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TWO-INTERVAL EVEN-ORDER DIFFERENTIAL OPERATORS IN MODIFIED HILBERT SPACES

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Abstract

By modifying the inner product in the direct sum of the Hilbert spaces associated with each of two underlying intervals on which an even-order equation is defined, we generate self-adjoint realisations for boundary conditions with any real coupling matrix which are much more general than the coupling matrices from the 'unmodified' theory.

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1. Introduction

Partly motivated by applications, in particular [1] (see also [4]), Everitt and Zettl in [2] developed a theory of self-adjoint realisations of Sturm–Liouville problems on two intervals in the direct sum of Hilbert spaces associated with these intervals. See [10, Ch. 13] for an exposition of this theory. This theory was extended in [3] to higher-order regular and singular equations and any number of intervals, finite or infinite.

As in the one-interval case the characterisation of [3] depends on maximal domain vectors. These vectors depend on the coefficients of each differential equation and this dependence is implicit and complicated. In [9] Wang *et al.* give an explicit characterisation of all self-adjoint domains for singular problems in terms of certain solutions for *real* λ for the one-interval case when one endpoint is regular and the other is singular. In analogy with the celebrated Weyl limit-point (LP), limit-circle (LC) theory in the second-order case, they construct LC and LP solutions and characterise the self-adjoint domains in terms of the LC solutions. Following [9], Hao *et al.* give a new characterisation in [5] by dividing (a_1, b_1) into two intervals (a_1, c_1) and (c_1, b_1) for some $c_1 \in (a_1, b_1)$ and using the LC solutions on each interval constructed in [9] when a_1 and b_1 are singular. In [7], Suo and Wang extend the characterisation in [5] to the two-interval case but the result reduces to the case when one, two, three or four endpoints are regular.

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As noted in [2], a simple way of getting self-adjoint operators in a direct-sum Hilbert space is to take the direct sum of self-adjoint operators from each of the separate Hilbert spaces. However, there are many self-adjoint operators which are not merely the sum of self-adjoint operators from each of the separate intervals. These 'new' self-adjoint operators involve interactions between the two intervals. Therefore in [2] the authors develop a 'two-interval' theory. Mukhtarov and Yakubov [6] observed that the set of two-interval self-adjoint realisations can be further enlarged by using different multiples of the usual inner products associated with each of the intervals. In [8] Sun *et al*. use the Mukhtarov–Yakubov modification of the Everitt–Zettl theory to characterise all self-adjoint realisations of regular two-interval problems. This characterisation is explicit and involves only the values of solutions and their quasiderivatives at the endpoints of the intervals and the multiple inner product parameters. In particular, for the second-order case with coupled boundary conditions and a real coupling matrix K, the method of [2] requires that $\det(K) = 1$ whereas with the Mukhtarov–Yakubov modification in [8] it is only required that $\det(K)$ is positive.

In this paper we develop a complete analogue of [7] when one endpoint of each interval $(a_1, b_1), (a_2, b_2)$ is regular using Hilbert spaces but with the usual inner products replaced by appropriate multiples. The interplay of these multiples with the boundary conditions generates self-adjoint problems of even order with real coupling matrices K which are much more general than the coupling matrices from the 'unmodified' theory. We give a number of examples to illustrate this additional generality, among other things.

From another perspective, instead of using multiples of the usual inner products, our approach can be described as using multiples of weight functions.

2. Notation and basic facts for one interval

Although we only consider even-order equations with real coefficients in this paper, we summarise some basic facts about general quasidifferential equations of even and odd order and real or complex coefficients for the convenience of the reader.

Let $J = (a, b)$ be an interval with $-\infty \le a < b \le \infty$ and let n be a positive integer (even or odd). For a given set S, $M_n(S)$ denotes the set of $n \times n$ complex matrices with entries from S.

Let $Z_n(J) := \{Q = (q_{is})_{i,s=1}^n\}$, where

$$\begin{cases} q_{i,i+1} \neq 0 \text{ almost everywhere on } J, \quad q_{i,i+1}^{-1} \in L_{\text{loc}}(J), \quad 1 \le i \le n-1, \\ q_{is} = 0 \text{ almost everywhere on } J, \quad 2 \le i+1 < s \le n, \\ q_{is} \in L_{\text{loc}}(J), \quad s \neq i+1, 1 \le i \le n-1. \end{cases} \tag{2.1}$$

Let $Q \in Z_n(J)$. We define

$$V_0 := \{y : J \to \mathbb{C}, y \text{ is measurable}\}$$

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and

$$y^{[0]} := y \quad (y \in V_0).$$

Inductively, for $i = 1, \ldots, n$, we define

$$V_i = \{y \in V_{i-1} : y^{[i-1]} \in AC_{\mathrm{loc}}(J)\},$$
$$y^{[i]} = q_{i,i+1}^{-1}\Big(y^{[i-1]\prime} - \sum_{s=1}^{i} q_{is} y^{[s-1]}\Big) \quad (y \in V_i),$$

where $q_{n,n+1} := 1$, and $AC_{\mathrm{loc}}(J)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of J. Finally we set

$$My = M_Q y := i^n y^{[n]} \quad (y \in V_n).$$

The expression $M = M_Q$ is called the quasidifferential expression associated with Q. For V_n we also write $V(M)$ and $D(Q)$. The function $y^{[i]}$ $(0 \le i \le n)$ is called the ith quasiderivative of y. Since the quasiderivative depends on Q, we sometimes write $y_Q^{[i]}$ instead of $y^{[i]}$.

Remark 2.1. The operator $M : D(Q) \to L_{\mathrm{loc}}(J)$ is linear.

Let $Z_n(J, \mathbb{R})$ denote the set of matrices $Q \in Z_n(J)$ which have real-valued components.

Definition 2.2. Let $Q \in Z_n(J, \mathbb{R})$ and let $M = M_Q$ be defined as above. Assume that

$$Q = -E^{-1} Q^* E, \quad \text{where } E = ((-1)^i \delta_{i,n+1-s})_{i,s=1}^n. \tag{2.2}$$

Then $M = M_Q$ is called a symmetric differential expression.

Let $w \in L_{\mathrm{loc}}(J)$ be positive almost everywhere on J. We consider the Hilbert space

$$H = L^2(J, w)$$

with its usual inner product

$$\langle y, v\rangle_w := \int_J y\bar{v}w.$$

The maximal and minimal operators associated with a symmetric expression Q and a positive weight function w in the Hilbert space H are defined as follows.

Definition 2.3. Assume $Q \in Z_n(J, \mathbb{R})$ satisfies (2.2) and let $M = M_Q$ be the associated symmetric expression. Let $w \in L_{\mathrm{loc}}(J)$ be positive almost everywhere on J. Define

$$D_{\max} = \{y \in L^2(J, w) : y \in D(Q),\ w^{-1}My \in L^2(J, w)\},$$
$$S_{\max} y = w^{-1} My, \quad y \in D_{\max},$$
$$S_{\min} = S_{\max}^*,$$
$$D_{\min} = D(S_{\min}).$$

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Lemma 2.4. *Let $S_{\min}$ and $S_{\max}$ be defined as above. Then $D_{\min}$ and $D_{\max}$ are dense in H, $S_{\min}$ and $S_{\max}$ are closed operators in H, $S^*_{\min} = S_{\max}$, $S_{\min} = S^*_{\max}$ and $S_{\min}$ is a symmetric operator in H.*

From Definition 2.3 and Lemma 2.4 we see that every self-adjoint extension S of the minimal operator is 'between' the minimal and maximal operators; that is,

$$S_{\min} \subset S = S^* \subset S_{\max}.$$

Thus these self-adjoint operators S are distinguished from one another only by their domains.

