BULL. AUSTRAL. MATH. SOC. 47H10
VOL. 68 (2003) [161–168]

APPROXIMATION AND FIXED POINT THEOREMS FOR COUNTABLE CONDENSING COMPOSITE MAPS

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This paper presents a multivalued version of an approximation result of Ky Fan (*Math. Z.* **112** (1969)) for $\mathcal{U}_c^\kappa$ maps.

1. INTRODUCTION

Ky Fan [**3**] proved the following result: Let S be a nonempty compact convex set in a normed space $X = (X, \|.\|)$. Then for any continuous map f from S into X there exists a point $x \in S$ with

$$\|x - f(x)\| = \inf_{y \in S} \|f(x) - y\|.$$

This result has been extended to other types of maps and other sets S; see for example [**2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17**]. In this paper we shall obtain a Ky Fan approximation type result for countably condensing $\mathcal{U}_c^\kappa(S, X)$ maps where S is a closed convex subset of a Banach space X and $0 \in \operatorname{int} S$. Also we deduce new fixed point theorems from our approximation result.

2. PRELIMINARIES

Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We shall look at maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F : X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values. $F : X \to K(Y)$ is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighbourhoods U and V of the origins in E_1 and E_2 repectively, a (U, V)*-approximate continuous selection* of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in \Big(F\big[(x + U) \cap X\big] + V\Big) \cap Y \text{ for every } x \in X.$$

Received 28th January, 2003

https://doi.org/10.1017/S0004972700037515 Published online by Cambridge University Press

We say $F : X \to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)–approximate continuous sélection for every open neighbourhood U and V of the origins in E_1 and E_2 repectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map if the following two conditions are satisfied:

(i) For each $x \in X$, the set $p^{-1}(x)$ is acyclic.

(ii) p is a proper map, that is, for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

DEFINITION 2.1: A multifunction $\phi : X \to K(Y)$ is (strongly) *admissible* in the sense of Gorniewicz, if $\phi : X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that

(i) p is a Vietoris map, and

(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

REMARK 2.1. It should be noted that ϕ upper semicontinuous is redundant in Definition 2.1.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

DEFINITION 2.2: $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps and the maps admissible in the sense of Gorniewicz.

Let Q be a subset of a Hausdorff topological space X and $x \in X$. The *inward set* $I_Q(x)$ is defined by

$$I_Q(x) = \{x + r(y - x) : y \in Q, r \geqslant 0\}.$$

We let $\overline{I_Q(x)}$ denote the closure of $I_Q(x)$ (in general we let $\overline{Q}$ (respectively ∂Q, $\operatorname{int} Q$) denote the closure (respectively, the boundary, the interior) of Q).

Let $X = (X,d)$ be a metric space. The *Kuratowski measure of noncompactness* is defined by

$$\alpha(A) = \inf\Big\{\varepsilon > 0 : A \subseteq \bigcup_{i=1}^{n} X_i \text{ for some } n \in \mathbb{N} \text{ and } \operatorname{diam}(X_i) \leqslant \varepsilon\Big\};$$

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here $A \subseteq X$. Let S be a nonempty subset of X, and for each $x \in X$ define $d(x, S) = \inf_{y \in S} d(x, y)$. Let $H : S \to 2^X$ (here 2^X denotes the family of nonempty subsets of X). H is called

(i) *countably k-set contractive* ($k \geqslant 0$) if $H(S)$ is bounded and $\alpha(H(Y)) \leqslant k\alpha(Y)$ for all countably bounded sets Y of S;

(ii) *countably condensing* if H is countably 1-set contractive and $\alpha(H(Y)) < \alpha(Y)$ for all countably bounded sets Y of S with $\alpha(Y) \neq 0$.

Let S be a convex subset of a Banach space X with $0 \in \operatorname{int} S$. We define the *Minkowski functional* on S, $p : X \to [0, \infty)$, as

$$p(x) = \inf\{r > 0 : x \in rS\}, x \in X.$$

The following properties are well known (see [**18**]):

(i) p is continuous;

(ii) $p(x + y) \leqslant p(x) + p(y)$ for $x, y \in X$;

(iii) $p(\lambda x) = \lambda p(x)$, $\lambda \geqslant 0$, $x \in X$;

(iv) $0 \leqslant p(x) < 1$ for $x \in \operatorname{int} S$;

(v) $p(x) > 1$ for $x \notin \overline{S}$;

(vi) $p(x) = 1$ for $x \in \partial S$.

