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Non-tangential Maximal Function Characterizations of Hardy Spaces Associated with Degenerate Elliptic Operators

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Abstract. Let w be either in the Muckenhoupt class of $A_2(\mathbb{R}^n)$ weights or in the class of $QC(\mathbb{R}^n)$ weights, and let $L_w := -w^{-1}\operatorname{div}(A\nabla)$ be the degenerate elliptic operator on the Euclidean space $\mathbb{R}^n$, $n \geq 2$. In this article, the authors establish the non-tangential maximal function characterization of the Hardy space $H^p_{L_w}(\mathbb{R}^n)$ associated with L_w for $p \in (0,1]$, and when $p \in (\frac{n}{n+1}, 1]$ and $w \in A_{q_0}(\mathbb{R}^n)$ with $q_0 \in [1, \frac{p(n+1)}{n})$, the authors prove that the associated Riesz transform $\nabla L_w^{-1/2}$ is bounded from $H^p_{L_w}(\mathbb{R}^n)$ to the weighted classical Hardy space $H^p_w(\mathbb{R}^n)$.

1 Introduction

Let w be a nonnegative weight function such that w is either in the Muckenhoupt class of $A_2(\mathbb{R}^n)$ weights or in the class of $QC(\mathbb{R}^n)$ weights with $n \geq 2$. Let $H^1_0(w, \mathbb{R}^n)$ be the *Sobolev space*, which is defined to be the closure of $C^\infty_c(\mathbb{R}^n)$ with respect to the *norm*

$$\|f\|_{H^1_0(w,\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[|f(x)|^2 + |\nabla f(x)|^2 \right] w(x)\, dx \right\}^{1/2}.$$

For all $f, g \in H^1_0(w, \mathbb{R}^n)$, the *sesquilinear form* $\mathfrak{a}$ is defined by setting

$$\mathfrak{a}(f, g) := \int_{\mathbb{R}^n} (A(x)\nabla f(x)) \cdot \overline{\nabla g(x)}\, dx, \tag{1.1}$$

where $A := (A_{ij}(x))^n_{i,j=1}$ is a matrix of complex-valued measurable functions on $\mathbb{R}^n$ satisfying the *degenerate elliptic condition*; namely, there exist constants $0 < \lambda \leq \Lambda < \infty$ such that for all ξ and $\eta \in \mathbb{C}^n$,

$$|\langle A\xi, \eta\rangle| \leq \Lambda w(x)|\xi||\eta| \tag{1.2}$$

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and

$$\Re\langle A\xi,\xi\rangle \geq \lambda w(x)|\xi|^2, \tag{1.3}$$

here and hereafter, $\Re z$ for any $z\in\mathbb{C}$ denotes the *real part* of z. Then the associated *degenerate elliptic operator* L_w is defined by setting

$$L_w f := -\frac{1}{w}\operatorname{div}(A\nabla f), \tag{1.4}$$

for all $f\in H_0^1(w,\mathbb{R}^n)$. This is interpreted in the usual weak sense via the sesquilinear form; namely, for all $f,g\in H_0^1(w,\mathbb{R}^n)$,

$$\mathfrak{a}(f,g) = (L_w f, g)_{L^2(w,\mathbb{R}^n)} := \int_{\mathbb{R}^n} L_w f(x)\overline{g(x)}w(x)\,dx. \tag{1.5}$$

From its form, it is easy to see that the degenerate elliptic operator L_w, with the degeneracy controlled by the weight w, is a generalization of the usual uniformly elliptic operator. One motivation to study the degenerate elliptic operator L_w comes from the fact that, for some quasi-conformal mapping f and nonnegative harmonic function u defined in the range of f, $u\circ f$ satisfies a weighted degenerate elliptic equation with the weight $w := |f'|^{1-\frac{2}{n}}$, where $|f'|$ denotes the absolute value of the determinant of the Jacobian matrix f' of f (see [21] for more details on this fact).

In recent years, the study of the degenerate elliptic operators and their associated equations has attracted considerable attention (see, for example, [8–10, 21, 29] and, especially, some recent articles by Cruz–Uribe and Rios [12–14]). We point out that in the study of degenerate elliptic operators it is natural to assume that the weights w are in the Muckenhoupt class of $A_2(\mathbb{R}^n)$ weights or in the class of $QC(\mathbb{R}^n)$ weights, since the weighted Sobolev embedding theorems and the Poincaré inequalities hold true in these cases.

Let L_w be a degenerate elliptic operator as in (1.4) with w either in the Muckenhoupt class of $A_2(\mathbb{R}^n)$ weights or in the class of $QC(\mathbb{R}^n)$ weights (see Subsection 2.1 for their exact definitions). The main purpose of this article is to complete the real-variable theory of the weighted Hardy space associated with L_w.

It is well known that the theory of classical real Hardy spaces $H^p(\mathbb{R}^n)$, introduced by Stein and Weiss [37] in the early 1960s and systematically developed by Fefferman and Stein [22], is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, for $p\in(0,1]$, and plays important roles in various fields of analysis and partial differential equations. Notice that $H^p(\mathbb{R}^n)$ is essentially associated with the Laplace operator $\Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$; see, for instance, [20, 25, 28].