Lemma 2.5 (Lagrange identity). *Assume $Q \in Z_n(J, \mathbb{R})$ satisfies (2.2) and let $M = M_Q$ be the corresponding differential expression. Then, for any $y, z \in D(Q)$,*

$$\bar{z}My - y\overline{Mz} = [y, z]',$$

where

$$[y, z] = (-1)^k \sum_{i=0}^{n-1} (-1)^{n+1-i} \bar{z}^{[n-i-1]} y^{[i]} = (-1)^k (Z^* E Y), \tag{2.3}$$

$$Y = \begin{pmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \quad Z = \begin{pmatrix} z \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}.$$

Definition 2.6 (Regular endpoints). Let $Q \in Z_n(J, \mathbb{R})$, $J = (a, b)$. The expression $M = M_Q$ is said to be regular at a if for some c, $a < c < b$, we have

$$q^{-1}_{i,i+1} \in L(a, c), \quad i = 1, \ldots, n-1;$$
$$q_{is} \in L(a, c), \quad 1 \le i, s \le n, s \ne i + 1.$$

Similarly the endpoint b is regular if for some c, $a < c < b$, we have

$$q^{-1}_{i,i+1} \in L(c, b), \quad i = 1, \ldots, n-1;$$
$$q_{is} \in L(c, b), \quad 1 \le i, s \le n, s \ne i + 1.$$

Note that, from (2.1) it follows that if the above hold for some $c \in J$ then they hold for any $c \in J$. We say that M is regular on J, or just M is regular, if M is regular at both endpoints.

3. Notation and Basic Assumptions for Two Intervals

Let

$$J_r = (a_r, b_r), \quad -\infty < a_r < b_r \le \infty, r = 1, 2.$$

Define two differential expressions with real-valued coefficients by

$$M_r y = M_{Q_r} y := i^n y^{[n]} \quad \text{on } J_r, r = 1, 2, n = 2k, \ k > 1.$$

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Let

$$H_r = L^2(J_r, w_r), \quad w_r > 0, r = 1, 2.$$

The two-interval maximal and minimal domains and operators are simply the direct sums of the corresponding one-interval domains and operators:

$$D_{\max} = D_{1\max} \dotplus D_{2\max}, \quad D_{\min} = D_{1\min} \dotplus D_{2\min},$$
$$S_{\max} = S_{1\max} \dotplus S_{2\max}, \quad S_{\min} = S_{1\min} \dotplus S_{2\min}.$$

Elements of $H_u = H_1 \dotplus H_2$ will be denoted in bold face type: $\mathbf{f} = \{f_1, f_2\}$ with $f_1 \in H_1$, $f_2 \in H_2$. As usual the inner product in H_u is defined by

$$(\mathbf{f}, \mathbf{g}) = (f_1, g_1)_1 + (f_2, g_2)_2, \tag{3.1}$$

where $(\cdot, \cdot)_r$ is the usual inner product in H_r:

$$(f_r, g_r)_r = \int_{J_r} f_r \bar{g}_r w_r.$$

In this paper, following [8] we replace the direct-sum inner product (3.1) by

$$\langle \mathbf{f}, \mathbf{g} \rangle = l(f_1, g_1)_1 + s(f_2, g_2)_2, \quad l > 0, s > 0, \tag{3.2}$$

and apply operator theory in the direct-sum space

$$H = (L^2(J_1, w_1) \dotplus L^2(J_2, w_2), \ \langle \cdot, \cdot \rangle). \tag{3.3}$$

REMARK 3.1. Note that (3.2) is an inner product in H for any positive numbers l and s. The elements of the Hilbert space H defined by (3.3) are the same as those of the usual direct-sum Hilbert spaces H_u; thus these spaces are differentiated from each other only by their inner products. As we will see below, the parameters l, s influence the boundary conditions which yield self-adjoint realisations of the equations in the two-interval case. Observe also that the Hilbert space (3.3) can be viewed as a 'usual' direct-sum space H_u with summands $H_r = L^2(J_r, w_r)$ but with each w_r replaced by an appropriate multiple.

As in the one-interval case the Lagrange sesquilinear form plays an important role. It is defined, for appropriate functions $\mathbf{f}$, $\mathbf{g}$, by

$$[\mathbf{f}, \mathbf{g}] = l[f_1, g_1]_1(b_1) - l[f_1, g_1]_1(a_1) + s[f_2, g_2]_2(b_2) - s[f_2, g_2]_2(a_2), \tag{3.4}$$

where

$$[f_r, g_r]_r = (-1)^k (G_r^* E F_r), \quad F_r = \begin{pmatrix} f_r \\ f_r^{[1]} \\ \vdots \\ f_r^{[n-1]} \end{pmatrix}, \quad G_r = \begin{pmatrix} g_r \\ g_r^{[1]} \\ \vdots \\ g_r^{[n-1]} \end{pmatrix}.$$

Note that the two-interval Lagrange form $[f, g]$ connects all four endpoints with each other and depends on the parameters l, s.

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4. Characterisation of all self-adjoint domains for two-interval problems

In this section we assume that $M_{Q_r}(r = 1, 2)$ are generated by $Q_r \in Z_{n(r)}(J_r, \mathbb{R})$ satisfying (2.2), $n = 2k$, $k > 1$. First, we give some preliminary lemmas.

LEMMA 4.1.

(1) *The following equalities hold.*

$$S^*_{\min} = S^*_{1\min} + S^*_{2\min} = S_{1\max} + S_{2\max} = S_{\max},$$
$$S^*_{\max} = S^*_{1\max} + S^*_{2\max} = S_{1\min} + S_{2\min} = S_{\min}.$$

In particular,

$$D_{\max} = D(S_{\max}) = D(S_{1\max}) + D(S_{2\max}),$$
$$D_{\min} = D(S_{\min}) = D(S_{1\min}) + D(S_{2\min}).$$

(2) *The minimal operator $S_{\min}$ is a closed, symmetric, densely defined operator in the Hilbert space H with deficiency index d given by $d = d_1 + d_2$.*

PROOF. The proof given in [2] for (3.1) extends readily to (3.2). □

DEFINITION 4.2. Assume that the endpoint a_r is regular and $S_{r\min}$ is defined as above. Then for each r the deficiency index d_r of $S_{r\min}$ is the number of linearly independent solutions of

$$M_r y = i w_r y \quad \text{on } J_r, \quad i = \sqrt{-1}, r = 1, 2,$$

which lie in H_r.

LEMMA 4.3. *The number d_r of linearly independent solutions of*

$$M_r y = \lambda_r w_r y \quad \text{on } J_r, \quad r = 1, 2, \tag{4.1}$$

lying in H_r is independent of $\lambda_r \in \mathbb{C}$, provided $\operatorname{Im}(\lambda_r) \neq 0$. *The inequalities*

$$k \le d_r \le 2k = n$$

hold. For $\lambda = \lambda_r \in \mathbb{R}$, the number of linearly independent solutions of $(4.1)_{r=1}$ is less than or equal to d_1, and the number of linearly independent solutions of $(4.1)_{r=2}$ is less than or equal to d_2.