For $R > 0$ let $B_R = \{x \in X : \|x\| \leqslant R\}$. Finally let $d_p(x, S) = \inf\{p(x - y) : y \in S\}$ for $x \in X$.

3. Results

The following fixed point result (see [**1, 13**] will be needed in this section.

Theorem 3.1. *Let S be a nonempty, closed, convex subset of a Banach space X and assume $F \in \mathcal{U}_c^\kappa(S, S)$ is a countably condensing map. Then F has a fixed point in S.*

We now prove our approximation result.

Theorem 3.2. *Let S be a closed, convex subset of a Banach space X with $0 \in \operatorname{int}(S)$. Suppose that $F \in \mathcal{U}_c^\kappa(S, X)$ is a countably condensing map. Then there exist $x_0 \in S$ and $y_0 \in F(x_0)$ with*

$$p(y_0 - x_0) = d_p(y_0, S) = d_p\big(y_0, \overline{I_S(x_0)}\big);$$

here p is the Minkowski functional on S. More precisely, either (i). F has a fixed point $x_0 \in S$, or (ii). there exist $x_0 \in \partial S$ and $y_0 \in F(x_0)$ with

$$0 < p(y_0 - x_0) = d_p(y_0, S) = d_p\big(y_0, \overline{I_S(x_0)}\big).$$

https://doi.org/10.1017/S0004972700037515 Published online by Cambridge University Press

PROOF: Define $r : X \to S$ by

$$r(x) = \begin{cases} x \text{ if } x \in S \\ \dfrac{x}{p(x)} \text{ if } x \notin S. \end{cases}$$

Now r is continuous and notice $r(A) \subseteq \overline{co}\big(\{0\} \cup A\big)$ for any subset A of S. As a result r is a 1–set–contractive map. This together with F is countably condensing implies $G = r \circ F$ is countably condensing. Also since $\mathcal{U}_c^\kappa$ is closed under compositions we have that $G \in \mathcal{U}_c^\kappa(S, S)$. Now Theorem 3.1 guarantees that G has a fixed point $x_0 \in S$, so there exists $y_0 \in F(x_0)$ with $x_0 = r(y_0)$. The proof is now broken up into two cases.

(i) Suppose $y_0 \in S$.

Then $x_0 = r(y_0) = y_0$. As a result

$$p(y_0 - x_0) = 0 = d_p(y_0, S)$$

and x_0 is a fixed point of F.

(ii) Suppose $y_0 \notin S$.

Then $x_0 = r(y_0) = y_0/\big(p(y_0)\big)$. Thus for any $x \in S$ we have

$$\begin{aligned} p(y_0 - x_0) &= p\Big(y_0 - \frac{y_0}{p(y_0)}\Big) = p\Big(\frac{p(y_0)y_0 - y_0}{p(y_0)}\Big) = \frac{\big(p(y_0) - 1\big)}{p(y_0)}p(y_0) \\ &= p(y_0) - 1 \leqslant p(y_0) - p(x) = p((y_0 - x) + x) - p(x) \\ &\leqslant p(y_0 - x) \leqslant \inf\big\{p(y_0 - z) : z \in S\big\} = d_p(y_0, S). \end{aligned}$$

As a result $p(y_0 - x_0) = d_p(y_0, S)$ and also $p(y_0 - x_0) > 0$ since $p(y_0 - x_0) = p(y_0) - 1$. It remains to show that

$$p(y_0 - x_0) = d_p\big(y_0, \overline{I_S(x_0)}\big).$$

Let $z \in I_S(x_0) \backslash S$. Then there exist $y \in S$ and $r > 1$ with $z = x_0 + r(y - x_0)$ (note if $0 \leqslant r \leqslant 1$ then $z = (1 - r)x_0 + ry \in S$). Assume that