The motivation to study the Hardy spaces associated with different operators (for example, divergence form elliptic operators $-\operatorname{div}(A\nabla)$ and Schrödinger operators $-\Delta+V$) comes from characterizing the boundedness of the associated Riesz transforms and the regularity of solutions of the associated equations; see, for example, [2, 6, 17–20, 25, 27, 28, 31, 38].

To state the main results of this article, we first introduce some definitions and notation. Let $w\in A_2(\mathbb{R}^n)\cup QC(\mathbb{R}^n)$, L_w be as in (1.4) and $f\in L^2(w,\mathbb{R}^n)$, where

$L^2(w,\mathbb{R}^n)$ denotes the *weighted Lebesgue space* with the *norm*

$$\|f\|_{L^2(w,\mathbb{R}^n)} := \Big\{ \int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx \Big\}^{\frac{1}{2}}.$$

It is well known that if $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, then $L^2(w,\mathbb{R}^n)$ is a space of homogeneous type in the sense of Coifman and Weiss, since $w(x)\,dx$ is a doubling measure. In what follows, let $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$. For any $f \in L^2(w,\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *square function* $\mathcal{S}_{L_w}(f)$ *associated with* L_w is defined by setting

$$\mathcal{S}_{L_w}(f)(x) := \Big[\iint_{\Gamma(x)} \big| t^2 L_w e^{-t^2 L_w}(f)(y) \big|^2 w(y) \frac{dy}{w(B(x,t))} \frac{dt}{t} \Big]^{1/2}, \tag{1.6}$$

where, for all $x \in \mathbb{R}^n$, $t \in (0,\infty)$, $\alpha \in (0,\infty)$ and balls $B(x,t)$,

$$w(B(x,t)) := \int_{B(x,t)} w(y)\,dy,$$

and

$$\Gamma_\alpha(x) := \big\{ (y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t \big\} \tag{1.7}$$

denotes the *cone of aperture* α *with vertex* x. In particular, if $\alpha = 1$, we write $\Gamma(x)$ instead of $\Gamma_\alpha(x)$.

Definition 1.1 Let $p \in (0,1]$, $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator as in (1.4) with the matrix A satisfying the degenerate elliptic conditions (1.2) and (1.3). The *Hardy space* $H^p_{L_w}(\mathbb{R}^n)$, *associated with* L_w, is defined as the completion of the space

$$\big\{ f \in L^2(w,\mathbb{R}^n) : \|S_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} < \infty \big\}$$

with respect to the *quasi-norm*

$$\|f\|_{H^p_{L_w}(\mathbb{R}^n)} := \|S_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)}. \tag{1.8}$$

Remark 1.2 (i) The definition of the above Hardy space $H^p_{L_w}(\mathbb{R}^n)$ uses the strategy that we first restrict the work space to $L^2(w,\mathbb{R}^n)$ and then extend the work space via the quasi-norm (1.8) defined by the square function. This strategy was first introduced by P. Auscher, X. T. Duong, and A. McIntosh in an unpublished manuscript (see also [2]) and has proved to be a very useful method in the study on the real-variable theory of function spaces associated with operators.

(ii) It is easy to see that in Definition 1.1 if $w \equiv 1$, then $H^p_{L_w}(\mathbb{R}^n)$ is the Hardy space associated with the second order divergence form elliptic operator studied in [27, 28, 31], and, moreover, if $L_w \equiv -\Delta$, then $H^p_{L_w}(\mathbb{R}^n)$ is just the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman and Stein [22].

(iii) In [12, 13], Cruz–Uribe and Rios proved that L_w is a sectorial operator in $L^2(w,\mathbb{R}^n)$ satisfying the so-called bounded H_∞ functional calculus and the *weighted Davies–Gaffney estimates* in $L^2(w,\mathbb{R}^n)$. Namely, there exist positive constants c and C such that for all closed subsets $E, F \subset \mathbb{R}^n$ and $f \in L^2(w,\mathbb{R}^n)$ with $\operatorname{supp} f \subset E$,

$$\|e^{-tL_w}(f)\|_{L^2(w,F)} \le C e^{-c\frac{[d(E,F)]^2}{t}} \|f\|_{L^2(w,E)}. \tag{1.9}$$

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Here and hereafter, for any measurable function g, define $\|g\|_{L^2(w,E)} := \|g\chi_E\|_{L^2(w,\mathbb{R}^n)}$. These results, together with Remark 2.10, show that L_w is a special case of the operators that were considered in [4], where a part of the real-variable theory of Hardy-type spaces associated with some abstract operators was established. Thus, by [4, Theorem 4.8], we know that $H^p_{L_w}(\mathbb{R}^n)$ has a molecular characterization (see Section 3 for more details on this characterization). However, the non-tangential maximal function characterization of $H^p_{L_w}(\mathbb{R}^n)$ is still missing and we will show that this non-tangential maximal function characterization strongly depends on the special structure of the operator L_w.