PROOF. For the proof of the statement in the last sentence see [9, Lemma 5]. The other statements are well known. □

LEMMA 4.4. *Suppose M_r is regular at c_r. Then for any $y = \{y_1, y_2\} \in D_{\max}$ the limits*

$$y_1^{[i]}(c_1) = \lim_{t \to c_1} y^{[i]}(t), \quad y_2^{[i]}(c_2) = \lim_{t \to c_2} y^{[i]}(t)$$

exist and are finite. In particular, this holds at any regular endpoint and at each interior point of J_r. At an endpoint the limit is the appropriate one-sided limit.

PROOF. This follows from the one-interval theory, see [9, Lemma 3]. □

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Lemma 4.5 (Naimark patching lemma). *Let $Q_r \in Z_{n(r)}(J_r, \mathbb{R})$ and assume that M_r is regular on J_r. Suppose that $w_r \in L(J_r)$, $w_r > 0$ on J_r, $r = 1, 2$. Let $\alpha_s, \beta_s, \delta_s, \eta_s \in \mathbb{C}$, $s = 0, \dots, n-1$. Then there is a function $y = \{y_1, y_2\} \in D_{\max}$ such that*

$$y_1^{[s]}(a_1) = \alpha_s, \quad y_1^{[s]}(b_1) = \beta_s, \quad y_2^{[s]}(a_2) = \delta_s, \quad y_2^{[s]}(b_2) = \eta_s \quad (s = 0, \dots, n-1).$$

Proof. This follows from the one-interval theory; see [9, Lemma 4]. □

Corollary 4.6. *Let $c_r < d_r \in J_r$, $r = 1, 2$, and $\alpha_s, \beta_s, \delta_s, \eta_s \in \mathbb{C}$, $s = 0, \dots, n-1$. Then there is a $y = \{y_1, y_2\} \in D_{\max}$ such that y_r has compact support in J_r and satisfies*

$$y_1^{[s]}(c_1) = \alpha_s, \quad y_1^{[s]}(d_1) = \beta_s, \quad y_2^{[s]}(c_2) = \delta_s, \quad y_2^{[s]}(d_2) = \eta_s \quad (s = 0, \dots, n-1).$$

Proof. This follows from the one-interval theory; see [9, Corollary 4]. □

Corollary 4.7. *Let $a_{1r} < \cdots < a_{kr} \in J_r$, $r = 1, 2$, where a_{1r} and a_{kr} can also be regular endpoints. Let $\alpha_{js}, \beta_{js} \in \mathbb{C}$, $j = 1, \dots, k$, $s = 0, \dots, n-1$. Then there is a $y = \{y_1, y_2\} \in D_{\max}$ such that*

$$y_1^{[s]}(a_{j1}) = \alpha_{js}, \quad y_2^{[s]}(a_{j2}) = \beta_{js} \quad (j = 1, \dots, k, s = 0, \dots, n-1).$$

Proof. This follows from repeated applications of the previous corollary. □

Lemma 4.8. *Suppose a_r, $r = 1, 2$ is regular. Then minimal domains $D_{\min}$ consist of all functions $y = \{y_1, y_2\} \in D_{\max}$ which satisfy the following two conditions.*

(1) $y_1^{[j]}(a_1) = y_2^{[j]}(a_2) = 0$, $j = 0, 1, \dots, n-1$.
(2) For all $z = \{z_1, z_2\} \in D_{\max}$,

$$[y_1, z_1]_1(b_1) = [y_2, z_2]_2(b_2) = 0.$$

Proof. This follows from the one-interval theory; see [9, Lemma 7]. A similar result holds when the endpoints b_r are regular. □

Next we give the decomposition of the maximal domain and the characterisation of all self-adjoint domains for the case where a_1 and a_2 are regular.

Let

$$a_1 < c < b_1, \quad a_2 < d < b_2,$$

and determine functions $z_j \in D_{1\max}$, $g_j \in D_{2\max}$, $j = 1, \dots, n$, such that $z_j(t) = 0$ for $t \geq c$ and $g_j(t) = 0$ for $t \geq d$, $j = 1, \dots, n$, and

$$\begin{pmatrix} [z_1, z_1]_1(a_1) & [z_2, z_1]_1(a_1) & \cdots & [z_{n-1}, z_1]_1(a_1) & [z_n, z_1]_1(a_1) \\ [z_1, z_2]_1(a_1) & [z_2, z_2]_1(a_1) & \cdots & [z_{n-1}, z_2]_1(a_1) & [z_n, z_2]_1(a_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ [z_1, z_n]_1(a_1) & [z_2, z_n]_1(a_1) & \cdots & [z_{n-1}, z_n]_1(a_1) & [z_n, z_n]_1(a_1) \end{pmatrix}$$

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$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & & . & & \vdots \\ 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} = E,$$

$$\begin{pmatrix} [g_1, g_1]_2(a_2) & [g_2, g_1]_2(a_2) & \cdots & [g_{n-1}, g_1]_2(a_2) & [g_n, g_1]_2(a_2) \\ [g_1, g_2]_2(a_2) & [g_2, g_2]_2(a_2) & \cdots & [g_{n-1}, g_2]_2(a_2) & [g_n, g_2]_2(a_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ [g_1, g_n]_2(a_2) & [g_2, g_n]_2(a_2) & \cdots & [g_{n-1}, g_n]_2(a_2) & [g_n, g_n]_2(a_2) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & & . & & \vdots \\ 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} = E.$$

Such functions exist by the patching lemma and the fact that for each $i = 1, \ldots, n$ the values $z_i^{[j]}(a_1)$ and $g_i^{[j]}(a_2)$ can be assigned arbitrarily.

THEOREM 4.9. *Let the endpoints a_r be regular while the endpoints b_r are singular, let d_r be the deficiency index of the minimal operator $S_{r\,\min}$, and let $m_r = 2d_r - 2k$, $r = 1, 2$. Assume there exists λ_1 and $\lambda_2 \in \mathbb{R}$ such that (4.1)$_{r=1}$ has d_1 linearly independent solutions lying in H_1 and (4.1)$_{r=2}$ has d_2 linearly independent solutions lying in H_2. Then there exist solutions u_j, $j = 1, \ldots, m_1$, of (4.1)$_{r=1}$ lying in H_1 and v_j, $j = 1, \ldots, m_2$, of (4.1)$_{r=2}$ lying in H_2 such that the $m_1 \times m_1$ and $m_2 \times m_2$ matrices*

$$U = ([u_i, u_j]_1(a_1)), \quad 1 \le i, j \le m_1, \quad V = ([v_i, v_j]_2(a_2)), \quad 1 \le i, j \le m_2,$$

are nonsingular and

$$D_{1\,\max} = D_{1\,\min} \dotplus \mathrm{span}\{z_1, z_2, \ldots, z_n\} \dotplus \mathrm{span}\{u_1, u_2, \ldots, u_{m_1}\},$$
$$D_{2\,\max} = D_{2\,\min} \dotplus \mathrm{span}\{g_1, g_2, \ldots, g_n\} \dotplus \mathrm{span}\{v_1, v_2, \ldots, v_{m_2}\}.$$

PROOF. This follows from the one-interval case; see [9, Theorem 3]. □

COROLLARY 4.10. *Assume that a_r is regular, $r = 1, 2$. Let d_r be the deficiency index of the minimal operator $S_{r\,\min}$ and let $m_r = 2d_r - 2k$, $r = 1, 2$. Assume that there exist λ_1 and $\lambda_2 \in \mathbb{R}$ such that (4.1)$_{r=1}$ has d_1 linearly independent solutions lying in H_1 and (4.1)$_{r=2}$ has d_2 linearly independent solutions lying in H_2. Then there exist d_1 and d_2 linearly independent real solutions $u_1, \ldots, u_{d_1}$ of (4.1)$_{r=1}$ and $v_1, \ldots, v_{d_2}$ of (4.1)$_{r=2}$ for λ_1 and λ_2 satisfying the following three conditions.*

(1) *The $m_1 \times m_1$ and $m_2 \times m_2$ matrices*

$$U = ([u_i, u_j]_1(a_1)), \quad 1 \le i, j \le m_1, \quad V = ([v_i, v_j]_2(a_2)), \quad 1 \le i, j \le m_2,$$

are given by

$$U = (-1)^{k+1} E_{m_1 \times m_1},$$
$$V = (-1)^{k+1} E_{m_2 \times m_2},$$

and are therefore nonsingular.