$$p(y_0 - z) < p(y_0 - x_0).$$

Clearly

$$\frac{1}{r}z + \Big(1 - \frac{1}{r}\Big)x_0 = y \in S,$$

so we have

$$\begin{aligned} p(y_0 - y) &= p\Big[\frac{1}{r}(y_0 - z) + \Big(1 - \frac{1}{r}\Big)(y_0 - x_0)\Big] \\ &\leqslant \frac{1}{r}p(y_0 - z) + \Big(1 - \frac{1}{r}\Big)p(y_0 - x_0) \\ &< p(y_0 - x_0), \end{aligned}$$

which contradicts the fact that $p(y_0 - x_0) = d_p(y_0, S)$. Thus

$$p(y_0 - x_0) \leqslant p(y_0 - z) \text{ for all } z \in I_S(x_0).$$

Furthermore (note p is continuous) we have

$$p(y_0 - x_0) \leqslant p(y_0 - z) \text{ for all } z \in \overline{I_S(x_0)}.$$

Thus $p(y_0 - x_0) \leqslant d_p\big(y_0, \overline{I_S(x_0)}\big)$ so we have equality since $x_0 \in \overline{I_S(x_0)}$. As a result

$$0 < p(y_0 - x_0) = d_p(y_0, S) = d_p\big(y_0, \overline{I_S(x_0)}\big).$$

If $x_0 \in \text{int}(S)$ it is well known that $\overline{I_S(x_0)} = X$ and so $d_p\big(y_0, \overline{I_S(x_0)}\big) = 0$. Thus $x_0 \in \partial S$. ∎

Corollary 3.3. *Let B_R be a closed ball with centre at the origin and radius R in a Banach space $X = \big(X, \|.\|\big)$. Suppose that $F \in \mathcal{U}_c^\kappa(B_R, X)$ is a countably condensing map. Then there exist $x_0 \in B_R$ and $y_0 \in F(x_0)$ with*

$$\|y_0 - x_0\| = d(y_0, B_R) = d\big(y_0, \overline{I_{B_R}(x_0)}\big);$$

here $d(y_0, B_R) = \inf_{z \in B_R} \|y_0 - z\|$. More precisely, either (i). F has a fixed point $x_0 \in B_R$, or (ii). there exist $x_0 \in \partial B_R$ and $y_0 \in F(x_0)$ with

$$0 < \|y_0 - x_0\| = d(y_0, B_R) = d\big(y_0, \overline{I_{B_R}(x_0)}\big).$$

Proof: It is clear that $p(x) = \|x\|/R$ is the Minkowski functional on B_R. Now apply Theorem 3.2. ∎

Remark 3.4. Theorem 3.2 and Corollary 3.3. extend [**10**, Theorem 1].

Now we apply our theorem to obtain the following fixed point theorem which contains Theorem 2 of [**10**] as a special case.

Theorem 3.5. *Let S be a closed, convex subset of a Banach space X with $0 \in \text{int}(S)$. Suppose that $F \in \mathcal{U}_c^\kappa(S, X)$ is a countably condensing map and assume any one of the following conditions hold for all $x \in \partial S \backslash F(x)$:*

(i) *For each $y \in F(x), p(y - z) < p(y - x)$ for some $z \in \overline{I_S(x)}$;*
(ii) *For each $y \in F(x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda) y \in \overline{I_S(x)}$;*
(iii) *$F(x) \subseteq \overline{I_S(x)}$;*
(iv) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(x)$;*
(v) *For each $y \in F(x)$, there exist $\alpha \in (1, \infty)$ such that $p^\alpha(y) - 1 \leqslant p^\alpha(y - x)$;*
(vi) *For each $y \in F(x)$, there exist $\beta \in (0, 1)$ such that $p^\beta(y) - 1 \geqslant p^\beta(y - x)$.*

Then F has a fixed point in S.

Proof: Theorem 3.2 guarantees that either

(1) F has a fixed point in S; or

(2) there exist $x_0 \in \partial S$ and $y_0 \in F(x_0)$ (note also that $x_0 = r(y_0) = y_0/(p(y_0)))$ with

$$0 < p(y_0) - 1 = p(y_0 - x_0) = d_p(y_0, S) = d_p\big(y_0, \overline{I_S(x_0)}\big)$$

holding.

