A NOTE ON THE MODULUS OF CONVEXITY

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In [**1**, Corollary 5], Figiel gives an elegant demonstration that the modulus of convexity δ in real Banach space X is nondecreasing, where

$$\delta(\varepsilon)=\inf\{1-\|x+y\|/2 : x, y \in S_X, \|x-y\|=\varepsilon\},$$
$$S_X=\{x:\|x\|=1\}.$$

It is deduced from this that in fact $\delta(\varepsilon)/\varepsilon$ is nondecreasing [Proposition 3]. During the course of the proof [Lemma 4] it is stated that if $v\in S_X$ is a *local* maximum on S_X of $\varphi\in S_{X^*}$, then v is a global maximum ($\varphi(v)=1$). This is false; it could be that v is a global minimum. It is easy to construct such an example in $\mathbb{R}^2$ endowed with the maximum norm. What is true is that v is a global maximum of $|\varphi|$.

To see this, choose $\varepsilon>0$ and $u\in S_X$ such that $\varphi(u)>1-\varepsilon$. Then, by local maximality, $\varphi((v+\lambda u)/\|v+\lambda u\|)\leq\varphi(v)$ for all sufficiently small positive λ. By rearrangement

$$\lambda(1-\varepsilon)<\lambda\varphi(u)\leq(\|v+\lambda u\|-\|v\|)\varphi(v)\leq\lambda\,|\varphi(v)|.$$

Hence, as required, $|\varphi(v)|=1$.

Making the necessary modifications to [**1**, Lemma 4], we complete this note by proving the following result.

LEMMA. $\delta(\varepsilon)=\inf\{1-\|x+y\|/2 : x, y\in B_X, \|x-y\|=\varepsilon\}$, *where* $B_X=\{x:\|x\|\leq 1\}$.

Monotonicity of δ then follows easily as in Corollary 5 of [**1**].

Proof of Lemma. Without loss in generality we may assume X is finite dimensional and $\varepsilon>0$. Let x and y be chosen in B_X such that $\|x+y\|$ is maximal, subject to $\|x-y\|=\varepsilon$. Assume $\|x\|\leq\|y\|$. Then $y\neq 0$.

In fact $\|y\|=1$. To see this, set $x_1=(x-c(x-y))/\|y\|$ and $y_1=(y+c(x-y))/\|y\|$, where $c=(1-\|y\|)/2$. Then x_1, $y_1\in B_X$, $\|x_1-y_1\|=\varepsilon$ and $\|x_1+y_1\|=\|x+y\|/\|y\|$. Hence, by maximality of $\|x+y\|$, $\|y\|=1$.

To complete the proof we show either $\|x\|=1$, or $z\in B_X\cap(y+\varepsilon S_X)$ implies $\|z+y\|=\|x+y\|$. Since S_X is connected ($\dim X\geq 2$), $S_X\cap(y+\varepsilon S_X)\neq\varnothing$ for $\varepsilon\leq 2$. Hence the lemma holds whichever case applies.

Assume $\|x\|<1$. Choose $\varphi\in S_{X^*}$ such that $\varphi(x+y)=\|x+y\|$. Let $z\in B_X\cap(y+\varepsilon S_X)$. By maximality of $\|x+y\|$,

$$\varphi(z+y)\leq\|z+y\|\leq\|x+y\|=\varphi(x+y). \tag{1}$$

So $\varphi(z)\leq\varphi(x)$, and hence, $(x-y)/\varepsilon$ is a local maximum of φ on S_X. Either (a) $\varphi(x-y)=\|x-y\|$ or (b) $\varphi(x-y)=-\|x-y\|$.

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If (a) holds,

$$1=\|y\|\leqslant(\|x+y\|+\|x-y\|)/2$$
$$=(\varphi(x+y)+\varphi(x-y))/2=\varphi(x),$$

implying $\|x\|=1$. Consequently, if $\|x\|<1$, (b) holds. Then $\varphi(z-y)\geqslant-\varepsilon=\varphi(x-y)$ and so, using (1), $\|z+y\|=\|x+y\|$, as required.

REFERENCE

1. T. Figiel, On the moduli of convexity and smoothness, *Studia Math.* **56** (1976), 121–155.

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