(2) *For every* $y_r \in D_{r\,\max}$,

$$[u_j, y_1]_1(b_1) = 0 \quad \textit{for } j = m_1 + 1, \ldots, d_1,$$
$$[v_j, y_2]_2(b_2) = 0 \quad \textit{for } j = m_2 + 1, \ldots, d_2.$$

(3) *Further,*

$$[u_i, u_j]_1(a_1) = [u_i, u_j]_1(b_1) = 0 \quad \textit{for } i = 1, \ldots, d_1,\ j = m_1 + 1, \ldots, d_1,$$
$$[v_i, v_j]_2(a_2) = [v_i, v_j]_2(b_2) = 0 \quad \textit{for } i = 1, \ldots, d_2,\ j = m_2 + 1, \ldots, d_2.$$

Proof. This follows from the one-interval case; see [9, Corollary 6]. □

Next we give the Everitt and Zettl [3] extension of the Glazman, Krein, Naimark theorem from the one-interval to the two-interval case.

Lemma 4.11. *Let $S_{\min}$ be the two-interval minimal operator in H and let d be the deficiency index of $S_{\min}$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ in $D_{\max}$ satisfying the following conditions.*

(i) $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ *are linearly independent modulo* $D_{\min}$.

(ii) $[\mathbf{w}_i, \mathbf{w}_j] = l[w_{i1}, w_{j1}]_1(b_1) - l[w_{i1}, w_{j1}]_1(a_1) + s[w_{i2}, w_{j2}]_2(b_2) - s[w_{i2}, w_{j2}]_2(a_2) = 0,\ i, j = 1, \ldots, d.$

(iii) $D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max} : [\mathbf{y}, \mathbf{w}_j] = l[y_1, w_{j1}]_1(b_1) - l[y_1, w_{j1}]_1(a_1) + s[y_2, w_{j2}]_2(b_2) - s[y_2, w_{j2}]_2(a_2) = 0,\ j = 1, \ldots, d\}.$

Proof. See Theorem 3.1 and Corollary 3.3 in Everitt and Zettl [3] for the case with inner product (3.1); the adaptation to inner product (3.2) is routine. □

Remark 4.12. As mentioned in the Introduction, the characterisation of Lemma 4.11 depends on the maximal domain vectors w_j, $j = 1, \ldots, d$. These vectors depend on the coefficients of each differential equation and this dependence is implicit and complicated. Based on Theorem 4.9, we next give more explicit equivalent conditions for (i)–(iii) of Lemma 4.11.

The next theorem is our main result in this paper.

Theorem 4.13. *Assume that a_r is regular, $r = 1, 2$. Let d_r be the deficiency index of the minimal operator $S_{r\,\min}$ and let $m_r = 2d_r - 2k$, $r = 1, 2$. Let the Lagrange form $[\cdot, \cdot]$ be given by (3.4). Assume there exist λ_1 and $\lambda_2 \in \mathbb{R}$ such that $(4.1)_{r=1}$ has d_1 linearly independent solutions lying in H_1 and $(4.1)_{r=2}$ has d_2 linearly independent solutions lying in H_2. Then there exist d_1 linearly independent real-valued solutions in H_1 and d_2 linearly independent real-valued solutions in H_2. A linear submanifold $D(S)$ of $D_{\max}$*

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is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist two complex $d\times n$ matrices A_1 and A_2 and two complex $d\times m_1$ and $d\times m_2$ matrices B_1 and B_2 such that the following three conditions hold.

(1) $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$.

(2) $sA_1E_nA_1^* - sB_1E_{m_1}B_1^* + lA_2E_nA_2^* - lB_2E_{m_2}B_2^* = 0$.

(3) $D(S) = \{y = \{y_1, y_2\} \in D_{\max}\}$, *where*

$$A_1\begin{pmatrix} y_1(a_1)\\ \vdots \\ y_1^{[n-1]}(a_1)\end{pmatrix} + B_1\begin{pmatrix} [y_1,u_1]_1(b_1)\\ \vdots \\ [y_1,u_{m_1}]_1(b_1)\end{pmatrix} + A_2\begin{pmatrix} y_2(a_2)\\ \vdots \\ y_2^{[n-1]}(a_2)\end{pmatrix} + B_2\begin{pmatrix} [y_2,v_1]_2(b_2)\\ \vdots \\ [y_2,v_{m_2}]_2(b_2)\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

PROOF. *Necessity.* Let $D(S)$ be the domain of a self-adjoint extension S of $S_{\min}$. By Lemma 4.11 there exist $w_1 = \{w_{11}, w_{12}\}, \ldots, w_d = \{w_{d1}, w_{d2}\} \in D_{\max}$ satisfying the conditions (i)–(iii) of Lemma 4.11. By Theorem 4.9, each w_{i1} and w_{i2} can be uniquely written as

$$w_{i1} = \widehat{y}_{i1} + \sum_{s=1}^{n} \tau_{is} z_s + \sum_{j=1}^{m_1} e_{ij} u_j, \tag{4.2}$$

$$w_{i2} = \widehat{y}_{i2} + \sum_{s=1}^{n} \sigma_{is} g_s + \sum_{j=1}^{m_2} h_{ij} v_j,$$

where $\widehat{y}_{i1} \in D_{1\min}$, $\widehat{y}_{i2} \in D_{2\min}$, $\tau_{ij}, e_{ij}, \sigma_{ij}, h_{ij} \in \mathbb{C}$. From (2.3),

$$([y_r, w_{jr}]_r(a_r))_{d\times 1} = (-1)^k V_r^* E_n \begin{pmatrix} y_r(a_r)\\ \vdots\\ y_r^{[n-1]}(a_r)\end{pmatrix},$$

where

$$V_r^* = \begin{pmatrix} \overline{w}_{1r}(a_r) & \cdots & \overline{w}_{1r}^{[n-1]}(a_r)\\ \cdots & \cdots & \cdots\\ \overline{w}_{dr}(a_r) & \cdots & \overline{w}_{dr}^{[n-1]}(a_r)\end{pmatrix}, \quad r = 1, 2,$$

$$\begin{aligned}([y_1, w_{i1}]_1(b_1))_{d\times 1} &= \Big(\Big[y_1, \widehat{y}_{i1} + \sum_{s=1}^{n}\tau_{is}z_s + \sum_{j=1}^{m_1} e_{ij}u_j\Big]_1(b_1)\Big)_{d\times 1}\\ &= (\bar{e}_{ij})_{d\times m_1}([y_1, u_j]_1(b_1))_{m_1\times 1},\\ ([y_2, w_{i2}]_2(b_2))_{d\times 1} &= \Big(\Big[y_2, \widehat{y}_{i2} + \sum_{s=1}^{n}\sigma_{is}g_s + \sum_{j=1}^{m_2} h_{ij}v_j\Big]_2(b_2)\Big)_{d\times 1}\\ &= (\bar{h}_{ij})_{d\times m_2}([y_2, v_j]_2(b_2))_{m_2\times 1}.\end{aligned}$$

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Let

$$A_1 = (-1)^{k+1} l V_1^* E_n, \quad B_1 = l(\bar{e}_{ij})_{d\times m_1}, \quad A_2 = (-1)^{k+1} s V_2^* E_n, \quad B_2 = s(\bar{h}_{ij})_{d\times m_2}.$$

Hence the boundary condition (iii) is equivalent to part (3) of Theorem 4.13.