Suppose F satisfies condition (i). Now suppose (2) holds (with x_0 and y_0 as described above). We show $x_0 \in F(x_0)$. Suppose $x_0 \notin F(x_0)$. Then condition (i) implies that

$$p(y_0 - z) < p(y_0 - x_0) \text{ for some } z \in \overline{I_S(x_0)}.$$

This contradicts $p(y_0 - x_0) = d_p\big(y_0, \overline{I_S(x_0)}\big)$.

Suppose F satisfies condition (ii). Now suppose (2) holds (with x_0 and y_0 as described above). We show $x_0 \in F(x_0)$. Suppose $x_0 \notin F(x_0)$. Then condition (ii) implies that there exists λ with $|\lambda| < 1$ such that

$$\lambda x_0 + (1 - \lambda) y_0 \in \overline{I_S(x)}.$$

By (2) we have

$$\begin{aligned} 0 < p(y_0 - x_0) &\leqslant p\Big(y_0 - \big[\lambda x_0 + (1 - \lambda) y_0\big]\Big) = p\big(\lambda(y_0 - x_0)\big) \\ &= |\lambda| p(y_0 - x_0) < p(y_0 - x_0), \end{aligned}$$

which is a contradiction.

If F satisfies condition (iii), then F satisfies condition (ii) by letting $\lambda = 0$.

Suppose F satisfies condition (iv). Now suppose (2) holds (with x_0 and y_0 as described above). We show $x_0 \in F(x_0)$. Suppose $x_0 \notin F(x_0)$. Notice that

$$x_0 = r(y_0) = \frac{y_0}{p(y_0)} \text{ and } p(y_0) > 1,$$

and this implies that

$$x_0 = \lambda_0 y_0 \text{ where } \lambda_0 = \frac{1}{p(y_0)} \in (0, 1).$$

This contractiocts condition (iv).

Suppose F satisfies condition (v). Now suppose (2) holds (with x_0 and y_0 as described above). We show $x_0 \in F(x_0)$. Suppose $x_0 \notin F(x_0)$. Then condition (v) implies that there exists $\alpha \in (1, \infty)$ with $p^\alpha(y_0) - 1 \leqslant p^\alpha(y_0 - x_0)$. Let $\lambda_0 = 1/p(y_0)$. Note $\lambda_0 \in (0, 1)$ and

$$\frac{(p(y_0) - 1)^\alpha}{p^\alpha(y_0)} = (1 - \lambda_0)^\alpha < 1 - \lambda_0^\alpha = \frac{p^\alpha(y_0) - 1}{p^\alpha(y_0)} \leqslant \frac{p^\alpha(y_0 - x_0)}{p^\alpha(y_0)}.$$

This implies

$$p(y_0 - x_0) > p(y_0) - 1,$$

https://doi.org/10.1017/S0004972700037515 Published online by Cambridge University Press

and this contradicts $p(y_0 - x_0) = p(y_0) - 1$.

Finally assume F satisfies condition (vi). Using an argument similar to that above (for condition (v)) we obtain the desired conclusion. □

COROLLARY 3.6. *Let B_R be a closed ball with centre at the origin and radius R in a Banach space $X = (X, \|.\|)$. Suppose that $F \in \mathcal{U}_c^\kappa(B_R, X)$ is a countably condensing map and assume any one of the following conditions hold for all $x \in \partial B_R \backslash F(x)$:*

(i) *For each $y \in F(x)$, $\|y - z\| < \|y - x\|$ for some $z \in \overline{I_{B_R}(x)}$;*

(ii) *For each $y \in F(x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{B_R}(x)}$;*

(iii) *$F(x) \subseteq \overline{I_{B_R}(x)}$;*

(iv) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(x)$;*

(v) *For each $y \in F(x)$, there exist $\alpha \in (1, \infty)$ such that $\|y\|^\alpha - R^\alpha \leqslant \|y - x\|^\alpha$;*

(vi) *For each $y \in F(x)$, there exist $\beta \in (0, 1)$ such that $\|y\|^\beta - R^\beta \geqslant \|y - x\|^\beta$.*

Then F has a fixed point in B_R.