Now, motivated by Hofmann and Mayboroda [27], for any $f \in L^2(w,\mathbb{R}^n)$, we define the *non-tangential maximal function* $\mathcal{N}_h(f)$ associated with the heat semigroup generated by L_w via setting

$$\mathcal{N}_h(f)(x) := \sup_{(y,t)\in\Gamma(x)} \left[\frac{1}{w(B(y,t))}\int_{B(y,t)} |e^{-t^2 L_w}(f)(z)|^2 w(z)\,dz\right]^{1/2}, \tag{1.10}$$

for all $x \in \mathbb{R}^n$. Then the *Hardy space* $H^p_{L_w,\mathcal{N}_h}(\mathbb{R}^n)$, associated with L_w, is defined as in Definition 1.1 with S_{L_w} replaced by the non-tangential maximal function $\mathcal{N}_h$.

The following theorem establishes the non-tangential maximal function characterization of $H^p_{L_w}(\mathbb{R}^n)$.

Theorem 1.3 *Let $p \in (0,1]$, $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator as in (1.4) satisfying the degenerate elliptic conditions (1.2) and (1.3). Then the weighted Hardy spaces $H^p_{L_w}(\mathbb{R}^n)$ and $H^p_{L_w,\mathcal{N}_h}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

We prove Theorem 1.3 borrowing some ideas from Hofmann and Mayboroda [27], where the authors considered the case when $w \equiv 1$ and $p = 1$. More precisely, to prove the inclusion

$$H^p_{L_w,\mathcal{N}_h}(\mathbb{R}^n) \subset H^p_{L_w}(\mathbb{R}^n),$$

we show that, for all $f \in L^2(w,\mathbb{R}^n) \cap H^p_{L_w,\mathcal{N}_h}(\mathbb{R}^n)$ and $p \in (0,1]$,

$$\|\mathcal{S}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\widetilde{\mathcal{S}}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\mathcal{N}_h(f)\|_{L^p(w,\mathbb{R}^n)},$$

(see Theorems 3.5 and 3.6), where $\mathcal{S}_{L_w}(f)$, $\widetilde{\mathcal{S}}_{L_w}(f)$ and $\mathcal{N}_h(f)$ are defined, respectively, as in (1.6), (3.1), and (1.10).

To prove the inclusion

$$H^p_{L_w}(\mathbb{R}^n) \subset H^p_{L_w,\mathcal{N}_h}(\mathbb{R}^n),$$

we use the weighted molecular characterization of $H^p_{L_w}(\mathbb{R}^n)$ (see Theorem 3.4 below) to prove that, for each weighted molecule m, $\mathcal{N}_h(m)$ is uniformly bounded in $L^p(w,\mathbb{R}^n)$ (see Theorem 3.7). The proof of Theorem 3.7 rests on the *weighted off-diagonal estimates on balls* of the heat semigroup generated by $-L_w$ (see Proposition 1.5).

We first recall from [1] the following notion of *weighted off-diagonal estimates on balls*. In what follows, for $p \in [1,\infty)$, the *space* $L^p_{\mathrm{loc}}(w,\mathbb{R}^n)$ denotes the set of all locally p-integrable functions on the measure $w(x)\,dx$ of $\mathbb{R}^n$.

Definition 1.4 ([1]) Let $p,q \in [1,\infty]$ with $p \le q$, $w \in A_\infty(\mathbb{R}^n)$ and let $\{T_t\}_{t>0}$ be a family of sublinear operators. The family $\{T_t\}_{t>0}$ is said to satisfy *weighted L^p-L^q off-diagonal estimates on balls*, denoted by $T_t \in \mathcal{O}_w(L^p\text{-}L^q)$, if there exist constants $\theta_1,\theta_2 \in [0,\infty)$, and $C,c \in (0,\infty)$ such that, for all $t \in (0,\infty)$ and all balls $B := B(x_B,r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and $f \in L^p_{\mathrm{loc}}(w,\mathbb{R}^n)$,

$$\left\{\frac{1}{w(B)}\int_B |T_t(\chi_B f)(x)|^q w(x)\,dx\right\}^{1/q} \le C\left[\Upsilon\left(\frac{r_B}{t^{1/2}}\right)\right]^{\theta_2}\left\{\frac{1}{w(B)}\int_B |f(x)|^p w(x)\,dx\right\}^{1/p}, \tag{1.11}$$

and, for all $j \in \mathbb{N}$ with $j \ge 3$,

$$\left\{\frac{1}{w(2^jB)}\int_{U_j(B)} |T_t(\chi_B f)(x)|^q w(x)\,dx\right\}^{1/q} \le C2^{j\theta_1}\left[\Upsilon\left(\frac{2^jr_B}{t^{1/2}}\right)\right]^{\theta_2} e^{-c\frac{(2^jr_B)^2}{t}}\left\{\frac{1}{w(B)}\int_B |f(x)|^p w(x)\,dx\right\}^{1/p} \tag{1.12}$$

