degree 1, $D_3(a, b) \notin \mathfrak{V}$. We have now exhausted the class $\mathfrak{D}(3)$.

Turning to $\mathfrak{D}(4)$, consider $D_4(0)$. No deletions are possible. Any new points must be added incident to y, yielding a degenerate incidence structure. $D_4(1)$ is isomorphic to $D_3(1, 1)$ which has already been considered. Now consider $D_4(2)$. In order to obtain a candidate for membership in $\mathfrak{V}$, each point of $D_4(2)$ must either be deleted, or its degree reduced to 0 by the deletion of lines. We may remove all three points x, x_1, x_2 to obtain V_6. By adding one point to each of the lines y_1 and y_2, we obtain a member $\mathfrak{W}$. Let $\mathfrak{V}(6) = \{V_6\}$. Now suppose that the degree of x is to be reduced to 0. Then y_1 and y_2 must be deleted. Now x_1 and x_2 have degree 1; thus, y must be deleted. Since there are no remaining lines, the resulting incidence structure is not in $\mathfrak{V}$.

Now consider $D_4(c)$, $c \geqq 3$. If any line y_i is deleted, then the degree of x_i is reduced to 1. In order to obtain a member of $\mathfrak{V}$, the degree of x_i must be further reduced to 0. This entails deleting the line y. Hence, the degree of y must be reduced to 2. This can be done only by removing all but two of the points x_i. Hence, we may assume that either all x_i are removed or all but two of the x_i are removed. We may delete the c points $x_1, \ldots, x_c$ to obtain $V_7(c)$. By adding a point to y, we obtain a member of $\mathfrak{W}$. Let

$$\mathfrak{V}(7) = \{V_7(c)\colon c \geqq 3\}.$$

On the other hand, we may start by deleting from $D_4(c)$ the $c - 2$ points $x_1, \ldots, x_{c-2}$, and then delete y. Then we delete the two lines y_{c-1} and y_c. In the case $c = 3$, the point x is now of degree 1; thus, we do not have a member of $\mathfrak{V}$. If $c = 4$, x now has degree 2, and we delete x to obtain V_8. V_8 consists of the points x_3, x_4 and the lines y_1, y_2 with no incidences. By adding a point to y_1 in V_8 we obtain a member of $\mathfrak{W}$; thus $V_8 \in \mathfrak{V}$. We let $\mathfrak{V}(8) = \{V_8\}$. If $c \geqq 5$, we let

$$V_9(c-2) = \{x, x_{c-1}, x_c, y_1, y_2, \ldots, y_{c-2}\}.$$

By adding a point to y_1 in $V_9(c-2)$ we obtain a member of $\mathfrak{W}$. Thus, $V_9(c-2) \in \mathfrak{V}$, and we let

$$\mathfrak{V}(9) = \{V_9(c)\colon c \geqq 3\}.$$

We now give two examples of the calculation of the $w_r(i)$. A member of $\mathfrak{W}(1)$ is obtained by adding points incident to lines of a member of $\mathfrak{V}(1)$. $\mathfrak{V}(1)$ contains only V_1, where V_1 consists of the two lines y and y_1. Suppose that we add k points to y and k_1 points to y_1 to obtain $W_1(k, k_1)$. Then $R(W(k, k_1)) = 4 + k + k_1$. In order that $W_1(k, k_1)$ be non-degenerate, we must have $k > 1$, $k_1 > 1$. A member of $\mathfrak{W}$ is a type U incidence structure and has no

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elements of degree 2. Since $d(y, W_1(k, k_1)) = k$ and $d(y_1, W_1(k, k_1)) = k_1$, we must have $k \geqq 3$ and $k_1 \geqq 3$. If $k = k_1$, then

$$|G(W_1(k, k_1))| = 2(k!)^2.$$

If $k \neq k_1$, then

$$|G(W_1(k, k_1))| = k!k_1!.$$

Taking the maximum of $|G(W_1(k, k_1))|$ subject to the restrictions $k + k_1 = r - 4$, $k \geqq 3$, and $k_1 \geqq 3$, we obtain:

$$w_r(1) = \begin{cases} 0 & \text{for } r = 8, 9, \\ 72 & \text{for } r = 10, \\ 1152 & \text{for } r = 12, \\ 3!(r-7)! & \text{otherwise.} \end{cases}$$

A member of $\mathfrak{W}(2)$ is obtained by adding points incident to lines of a member $V_2(b)$, $b \geqq 2$, of $\mathfrak{V}(2)$. Let $W_2(f_0, f_1, \ldots)$ be the member of $\mathfrak{W}(2)$ obtained by adding k points to each of f_k of the lines of $V_2(b)$, $k = 0, 1, \ldots$. Then $f_0 + f_1 + \ldots = b + 1$, and

$$R(W_2(f_0, f_1, \ldots)) = 2 + \sum\{(k+1)f_k \colon k \leqq 0\}.$$

Suppose that $f_1 > 0$. Then exactly one new point was added incident to some line y_j. Thus, $d(y_j, W_2(f_0, f_1, \ldots)) = 2$. This is a contradiction since $W_2(f_0, f_1, \ldots)$ is a type U incidence structure. We conclude that $f_1 = 0$. In order that $W_2(f_0, f_1, \ldots)$ be non-degenerate we must have that