Next we prove that A_1, B_1, A_2, B_2 satisfy the conditions (1) and (2) of Theorem 4.13.

Clearly $\operatorname{rank}(A_1, B_1, A_2, B_2) \le d$. If $\operatorname{rank}(A_1, B_1, A_2, B_2) < d$, then there exist constants $h_1, \ldots, h_d$, not all zero, such that

$$(h_1 \cdots h_d)(A_1, B_1, A_2, B_2) = 0. \tag{4.3}$$

Hence $(h_1 \cdots h_d)A_1 = (-1)^{k+1}(h_1 \cdots h_d) l V_1^* E_n = 0$ and

$$(h_1 \cdots h_d)A_2 = (-1)^{k+1}(h_1 \cdots h_d) s V_2^* E_n = 0.$$

Note that, since E is nonsingular and $l > 0$, $s > 0$,

$$V_1 \begin{pmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_d \end{pmatrix} = 0, \quad V_2 \begin{pmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_d \end{pmatrix} = 0.$$

Let $f = \{f_1, f_2\} = \sum_{i=1}^d \bar{h}_i w_i$, that is, $f_1 = \sum_{i=1}^d \bar{h}_i w_{i1}$, $f_2 = \sum_{i=1}^d \bar{h}_i w_{i2}$. Then

$$\begin{pmatrix} f_1(a_1) \\ \vdots \\ f_1^{[n-1]}(a_1) \end{pmatrix} = V_1 \begin{pmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_d \end{pmatrix} = 0, \quad \begin{pmatrix} f_2(a_2) \\ \vdots \\ f_2^{[n-1]}(a_2) \end{pmatrix} = V_2 \begin{pmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_d \end{pmatrix} = 0. \tag{4.4}$$

From (4.2),

$$f_1 = \sum_{i=1}^{d} \bar{h}_i \widehat{y}_{i1} + \sum_{i=1}^{d} \sum_{s=1}^{n} \bar{h}_i \tau_{is} z_s + \sum_{i=1}^{d} \sum_{j=1}^{m_1} \bar{h}_i e_{ij} u_j.$$

By (4.3), we have $(h_1 \cdots h_d)B_1 = 0$. Hence

$$f_1 = \sum_{i=1}^{d} \bar{h}_i \widehat{y}_{i1} + \sum_{i=1}^{d} \sum_{s=1}^{n} \bar{h}_i \tau_{is} z_s.$$

Similarly,

$$f_2 = \sum_{i=1}^{d} \bar{h}_i \widehat{y}_{i2} + \sum_{i=1}^{d} \sum_{s=1}^{n} \bar{h}_i \sigma_{is} g_s.$$

For any $y = \{y_1, y_2\} \in D_{\max}$,

$$[f_1, y_1]_1(b_1) = 0, \quad [f_2, y_2]_2(b_2) = 0. \tag{4.5}$$

By (4.4) and (4.5), $f_1 \in D_{1\,\min}$,$f_2 \in D_{2\,\min}$. Thus $f = \{f_1, f_2\} \in D_{\min}$. This contradicts the fact that the functions $w_1, w_2, \ldots, w_d$ are linearly independent modulo $D_{\min}$. Therefore $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$.

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Now we verify part (2). By (4.2),

$$l[w_{i1}, w_{j1}]_1(b_1) = l\left[\sum_{k=1}^{m_1} e_{ik}u_k, \sum_{s=1}^{m_1} e_{js}u_s\right]_1 (b_1)$$
$$= l\sum_{k=1}^{m_1}\sum_{s=1}^{m_1} e_{ik}\overline{e}_{js}[u_k, u_s]_1(b_1), \quad i, j = 1, \ldots, d.$$

Hence

$$(l[w_{i1}, w_{j1}]_1(b_1))^T_{d\times d} = \frac{1}{l}B_1U^TB_1^* = (-1)^k\frac{1}{l}B_1E_{m_1}B_1^*,$$

where the matrix U is defined in Corollary 4.10. Note that $(E_n)^* = -E_n$, $(E_n)^{-1} = -E_n$, and $A_1 = (-1)^{k+1}lV_1^*E_n$, and we have

$$(l[w_{i1}, w_{j1}]_1(a_1))^T_{d\times d} = (-1)^klV_1^*E_nV_1 = (-1)^klV_1^*E_nE_n^{-1}E_nV_1$$
$$= (-1)^kl(V_1^*E_n)(-E_n)(-(E_n)^*V_1) = (-1)^k\frac{1}{l}A_1E_nA_1^*.$$

Similarly,

$$(s[w_{i2}, w_{j2}]_2(b_2))^T_{d\times d} = (-1)^k\frac{1}{s}B_2E_{m_2}B_2^*,$$
$$(s[w_{i2}, w_{j2}]_2(a_2))^T_{d\times d} = (-1)^k\frac{1}{s}A_2E_nA_2^*.$$

Hence condition (ii) of Lemma 4.11 becomes

$$sA_1E_nA_1^* - sB_1E_{m_1}B_1^* + lA_2E_nA_2^* - lB_2E_{m_2}B_2^* = 0.$$

Sufficiency. Let the matrices A_1, B_1, A_2 and B_2 satisfy the conditions (1) and (2). We prove that $D(S)$ defined by (3) is the domain of a self-adjoint extension S of $S_{\min}$.

Let

$$B_1 = l(\overline{b}_{ij})_{d\times m_1}, \quad (-1)^{k+1}E_nA_1^* = \overline{l}(\rho_{ij})_{n\times d}, \tag{4.6}$$
$$s_{i1} = \sum_{j=1}^{m_1} b_{ij}u_j, \quad i = 1, \ldots, d. \tag{4.7}$$

By the Naimark patching lemma, we may choose functions $w_{11}, \ldots, w_{d1}$ in $D_{1\max}$ such that

$$w_{i1}^{[j-1]}(a_1) = \rho_{ij}, \quad w_{i1}^{[j-1]}(c) = s_{i1}^{[j-1]}(c),$$

and

$$w_{i1}(x) = s_{i1}(x), \quad x \geq c, \tag{4.8}$$

where $a_1 < c < b_1$, $i = 1, \ldots, d$, $j = 1, \ldots, n$.