and

$$\left\{\frac{1}{w(B)}\int_B |T_t(\chi_{U_j(B)} f)(x)|^q w(x)\,dx\right\}^{1/q} \le C2^{j\theta_1}\left[\Upsilon\left(\frac{2^jr_B}{t^{1/2}}\right)\right]^{\theta_2} e^{-c\frac{(2^jr_B)^2}{t}}\left\{\frac{1}{w(2^jB)}\int_{U_j(B)} |f(x)|^p w(x)\,dx\right\}^{1/p}, \tag{1.13}$$

where $U_j(B)$ is as in (1.15), and for all $s \in (0,\infty)$,

$$\Upsilon(s) := \max\left\{s,\frac{1}{s}\right\}. \tag{1.14}$$

The following weighted off-diagonal estimates on balls play a key role in proving Theorem 3.7. In what follows, for any $p \in [1,\infty]$, we denote by p' its *conjugate exponent*, namely, $1/p + 1/p' = 1$.

Proposition 1.5 *Let $l \in \mathbb{Z}_+$, $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator satisfying the degenerate elliptic conditions (1.2) and (1.3). Then there exists a number $k_0 \in (1,\infty)$ such that, for all $(2k_0)' \le p \le q \le 2k_0$ and $t \in (0,\infty)$, the family $(tL_w)^l e^{-tL_w} \in \mathcal{O}_w(L^p\text{-}L^q)$. Moreover, when $w \in A_2(\mathbb{R}^n)$, $k_0 = \frac{n}{n-1}$.*

Recall that, in [12, Theorem 1.6], Cruz–Uribe and Rios established some weighted Davies–Gaffney estimates for L_w, which are equivalent to $(tL_w)^k e^{-tL_w} \in \mathcal{O}_w(L^2\text{-}L^2)$ (see also [1]). Thus, Proposition 1.5 extends the corresponding result of Cruz-Uribe and Rios [12]. Moreover, the proof of Proposition 1.5 is totally different from that of [12, Theorem 1.6]. The proof of [12, Theorem 1.6] reduced the desired weighted Davies–Gaffney estimates into the corresponding estimates of the resolvent, while the proof of Proposition 1.5 strongly depends on the local weighted Sobolev embedding theorems in [21], for both $A_2(\mathbb{R}^n)$ and $QC(\mathbb{R}^n)$ weights, and the weighted Davies–Gaffney estimates for $\{\sqrt{t}\nabla e^{-tL_w}\}_{t>0}$ (see Proposition 2.6), whose proof depends on the exponential perturbation method from [15].

Finally, as an application of $H^p_{L_w}(\mathbb{R}^n)$, we establish the following boundedness of the associated Riesz transforms $\nabla L_w^{-1/2}$.

Theorem 1.6 *Let $p \in (\frac{n}{n+1}, 1]$, $w \in A_{q_0}(\mathbb{R}^n)$ with $q_0 \in [1, \frac{p(n+1)}{n})$ and L_w be the degenerate elliptic operator as in (1.4) satisfying the degenerate elliptic conditions (1.2) and (1.3). Then the Riesz transform $\nabla L_w^{-1/2}$ is bounded from $H^p_{L_w}(\mathbb{R}^n)$ to $H^p_w(\mathbb{R}^n)$.*

Recall that the boundedness of operated-adapted Riesz transforms on the associated Hardy spaces was first established by Hofmann et al. [25] in the case $p = 1$. To prove Theorem 1.6, we borrow some ideas from [3, 4, 18, 25, 28, 31, 32]. In particular, we need some off-diagonal estimates of the following families of operators

$$\{\nabla L_w^{-1/2}(I - e^{-tL_w})^M\}_{t>0} \quad \text{and} \quad \{\nabla L_w^{-1/2}(tL_w e^{-tL_w})^M\}_{t>0}$$

(see Proposition 4.1), whose proofs rest on the weighted off-diagonal estimates of the gradient semigroup $\{\sqrt{t}\nabla e^{-tL_w}\}_{t>0}$ (see Proposition 2.7). We point out that, since we can only show that, for each $(p, 2, M, \epsilon)_{L_w}$-molecule m (see Definition 3.1), $\nabla L_w^{-1/2}(m)$ is a classical weighted Hardy molecule (see Definition 4.6), which only has the zero-order vanishing moment, this forces us to restrict the range of the weights to a smaller Muckenhoupt weight class $A_{q_0}(\mathbb{R}^n)$, with $q_0 \in [1, \frac{p(n+1)}{n})$, than $A_2(\mathbb{R}^n)$.

This article is organized as follows. In Subsection 2.1, we first recall some notions and results on Muckenhoupt weights and $QC(\mathbb{R}^n)$ weights; then, in Subsection 2.2, we establish the weighted off-diagonal estimates of L_w and prove Proposition 1.5. Section 3 is devoted to the proof of Theorem 1.3, while Theorem 1.6 is proved in Section 4.