Using the ideas in Theorem 3.2 (here r in Theorem 3.2 is replaced by the nearest point projection) it is immediate that the analogue of Theorem 3 in [**10**] holds for countably condensing maps; we leave the obvious details to the reader. Thus we have the following theorem.

THEOREM 3.7. *Let S be a closed, convex subset of a Hilbert space X. Suppose that $F \in \mathcal{U}_c^\kappa(S, X)$ is a countably condensing map. Then there exist $x_0 \in S$ and $y_0 \in F(x_0)$ with*

$$\|y_0 - x_0\| = d(y_0, S) = d\big(y_0, \overline{I_S(x_0)}\big);$$

here $\|.\|$ is the norm induced by the inner product. More precisely, either

(i) *F has a fixed point $x_0 \in S$, or*

(ii) *there exist $x_0 \in \partial S$ and $y_0 \in F(x_0)$ with*

$$0 < \|y_0 - x_0\| = d(y_0, S) = d\big(y_0, \overline{I_S(x_0)}\big).$$

REFERENCES

[**1**] R.P. Agarwal and D. O'Regan, 'Fixed points for admissible multimaps', *Dynamic Systems Appl.* **11** (2002), 437–447.

[**2**] A. Carbone and G. Conti, 'Multivalued maps and existence of best approximations', *J. Approx. Theory* **64** (1991), 203–208.

[**3**] K. Fan, 'Extensions of two fixed point theorems of F.E. Browder', *Math. Z.* **112** (1969), 234–240.

[**4**] K. Fan, 'Some properties of convex sets related to fixed point theorems', *Math. Ann.* **266** (1984), 519–537.

https://doi.org/10.1017/S0004972700037515 Published online by Cambridge University Press

[5] C.W. Ha, 'On a minimax inequality of Ky Fan', *Proc. Amer. Math. Soc.* **99** (1987), 680–682.

[6] W.B. Kong and X.P. Ding, 'Approximation theorems and fixed point theorems for multivalued condensing mappings in wedges', *J. Math. Anal. Appl.* **167** (1992), 468–481.

[7] T.C. Lin, 'A note on a theorem of Ky Fan', *Canad. Math. Bull.* **22** (1979), 513–515.

[8] T.C. Lin, Approximation and fixed points for condensing non self maps defined on a sphere, *Proc. Amer. Math. Soc.* **105** (1989), 66–69.

[9] T.C. Lin and C.L. Yen, 'Applications of the proximity map to fixed point theorems in Hilbert spaces', *J. Approx. Theory* **52** (1988), 141–148.

[10] T.C. Lin and S. Park, 'Approximation and fixed point theorems for condensing composites of multifunctions', *J. Math. Anal. Appl.* **223** (1998), 1–8.

[11] L.S. Liu, 'Approximation theorems and fixed point theorems for various classes of 1–set contractive mappings in Banach spaces', *Acta Math. Sinica* **17** (2001), 103–112.

[12] L.S. Liu, 'On approximation theorems and fixed point theorems for non self mappings in infinite dimensional Banach spaces', *J. Math. Anal. Appl.* **188** (1994), 541–551.

[13] D. O'Regan, 'A unified fixed point theory for countably P–concentrative mappings', *Appl. Anal.* **81** (2002), 565–574.

[14] S. Park, S.P. Singh and B. Watson, 'Some fixed point theorems for composites of acyclic maps', *Proc. Amer. Math. Soc.* **121** (1994), 1151–1158.

[15] S. Reich, Approximate selections, 'best approximations, fixed points and invariant sets', *J. Math. Anal. Appl.* **62** (1978), 104–113.

[16] V.M. Sehgal and S.P. Singh, 'A theorem on the minimization of a condensing multifunction and fixed points', *J. Math. Anal. Appl.* **107** (1985), 96–102.

[17] V.M. Sehgal and S.P. Singh, 'A generalization to multifunctions of Fan's best approximation theorem', *Proc. Amer. Math. Soc.* **102** (1988), 534–537.

[18] D.R. Smart, *Fixed point theory* (Cambridge University Press, London, 1974).

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