By (4.6) and (2.3),

$$-A_1\begin{pmatrix} y_1(a_1) \\ \vdots \\ y_1^{[n-1]}(a_1)\end{pmatrix} = (-1)^kl(\rho_{ij})^*E_n\begin{pmatrix} y_1(a_1) \\ \vdots \\ y_1^{[n-1]}(a_1)\end{pmatrix} = \begin{pmatrix} l[y_1, w_{11}]_1(a_1) \\ \vdots \\ l[y_1, w_{d1}]_1(a_1)\end{pmatrix}.$$

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By (4.7) and (4.8),

$$B_1 \begin{pmatrix} [y_1, u_1]_1(b_1) \\ \vdots \\ [y_1, u_{m_1}]_1(b_1) \end{pmatrix} = \begin{pmatrix} [y_1, \bar{l} \sum_{j=1}^{m_1} b_{1j} u_j]_1(b_1) \\ \vdots \\ [y_1, \bar{l} \sum_{j=1}^{m_1} b_{dj} u_j]_1(b_1) \end{pmatrix} = \begin{pmatrix} l[y_1, w_{11}]_1(b_1) \\ \vdots \\ l[y_1, w_{d1}]_1(b_1) \end{pmatrix}.$$

Similarly,

$$-A_2 \begin{pmatrix} y_2(a_2) \\ \vdots \\ y_2^{[n-1]}(a_2) \end{pmatrix} = \begin{pmatrix} s[y_2, w_{12}]_2(a_2) \\ \vdots \\ s[y_2, w_{d2}]_2(a_2) \end{pmatrix},$$

$$B_2 \begin{pmatrix} [y_2, v_1]_2(b_2) \\ \vdots \\ [y_2, v_{m_2}]_2(b_2) \end{pmatrix} = \begin{pmatrix} s[y_2, w_{12}]_2(b_2) \\ \vdots \\ s[y_2, w_{d2}]_2(b_2) \end{pmatrix}.$$

Therefore the boundary condition (3) becomes the boundary condition (iii), that is,

$$l[y_1, w_{i1}]_1(b_1) - l[y_1, w_{i1}]_1(a_1) + s[y_2, w_{i2}]_2(b_2) - s[y_2, w_{i2}]_2(a_2) = 0, \quad i = 1, \ldots, d.$$

It remains to show that w_i, $i = 1, \ldots, d$, satisfy the conditions (i) and (ii) of Lemma 4.11.

The condition (i) holds. If not, then there exist constants $c_1, \ldots, c_d$, not all zero, such that

$$\gamma = \sum_{i=1}^{d} c_i w_i \in D_{\min}.$$

that is,

$$\gamma_1 = \sum_{i=1}^{d} c_i w_{i1} \in D_{1\,\min}, \quad \gamma_2 = \sum_{i=1}^{d} c_i w_{i2} \in D_{2\,\min}.$$

Hence

$$\begin{pmatrix} \gamma_1(a_1) \\ \vdots \\ \gamma_1^{[n-1]}(a_1) \end{pmatrix} = (\rho_{ij})_{n\times d} \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = (-1)^{k+1} \frac{1}{l} E_n A_1^* \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since E_n is nonsingular and $l > 0$, we conclude that

$$(\bar{c}_1 \cdots \bar{c}_d) A_1 = 0.$$

Similarly,

$$(\bar{c}_1 \cdots \bar{c}_d) A_2 = 0.$$

By (4.7) and (4.8),

$$\gamma_1(x) = \sum_{i=1}^{d} c_i w_{i1}(x) = \sum_{i=1}^{d} c_i \sum_{j=1}^{m_1} b_{ij} u_j(x), \quad x \geq c.$$

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Since $\gamma_1 \in D_{1\,\min}$, it follows that $[\gamma_1, y_1]_1(b_1) = 0$ for any $y = \{y_1, y_2\} \in D_{\max}$. Hence

$$(0 \cdots 0) = ([\gamma_1, u_1]_1(b_1) \cdots [\gamma_1, u_{m_1}]_1(b_1)) = (c_1 \cdots c_d)(b_{ij})_{d\times m_1} U,$$

Since U is nonsingular, $(\overline{c}_1 \cdots \overline{c}_d)B_1 = 0$.

Similarly, $(\overline{c}_1 \cdots \overline{c}_d)B_2 = 0$.

Hence

$$(\overline{c}_1 \cdots \overline{c}_d)(A_1, B_1, A_2, B_2) = 0.$$

This contradicts the fact that $\mathrm{rank}(A_1, B_1, A_2, B_2) = d$.

Next we show that (ii) holds. By (4.7) and (4.8),

$$l[w_{i1}, w_{j1}]_1(b_1) = l\Big[\sum_{s=1}^{m_1} b_{is}u_s, \sum_{k=1}^{m_1} b_{jk}u_k\Big]_1(b_1) = l\sum_{s=1}^{m_1}\sum_{k=1}^{m_1} b_{is}\overline{b}_{jk}[u_s, u_k]_1(b_1).$$

Hence

$$(l[w_{i1}, w_{j1}]_1(b_1))^T_{d\times d} = \frac{1}{l}B_1 U^T B_1^* = (-1)^k \frac{1}{l} B_1 E_{m_1} B_1^*.$$

Similarly,

$$(s[w_{i2}, w_{j2}]_2(b_2))^T_{d\times d} = \frac{1}{s}B_2 V^T B_2^* = (-1)^k \frac{1}{s} B_2 E_{m_2} B_2^*.$$

Moreover,

$$l([w_{i1}, w_{j1}]_1(a_1))^T_{d\times d} = (-1)^k l(\rho_{ij})^* E_n(\rho_{ij})_{n\times d} = (-1)^k \frac{1}{l} A_1 E_n A_1^*.$$

Similarly,

$$(s[w_{i2}, w_{j2}]_2(a_2))^T_{d\times d} = (-1)^k \frac{1}{s} A_2 E_n A_2^*.$$

Therefore

$$\begin{aligned}(l[w_{i1}, w_{j1}]_1(b_1) - l[w_{i1}, w_{j1}]_1(a_1) + s[w_{i2}, w_{j2}]_2(b_2) - s[w_{i2}, w_{j2}]_2(a_2))^T \\ = (-1)^k \frac{1}{l} B_1 E_{m_1} B_1^* - (-1)^k \frac{1}{l} A_1 E_n A_1^* \\ + (-1)^k \frac{1}{s} B_2 E_{m_2} B_2^* - (-1)^k \frac{1}{s} A_2 E_n A_2^* = 0.\end{aligned}$$

By Lemma 4.11, we conclude that $D(S)$ is a self-adjoint domain. □

Remark 4.14. We call $u_1, \ldots, u_{m_1}$ and $v_1, \ldots, v_{m_2}$ LC solutions at b_1 and b_2, respectively. They are used to characterise the self-adjoint boundary conditions at the singular endpoints b_1 and b_2, while the remaining $d_1 - m_1$ and $d_2 - m_2$ LP solutions $u_{m_1+1}, \ldots, u_{d_1}$ and $v_{m_2+1}, \ldots, v_{d_2}$ do not contribute to the singular boundary conditions. In fact, for any $y = \{y_1, y_2\} \in D_{\max}$, we have $[y_1, u_j]_1(b_1) = 0$, $j = m_1 + 1, \ldots, d_1$, and $[y_2, v_j]_2(b_2) = 0$, $j = m_2 + 1, \ldots, d_2$.

In Theorem 4.13 it is assumed that the endpoints a_1 and a_2 are regular and b_1, b_2 are singular. The proof of Theorem 4.13 can easily be modified to prove analogues of Theorem 4.13 as long as at least one endpoint of each interval (a_1, b_1), (a_2, b_2) is regular. Thus we have variants of Theorem 4.13 for a_1 singular, b_1 regular, a_2 singular,

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b_2 regular as well as for a_1 singular, b_1 regular, a_2 regular, b_2 singular and a_1 regular, b_1 singular, a_2 singular, b_2 regular. In the next theorem we state one of these cases explicitly as an illustration.