We end this section by making some conventions on notation. Throughout this article, L_w always denotes a degenerate elliptic operator as in (1.4). We denote by C a positive constant that is independent of the main parameters, but which may vary from line to line. We also use $C_{(\alpha,\beta,\dots)}$ to denote a positive constant depending on the parameters $\alpha, \beta, \dots$. The *symbol* $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^\complement$ the *set* $\mathbb{R}^n \setminus E$. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, $\alpha \in (0, \infty)$, and $j \in \mathbb{N}$, we let $\alpha B := B(x_B, \alpha r_B)$,

$$U_0(B) := B \quad \text{and} \quad U_j(B) := (2^j B) \setminus (2^{j-1} B). \tag{1.15}$$

2 Preliminaries

In this section, we first recall the definition of the *Muckenhoupt weights*, the $QC(\mathbb{R}^n)$ *weights*, and some of their properties. Then we establish the weighted off-diagonal estimates on balls of the operator L_w, which play a key role in the proofs of our main results.

2.1 Muckenhoupt Weights and $QC(\mathbb{R}^n)$ Weights

Let $q \in [1, \infty)$. A nonnegative locally integrable function w on $\mathbb{R}^n$ is said to belong to the *Muckenhoupt class* $A_q(\mathbb{R}^n)$ if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|} \int_B w(x)\, dx \Big\{ \frac{1}{|B|} \int_B [w(x)]^{-\frac{1}{q-1}}\, dx \Big\}^{q-1} \le C$$

when $q \in (1, \infty)$, and, when $q = 1$,

$$\frac{1}{|B|} \int_B w(x)\, dx \le C \operatorname*{ess\,inf}_{x \in B} w(x).$$

We also let $A_\infty(\mathbb{R}^n) := \bigcup_{q \in [1,\infty)} A_q(\mathbb{R}^n)$ and $w(E) := \int_E w(x)\, dx$ for any measurable set $E \subset \mathbb{R}^n$.

Let $r \in (1, \infty]$. A nonnegative locally integrable function w is said to belong to the *reverse Hölder class* $RH_r(\mathbb{R}^n)$ if there exists a positive constant C such that, when $r \in (1, \infty)$, for all balls $B \subset \mathbb{R}^n$,

$$\Big\{ \frac{1}{|B|} \int_B [w(x)]^r\, dx \Big\}^{1/r} \le C \frac{1}{|B|} \int_B w(x)\, dx,$$

where we replace $\{\frac{1}{|B|} \int_B [w(x)]^r\, dx\}^{1/r}$ by $\|w\|_{L^\infty(B)}$ when $r = \infty$.

To define the $QC(\mathbb{R}^n)$ weights, for $n \ge 2$, let $f := (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism whose components $\{f_i\}_{i=1}^n$ have distributional derivatives in $L^n_{\mathrm{loc}}(\mathbb{R}^n)$. Then f is called a *quasi-conformal mapping* if there exists a positive constant k such that, for almost every $x \in \mathbb{R}^n$,

$$\Big[\sum_{i,j=1}^n \Big| \frac{\partial f_i}{\partial x_j}(x) \Big|^2 \Big]^{1/2} \le k |f'(x)|^{1/n},$$

where

$$f'(x) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{2.1}$$

denotes the determinant of the *Jacobian matrix* of f. Given such an f, the locally integrable function $w(x) := |f'(x)|^{1-2/n}$ (specifically, when $n = 2$, $w(x) \equiv 1$) for almost every $x \in \mathbb{R}^n$ is called a $QC(\mathbb{R}^n)$ *weight*, denoted by $w \in QC(\mathbb{R}^n)$.

Recall that $QC(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$ (see [21, p. 107]).

We recall some properties of the Muckenhoupt classes and the reverse Hölder classes in the following two lemmas (see, for example, [16] for their proofs).

Lemma 2.1

(i) *If* $1 \le p \le q \le \infty$, *then* $A_1(\mathbb{R}^n) \subset A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$.
(ii) $A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1,\infty)} A_p(\mathbb{R}^n) = \bigcup_{r \in (1,\infty]} RH_r(\mathbb{R}^n)$.

Lemma 2.2 *Let* $q \in [1, \infty)$ *and* $r \in (1, \infty]$. *If a nonnegative measurable function* $w \in A_q(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$, *then there exists a constant* $C \in (1, \infty)$ *such that, for all balls*

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$B \subset \mathbb{R}^n$ and any measurable subset E of B,

$$C^{-1}\Big(\frac{|E|}{|B|}\Big)^q \le \frac{w(E)}{w(B)} \le C\Big(\frac{|E|}{|B|}\Big)^{\frac{r-1}{r}}.$$

2.2 Weighted Off-diagonal Estimates for L_w

In this subsection, we establish some weighted off-diagonal estimates for L_w. To this end, by using the method of Davies [15], we need to introduce a twist sesquilinear form of $\mathfrak{a}$ in (1.1) under exponential perturbation. More precisely, let $\mathcal{E}(\mathbb{R}^n)$ be the set of all bounded real-valued functions $\phi \in C^\infty(\mathbb{R}^n)$ such that, for all multi-indices $\alpha \in (\mathbb{Z}_+)^n$ and $|\alpha| = 1$, $\|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \le 1$. The set $\mathcal{E}(\mathbb{R}^n)$ of functions plays an important role when we consider the distance between two closed sets in $\mathbb{R}^n$.