$$\sum\{f_k \colon k > 0\} \geqq 2.$$

That is, the total number of points added must be at least four, and they must not all lie on the same line. Furthermore,

$$|G(W_2(f_0, f_1, \ldots))| = \prod\{f_k!(k!)^{f_k} \colon k \geqq 0\}.$$

Taking the maximum over the rank r members of $\mathfrak{W}(2)$, we obtain

$$w_r(2) = \begin{cases} 0 & \text{for } r = 8, \\ 8 & \text{for } r = 9, \\ 16 & \text{for } r = 10, \\ 72 & \text{for } r = 11, \\ 2(r-7)! & \text{otherwise.} \end{cases}$$

Calculation of the remaining $w_r(i)$ is similar, and we omit the details. The values of $w_r(i)$ are given in the following table.

We see that

$$w_r = \begin{cases} 2 & \text{for } r = 8, \\ 72 & \text{for } r = 10, \\ 2(r-6)! & \text{otherwise.} \end{cases}$$

$w_r(i)$

i	$r = 8$	$r = 9$	$r = 10$	$r = 11$	$r = 12$	$r \geqq 13$
1	0	0	72	144	1152	$3!(r-7)!$
2	0	8	16	72	240	$2(r-7)!$
3	2	6	24	120	720	$(r-6)!$
4	0	4	12	48	240	$2(r-7)!$
5	0	12	48	240	1440	$2(r-6)!$
6	2	6	6	24	120	$(r-7)!$
7	0	4	12	48	240	$2(r-7)!$
8	0	2	4	12	48	$2(r-8)!$
9	0	0	0	8	24	$4(r-9)!$

Since $u_r = \max\{z_r, w_r\}$ we have that $u_8 = z_8 = 4!$, $u_9 = w_9 = 2 \cdot 3!$, $u_{10} = z_{10} = 5!$, and $u_r = w_r = 2(r-6)!$ for $r \geqq 11$. Thus,

$$u_r = \begin{cases} 4! & \text{for } r = 8, \\ 5! & \text{for } r = 10, \\ 2(r-6)! & \text{otherwise.} \end{cases}$$

We now calculate m_n. Let

$$x_n = \max\{t!u_{n-t+4}: 0 \leqq t \leqq n-7\}$$

and

$$y_n = \max\{t!u_{n-t+4}: n-6 \leqq t \leqq n-4\}.$$

Then $m_n = \max\{x_n, y_n\}$. For $4 \leqq n \leqq 6$ we have that $x_n = 0$. Thus, $m_4 = y_4 = u_8$, $m_5 = y_5 = u_8$ and $m_6 = y_6 = u_{10}$. Now suppose that $n \geqq 7$. Then $x_n = 2(n-2)!$ and $y_n = 4(n-4)!$. Hence $u_n = 2(n-2)!$. We have proved the following theorem.

THEOREM 5.4. *If H is a finite subgroup of G_n, then $|H| \leqq m_n$, where*

$$m_n = \begin{cases} 4! & \textit{for } n = 4, 5, \\ 5! & \textit{for } n = 6, \\ 2(n-2)! & \textit{otherwise.} \end{cases}$$

We now establish the fact that for $n \neq 5$, m_n is the best upper bound for the orders of the finite subgroups of G_n. We shall exhibit a subgroup of G_n of order m_n, for $n \neq 5$.

Suppose that $n = 4$. Let A be a free generator of F_4 consisting of four points. Now $G(A)$ is isomorphic to S_4; thus $|G(A)| = 4! = m_4$. Since each member of $G(A)$ has a unique extension of F_4, $G(A)$ is a subgroup of G_4. Suppose that $n = 6$. Let A be a free generator of F_6 consisting of five points. Now $G(A)$ is isomorphic to S_5; thus $|G(A)| = 5! = m_6$. As before, $G(A)$ is actually a subgroup of G_6. Suppose that $n \geqq 7$. Let A be a free generator of F_n consisting of n points $x_1, x_2, \ldots, x_n$ and one line y with x_i incident to y for $1 \leqq i \leqq n-2$. Now $G(A)$ is isomorphic to $S_2 \times S_{n-2}$; thus $|G(A)| = 2(n-2)! = m_n$. Again, $G(A)$ is a subgroup of G_n.

COROLLARY 5.1. *For $n \neq 5$, the upper bound m_n of Theorem 5.4 is in fact the best upper bound.*

The best upper bound for the orders of the finite subgroups of G_5 is not known. From Theorem 5.4 we conclude that 24 is an upper bound. It is easily shown that the best upper bound is greater than or equal to 12. Let A be the free generator of F_5 consisting of the five points $x_1, x_2, \ldots, x_5$ and one line y, with x_1, x_2, and x_3 incident to y. Then $|G(A)| = 12$ and $G(A)$ is a subgroup of G_5. The author conjectures that 12 is in fact the best upper bound for $n = 5$.

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