THEOREM 4.15. *Assume that a_1 is singular, b_1 is regular, a_2 is singular and b_2 is regular. Then a linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist a complex $d \times m_1$ matrix A_1, a $d \times n$ matrix B_1, a $d \times m_2$ matrix A_2 and a $d \times n$ matrix B_2 such that the following three conditions hold.*

(1) $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$.
(2) $sA_1E_{m_1}A_1^* - sB_1E_nB_1^* + lA_2E_{m_2}A_2^* - lB_2E_nB_2^* = 0$.
(3) $D(S) = \{y = \{y_1, y_2\} \in D_{\max}\}$, *where*

$$A_1\begin{pmatrix}[y_1, u_1]_1(a_1)\\ \vdots\\ [y_1, u_{m_1}]_1(a_1)\end{pmatrix} + B_1\begin{pmatrix}y_1(b_1)\\ \vdots\\ y_1^{[n-1]}(b_1)\end{pmatrix} + A_2\begin{pmatrix}[y_2, v_1]_2(a_2)\\ \vdots\\ [y_2, v_{m_2}]_2(a_2)\end{pmatrix} + B_2\begin{pmatrix}y_2(b_2)\\ \vdots\\ y_2^{[n-1]}(b_2)\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

In (2), E_j is the symplectic matrix (2.2) of order j.

PROOF. The proof is similar to that of Theorem 4.13 and therefore omitted. □

5. Examples

To illustrate the self-adjoint boundary conditions given by Theorem 4.13 and its variants we give a number of examples. Here we give some examples for

$$n = 4, \quad 4 \le d \le 8.$$

Similar examples can easily be constructed for all higher-order cases $n = 2k$, $k > 2$.

EXAMPLE 5.1. Assume the endpoint a_1 is singular, b_1 is regular, a_2 is regular and b_2 is singular. In the minimal deficiency case $d = 4$, we have $d_1 = 2$, $d_2 = 2$, $m_1 = 2d_1 - 4 = 0$, $m_2 = 2d_2 - 4 = 0$. Suppose that the boundary conditions at b_1, a_2 are coupled:

$$\begin{pmatrix}y_1(b_1)\\ y_1^{[1]}(b_1)\\ y_1^{[2]}(b_1)\\ y_1^{[3]}(b_1)\end{pmatrix} = K\begin{pmatrix}y_2(a_2)\\ y_2^{[1]}(a_2)\\ y_2^{[2]}(a_2)\\ y_2^{[3]}(a_2)\end{pmatrix}, \tag{5.1}$$

$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, i, j = 1, 2, 3, 4, \quad M_{14} - N_{14} < 0, M_{23} - N_{23} > 0,$$

$$M_{ij} = \begin{vmatrix}k_{i2} & k_{i3}\\ k_{j2} & k_{j3}\end{vmatrix}, \quad N_{ij} = \begin{vmatrix}k_{i1} & k_{i4}\\ k_{j1} & k_{j4}\end{vmatrix}, \quad i < j,\ i = 1, 2, 3,\ j = 2, 3, 4.$$

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Let

$$B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_2 = K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}.$$

Then $\operatorname{rank}(B_1, A_2) = 4$. From a straightforward computation, it follows that

$$sB_1EB_1^* = lA_2EA_2^*, \quad E = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is equivalent to:

(1) $M_{12} = N_{12}$;
(2) $M_{13} = N_{13}$;
(3) $M_{24} = N_{24}$;
(4) $M_{34} = N_{34}$;
(5) $l(M_{14} - N_{14}) = -s$;
(6) $l(M_{23} - N_{23}) = s$.

Therefore, if $l = 1$ and $s > 0$ so $M_{14} - N_{14} = -s$, $M_{23} - N_{23} = s$, and (1), (2), (3), (4) are satisfied, then the boundary conditions (5.1) are self-adjoint.

Remark 5.2. Note that $s > 0$ is needed to preserve the positivity of the inner product (3.2). Using appropriate multiples of the usual inner product, or changing the weight function w_2 to sw_2, we can generate self-adjoint operators for any real coupling matrix K satisfying $M_{14} - N_{14} = -s < 0$, $M_{23} - N_{23} = s > 0$ and (1), (2), (3), (4). This contrasts with the results in [7], where using the weight function w_2 requires $M_{14} - N_{14} = -1$, $M_{23} - N_{23} = 1$ and (1), (2), (3), (4) for self-adjointness. We see that the parameter s plays a role in establishing the self-adjoint boundary conditions.

Example 5.3. Let the endpoint a_1 be regular, b_1 singular, a_2 regular and b_2 singular. Assume $d_1 = 3$, $d_2 = 2$. Then $d = 5$ and $m_1 = 2$, $m_2 = 0$. Consider a separated condition at b_1 and coupled conditions at a_1, a_2:

$$C_1[y_1, u_1]_1(b_1) + C_2[y_1, u_2]_1(b_1) = 0, \quad C_1, C_2 \in R, (C_1, C_2) \neq (0, 0),$$

$$\begin{pmatrix} y_2(a_2) \\ y_2^{[1]}(a_2) \\ y_2^{[2]}(a_2) \\ y_2^{[3]}(a_2) \end{pmatrix} = K \begin{pmatrix} y_1(a_1) \\ y_1^{[1]}(a_1) \\ y_1^{[2]}(a_1) \\ y_1^{[3]}(a_1) \end{pmatrix}, \tag{5.2}$$

$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, i, j = 1, 2, 3, 4, \quad M_{14} - N_{14} > 0, M_{23} - N_{23} < 0,$$

$$M_{ij} = \begin{vmatrix} k_{i2} & k_{i3} \\ k_{j2} & k_{j3} \end{vmatrix}, \quad N_{ij} = \begin{vmatrix} k_{i1} & k_{i4} \\ k_{j1} & k_{j4} \end{vmatrix}, \quad i < j, \ i = 1, 2, 3, \ j = 2, 3, 4.$$

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Let

$$A_1 = K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case $\operatorname{rank}(A_1, B_1, A_2) = 5$ and $B_1 E_2 B_1^* = 0$. Then in terms of Theorem 4.13, we obtain the equivalence of the conditions for self-adjointness:

(1) $M_{12} = N_{12}$;
(2) $M_{13} = N_{13}$;
(3) $M_{24} = N_{24}$;
(4) $M_{34} = N_{34}$;
(5) $s(M_{14} - N_{14}) = l$;
(6) $s(M_{23} - N_{23}) = -l$.

Therefore, if $s = 1$ and $l > 0$ so $M_{14} - N_{14} = l$ and $M_{23} - N_{23} = -l$, and (1), (2), (3), (4) are satisfied, then the boundary conditions (5.2) are self-adjoint.

Note that by studying the two-interval theory in direct-sum spaces with inner product multiples we obtain self-adjoint operators for any real coupling matrix K satisfying $M_{14} - N_{14} = l > 0$, $M_{23} - N_{23} = -l < 0$ and (1), (2), (3), (4). This contrasts with the results in [7] which require $M_{14} - N_{14} = 1$, $M_{23} - N_{23} = -1$ and (1), (2), (3), (4).