Let E and F be two disjoint closed subsets of $\mathbb{R}^n$. Let $d(E,F)$ be the *Euclidean distance* between E and F, namely,

$$d(E,F) := \inf\{|x-y| : x \in E,\ y \in F\}.$$

Define

$$\widetilde{d}(E,F) := \sup_{\phi \in \mathcal{E}(\mathbb{R}^n)} \Big[\inf\big\{\phi(x) - \phi(y) : x \in E, y \in F\big\}\Big].$$

The following result implies that $d(E,F)$ and $\widetilde{d}(E,F)$ are comparable. Notice that Davies [15, Lemma 4] proved a similar result, in a different way, by requiring the sets E and F to be compact and convex. Lemma 2.3 is more general, and its proof is simpler than that of [15, Lemma 4].

Lemma 2.3 *There exists a positive constant C such that, for any two disjoint closed subsets $\{E,F\}$ of $\mathbb{R}^n$,*

$$\frac{1}{C}\widetilde{d}(E,F) \le d(E,F) \le C\widetilde{d}(E,F). \tag{2.2}$$

Proof Let $\phi \in \mathcal{E}(\mathbb{R}^n)$. The fact that $\|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \le 1$ for all $\alpha \in (\mathbb{Z}_+)^n$ and $|\alpha| = 1$ implies that, for all $x \in E$ and $y \in F$,

$$|\phi(x) - \phi(y)| \lesssim |x-y|,$$

which further yields $\widetilde{d}(E,F) \lesssim d(E,F)$.

Let us prove the second inequality of (2.2). If $d(E,F) = 0$, then the required inequality is obvious. Suppose now that $d(E,F) > 0$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp}\phi \subset B(0,1)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. Let

$$\widetilde{E} := \Big\{x \in \mathbb{R}^n :\ d(x,E) < \frac{1}{4}d(E,F)\Big\}.$$

For $\epsilon := \frac{1}{4}d(E,F)$ and $\phi_\epsilon(\cdot) := \epsilon^{-n}\phi(\frac{\cdot}{\epsilon})$, let

$$\psi := \frac{\epsilon}{C_{(\phi)}}\chi_{\widetilde{E}} * \phi_\epsilon,$$

where $C_{(\phi)} := \int_{\mathbb{R}^n} |\nabla\phi(x)|\,dx > 0$. The choice of ϕ implies that $\psi \in \mathcal{E}(\mathbb{R}^n)$.

Moreover, for all $x \in E$, by the definition of $\widetilde{E}$, we know that $B(x, \frac{1}{4}d(E,F)) \subset \widetilde{E}$. Thus, for all $x \in E$ and $y \in F$, it holds true that

$$\psi(x) - \psi(y) = \psi(x) = \frac{1}{4C_{(\phi)}} d(E,F) \int_{B(x,\frac{1}{4}d(E,F))} \epsilon^n \phi\Big(\frac{x-z}{\epsilon}\Big)\, dz = \frac{1}{4C_{(\phi)}} d(E,F),$$

which implies the second inequality of (2.2). This completes the proof of Lemma 2.3. ∎

Now, for $\nu \in \mathbb{R}_+ := (0,\infty)$ and $\phi \in \mathcal{E}(\mathbb{R}^n)$, let

$$L_{\nu,\phi} := e^{\nu\phi} L_w e^{-\nu\phi}. \tag{2.3}$$

For all $f, g \in H_0^1(w,\mathbb{R}^n)$, the *twist sesquilinear form* $\mathfrak{a}_{\nu,\phi}$ is defined by setting

$$\mathfrak{a}_{\nu,\phi}(f,g) := \int_{\mathbb{R}^n} \big(A(x)\nabla(e^{-\nu\phi} f)(x)\big) \cdot \nabla(e^{\nu\phi} g)(x)\, dx. \tag{2.4}$$

Then, by the definition of L_w, we know that

$$\mathfrak{a}_{\nu,\phi}(f,g) = \big(L_{\nu,\phi}(f), g\big)_{L^2(w,\mathbb{R}^n)}. \tag{2.5}$$

Namely, $L_{\nu,\phi}$ is the operator associated with $\mathfrak{a}_{\nu,\phi}$. Let also $\{e^{-tL_{\nu,\phi}}\}_{t>0}$ be the heat semigroup generated by $L_{\nu,\phi}$.

Notice that conditions (1.2) and (1.3) imply that L_w is of type $\omega := \arctan(\Lambda/\lambda) \in [0, \frac{\pi}{2})$; see [33] (also [12, p. 293]) for details. Hence, for $z \in \Sigma(\pi/2 - \omega)$, where

$$\Sigma(\pi/2-\omega) := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi/2 - \omega\},$$

it holds true that

$$e^{-zL_w}(f) = \frac{1}{2\pi i} \int_\Gamma e^{z\xi} (\xi I + L_w)^{-1}(f)\, d\xi, \tag{2.6}$$

where $\theta \in (\pi/2 + |\arg(z)|, \pi - \omega)$ and

$$\Gamma := \gamma^+ \cup \gamma^- := \big\{z \in \mathbb{C} : z = r^{i\theta}, r \in (0,\infty)\big\} \cup \big\{z \in \mathbb{C} : z = r^{-i\theta}, r \in (0,\infty)\big\}.$$

This, together with (2.3), implies that, for all $t \in (0,\infty)$,

$$e^{-tL_{\nu,\phi}} = e^{\nu\phi} e^{-tL_w} e^{-\nu\phi}. \tag{2.7}$$

We have the following perturbation estimate.

Lemma 2.4 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator satisfying the degenerate elliptic conditions (1.2) and (1.3). Then there exists a positive constant C such that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, and $f \in H_0^1(w,\mathbb{R}^n)$,*

$$|\mathfrak{a}_{\nu,\phi}(f,f) - \mathfrak{a}(f,f)| \le \frac{1}{4}\Re\big\{\mathfrak{a}(f,f)\big\} + C\nu^2 \|f\|^2_{L^2(w,\mathbb{R}^n)}. \tag{2.8}$$

Proof Let $f \in H_0^1(w, \mathbb{R}^n)$. By (2.4) and an elementary calculation, we see that

$$\mathfrak{a}_{v,\phi}(f,f) = -v^2 \int_{\mathbb{R}^n} (A(x)f(x)\nabla\phi(x)) \cdot \overline{f(x)\nabla\phi(x)}\,dx - v \int_{\mathbb{R}^n} (A(x)f(x)\nabla\phi(x)) \cdot \overline{\nabla f(x)}\,dx + v \int_{\mathbb{R}^n} (A(x)\nabla f(x)) \cdot \overline{f(x)\nabla\phi(x)}\,dx + \int_{\mathbb{R}^n} (A(x)\nabla f(x)) \cdot \overline{\nabla f(x)}\,dx, \tag{2.9}$$

which, together with (1.1), implies that

$$|\mathfrak{a}_{v,\phi}(f,f) - \mathfrak{a}(f,f)| \le \left| v^2 \int_{\mathbb{R}^n} (A(x)f(x)\nabla\phi(x)) \cdot \overline{f(x)\nabla\phi(x)}\,dx \right| + \left| v \int_{\mathbb{R}^n} (A(x)f(x)\nabla\phi(x)) \cdot \overline{\nabla f(x)}\,dx \right| + \left| v \int_{\mathbb{R}^n} (A(x)\nabla f(x)) \cdot \overline{f(x)\nabla\phi(x)}\,dx \right| =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \tag{2.10}$$

For I_1, by the condition that $\phi \in \mathcal{E}(\mathbb{R}^n)$ and the degenerate elliptic condition (1.2), we know that

$$\mathrm{I}_1 \lesssim v^2 \int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx \sim v^2 \|f\|^2_{L^2(w,\mathbb{R}^n)}. \tag{2.11}$$

For I_2, using again the condition that $\phi \in \mathcal{E}(\mathbb{R}^n)$, the degenerate elliptic conditions (1.2) and (1.3), and the Young inequality with ϵ, we see that

$$\begin{aligned} \mathrm{I}_2 &\lesssim v \int_{\mathbb{R}^n} |f(x)||\nabla f(x)| w(x)\,dx \\ &\lesssim \epsilon \int_{\mathbb{R}^n} |\nabla f(x)|^2 w(x)\,dx + \frac{v^2}{4\epsilon} \int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx \\ &\lesssim \epsilon \Re\Big\{ \int_{\mathbb{R}^n} (A(x)\nabla f(x)) \cdot \overline{\nabla f(x)}\,dx \Big\} + \frac{v^2}{4\epsilon} \|f\|^2_{L^2(w,\mathbb{R}^n)} \\ &\sim \epsilon \Re\big\{ \mathfrak{a}(f,f) \big\} + \frac{v^2}{4\epsilon} \|f\|^2_{L^2(w,\mathbb{R}^n)}. \end{aligned} \tag{2.12}$$

Similar to (2.12), we also have

$$\mathrm{I}_3 \lesssim \epsilon \Re\big\{ \mathfrak{a}(f,f) \big\} + \frac{v^2}{4\epsilon} \|f\|^2_{L^2(w,\mathbb{R}^n)},$$

which, combined with (2.9)–(2.12), and a suitable choice of ϵ, implies that (2.8) holds true. This finishes the proof of Lemma 2.4. ∎

We also need the following technical lemma. Recall that for all $f, g \in L^2(w, \mathbb{R}^n)$,

$$(f,g)_{L^2(w,\mathbb{R}^n)} := \int_{\mathbb{R}^n} f(x)\overline{g(x)} w(x)\,dx.$$

Lemma 2.5 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, $k \in \mathbb{Z}_+$ and let L_w be the degenerate elliptic operator satisfying the degenerate elliptic conditions (1.2) and (1.3). Then there exist*

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positive constants C_0 and C_1 such that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\|(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} \le C_0 e^{C_1\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)}. \tag{2.13}$$