EXAMPLE 5.4. Assume $d_1 = 3, d_2 = 3$. Then $d = 6$, and $m_1 = 2, m_2 = 2$. Let a_1 be regular, b_1 singular, a_2 singular and b_2 regular. Consider two pairs of coupled conditions:

$$\begin{pmatrix} [y_2, v_1]_2(a_2) \\ [y_2, v_2]_2(a_2) \end{pmatrix} = G \begin{pmatrix} [y_1, u_1]_1(b_1) \\ [y_1, u_2]_1(b_1) \end{pmatrix}, \tag{5.3}$$
$$G = (g_{ij}), \quad g_{ij} \in \mathbb{R}, \quad i, j = 1, 2, \quad \det G > 0,$$
$$\begin{pmatrix} y_2(b_2) \\ y_2^{[1]}(b_2) \\ y_2^{[2]}(b_2) \\ y_2^{[3]}(b_2) \end{pmatrix} = K \begin{pmatrix} y_1(a_1) \\ y_1^{[1]}(a_1) \\ y_1^{[2]}(a_1) \\ y_1^{[3]}(a_1) \end{pmatrix}, \tag{5.4}$$
$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, i, j = 1, 2, 3, 4, \quad M_{14} - N_{14} < 0, M_{23} - N_{23} > 0,$$
$$M_{ij} = \begin{vmatrix} k_{i2} & k_{i3} \\ k_{j2} & k_{j3} \end{vmatrix}, \quad N_{ij} = \begin{vmatrix} k_{i1} & k_{i4} \\ k_{j1} & k_{j4} \end{vmatrix}, \quad i < j, \ i = 1, 2, 3, j = 2, 3, 4.$$

Proceeding as in the previous example we obtain the equivalence of conditions for self-adjointness:

$$sGE_2G^* = lE_2 \quad \text{and} \quad sKE_4K^* = lE_4,$$

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$s(\det G) = l$ and the following equations.

(1) $M_{12} = N_{12}$.
(2) $M_{13} = N_{13}$.
(3) $M_{24} = N_{24}$.
(4) $M_{34} = N_{34}$.
(5) $s(M_{14} - N_{14}) = -l$.
(6) $s(M_{23} - N_{23}) = l$.

If we choose $s = 1$ and $l > 0$ so $\det G = l > 0$ and $M_{14} - N_{14} = -l < 0$, $M_{23} - N_{23} = l > 0$, and (1), (2), (3), (4) are satisfied, then boundary conditions (5.3) and (5.4) are self-adjoint.

REMARK 5.5. Using appropriate multiples of the usual inner product, we establish self-adjoint operators for any real coupling matrix K satisfying $\det G = l > 0$ and $M_{14} - N_{14} = -l < 0$, $M_{23} - N_{23} = l > 0$ and (1), (2), (3), (4). This contrasts with the results in [7] which require $\det G = 1$ and $M_{14} - N_{14} = -1$, $M_{23} - N_{23} = 1$ and (1), (2), (3), (4).

EXAMPLE 5.6. Assume $d_1 = 3, d_2 = 4$. Then $d = 7$ and $m_1 = 2, m_2 = 4$. Let a_1 be regular, b_1 singular, a_2 regular and b_2 singular. Consider separated conditions at a_1 and at b_1 and coupled conditions at a_2, b_2:

$$\begin{gathered} y_1(a_1) + iy_1^{[1]}(a_1) = 0, \quad iy_1^{[2]}(a_1) + y_1^{[3]}(a_1) = 0, \\ C_1[y_1, u_1]_1(b_1) + C_2[y_1, u_2]_1(b_1) = 0, \quad C_1, C_2 \in R, (C_1, C_2) \neq (0, 0), \end{gathered} \tag{5.5}$$

$$A_2 \begin{pmatrix} y_2(a_2) \\ y_2^{[1]}(a_2) \\ y_2^{[2]}(a_2) \\ y_2^{[3]}(a_2) \end{pmatrix} + B_2 \begin{pmatrix} [y_2, v_1]_2(b_2) \\ [y_2, v_2]_2(b_2) \\ [y_2, v_3]_2(b_2) \\ [y_2, v_4]_2(b_2) \end{pmatrix} = 0. \tag{5.6}$$

Then $sA_1E_4A_1^* - sB_1E_2B_1^* = 0$ for any s since $A_1E_4A_1^* = 0 = B_1E_2B_1^*$. In terms of Theorem 4.13, the boundary conditions (5.5) and (5.6) are self-adjoint if and only if $\operatorname{rank}(A_2, B_2) = 4$ and

$$A_2E_4A_2^* - B_2E_4B_2^* = 0.$$

Note that these conditions are independent of l and s and are simply the one-interval self-adjointness conditions for each of the two intervals separately. Thus the above example just gives the two-interval self-adjointness conditions which are generated by the direct sum of self-adjoint operators from each of the two intervals separately.

EXAMPLE 5.7. Assume a_1 is regular, b_1 is singular, a_2 is regular and b_2 is singular. In the maximal deficiency case $d = 8$, we have $d_1 = 4$, $d_2 = 4$ and $m_1 = 4$, $m_2 = 4$.

Consider separated conditions at a_1 and at a_2 and coupled conditions at b_1, b_2:

$$\begin{cases} y_1(a_1) + iy_1^{[1]}(a_1) = 0, \ iy_1^{[2]}(a_1) + y_1^{[3]}(a_1) = 0, \\ y_2(a_2) + iy_2^{[1]}(a_2) = 0, \ iy_2^{[2]}(a_2) + y_2^{[3]}(a_2) = 0, \end{cases} \tag{5.7}$$

$$\begin{pmatrix} [y_1, u_1]_1(b_1) \\ [y_1, u_2]_1(b_1) \\ [y_1, u_3]_1(b_1) \\ [y_1, u_4]_1(b_1) \end{pmatrix} = K \begin{pmatrix} [y_2, v_1]_2(b_2) \\ [y_2, v_2]_2(b_2) \\ [y_2, v_3]_2(b_2) \\ [y_2, v_4]_2(b_2) \end{pmatrix}, \tag{5.8}$$

$$K = (k_{ij}), \quad k_{ij} \in \mathbb{R}, i, j = 1, 2, 3, 4, \quad M_{14} - N_{14} > 0, M_{23} - N_{23} < 0,$$

$$M_{ij} = \begin{vmatrix} k_{i2} & k_{i3} \\ k_{j2} & k_{j3} \end{vmatrix}, \quad N_{ij} = \begin{vmatrix} k_{i1} & k_{i4} \\ k_{j1} & k_{j4} \end{vmatrix}, \quad i < j, \ i = 1, 2, 3, \ j = 2, 3, 4.$$

Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case $\operatorname{rank}(A_1, B_1, A_2, B_2) = 8$ and $sA_1EA_1^* + lA_2EA_2^* = 0$ for any s, l since $A_1EA_1^* = 0 = A_2EA_2^*$. Therefore the boundary conditions (5.7) and (5.8) are self-adjoint if and only if:

(1) $M_{12} = N_{12}$;
(2) $M_{13} = N_{13}$;
(3) $M_{24} = N_{24}$;
(4) $M_{34} = N_{34}$;
(5) $l(M_{14} - N_{14}) = s$;
(6) $l(M_{23} - N_{23}) = -s$.

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If we choose $l = 1$ and $s > 0$ so $M_{14} - N_{14} = s > 0$ and $M_{23} - N_{23} = -s < 0$, and (1), (2), (3), (4) are satisfied, then the boundary conditions (5.7) and (5.8) are self-adjoint. Note that the matrices generating this self-adjoint operator are nonreal.

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