Proof We first prove Lemma 2.5 in the case $k=0$. Let $f \in L^2(w,\mathbb{R}^n)$ and $f_t := e^{-tL_{\nu,\phi}}(f)$. Using (2.5), Lemma 2.4, and the degenerate elliptic condition (1.3), we conclude that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
\frac{d}{dt}\|f_t\|^2_{L^2(w,\mathbb{R}^n)} &= \frac{d}{dt}(e^{-tL_{\nu,\phi}}(f), e^{-tL_{\nu,\phi}}(f))_{L^2(w,\mathbb{R}^n)}\\
&= -\big\{(L_{\nu,\phi}(f_t),f_t) + (f_t, L_{\nu,\phi}(f_t))\big\} = -2\Re\big\{\mathfrak{a}_{\nu,\phi}(f_t,f_t)\big\}\\
&= -2\Re\big\{[\mathfrak{a}_{\nu,\phi}(f_t,f_t) - \mathfrak{a}(f_t,f_t)]\big\} - 2\Re\big\{\mathfrak{a}(f_t,f_t)\big\}\\
&\le 2|\mathfrak{a}_{\nu,\phi}(f_t,f_t) - \mathfrak{a}(f_t,f_t)| - 2\Re\big\{\mathfrak{a}(f_t,f_t)\big\}\\
&\le C\nu^2\|f\|_{L^2(w,\mathbb{R}^n)} - \frac{3}{2}\Re\big\{\mathfrak{a}(f_t,f_t)\big\} \lesssim \nu^2\|f_t\|^2_{L^2(w,\mathbb{R}^n)},
\end{aligned}$$

where C is as in Lemma 2.4. By solving the above differential inequality, we see that there exists a positive constant $\widetilde{C}$ such that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\|e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} \le e^{\widetilde{C}\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)}, \tag{2.14}$$

which finishes the proof of Lemma 2.5 in the case $k=0$.

Next we prove Lemma 2.5 in the case $k \in \mathbb{N}$. For $0<\lambda\le\Lambda<\infty$ as in (1.2) and (1.3), let $\tau := \arctan\frac{\lambda}{\sqrt{\Lambda^2-\lambda^2}}$. From [12, Lemma 3.3], we deduce that for all $\theta \in (-\tau,\tau)$, $e^{i\theta}A$ also satisfies the degenerate elliptic conditions (1.2) and (1.3) with λ and Λ therein replaced by two other positive constants $\lambda_{(\theta)}$ and $\Lambda_{(\theta)}$, depending on θ, respectively. Let $L_\theta := e^{i\theta}L_w$ be the degenerate elliptic operator associated with the matrix $e^{i\theta}A$.

Let $\widetilde{\tau} := \min\{\pi/2 - \arctan(\Lambda/\lambda), \tau\}$. By (2.3) and (2.6), we see that for all $z \equiv re^{i\theta}$ with $r \in (0,\infty)$ and $\theta \in (-\widetilde{\tau},\widetilde{\tau})$, and $\phi \in \mathcal{E}(\mathbb{R}^n)$, $(L_\theta)_{\nu,\phi} = e^{i\theta}L_{\nu,\phi}$ and

$$e^{-zL_{\nu,\phi}} = e^{\nu\phi}e^{-zL_w}(e^{-\nu\phi}) = e^{\nu\phi}e^{-rL_\theta}(e^{-\nu\phi}) = e^{-r(L_\theta)_{\nu,\phi}}.$$

Similar to the proof of (2.14) with L_w replaced by L_θ, we see that there exists a positive constant $C_2 := \widetilde{C}/\cos\widetilde{\tau}$, where $\widetilde{C}$ is as in (2.14), such that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $z \equiv re^{i\theta}$ with $r \in (0,\infty)$ and $\theta \in (-\widetilde{\tau},\widetilde{\tau})$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
\|e^{-z(L_{\nu,\phi}+C_2\nu^2)}(f)\|_{L^2(w,\mathbb{R}^n)} &= \|e^{-r(L_\theta)_{\nu,\phi}}(e^{-re^{i\theta}C_2\nu^2}f)\|_{L^2(w,\mathbb{R}^n)}\\
&\le e^{\widetilde{C}\nu^2 r}e^{-r\cos\theta C_2\nu^2}\|f\|_{L^2(w,\mathbb{R}^n)} \le \|f\|_{L^2(w,\mathbb{R}^n)}.
\end{aligned} \tag{2.15}$$

Since e^{-zL_w} is holomorphic with respect to $z \in \Sigma(\widetilde{\tau})$ (see [34, Theorem 1.53] or [12, p. 293]), it is easy to show that $e^{-z(L_{\nu,\phi}+C_2\nu^2)}$ is also holomorphic with respect to $z \in \Sigma(\widetilde{\tau})$. For all $k \in \mathbb{N}$, by the Cauchy formula, we see that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $t \in (0,\infty)$,

$$[t(L_{\nu,\phi}+C_2\nu^2)]^k e^{-t(L_{\nu,\phi}+C_2\nu^2)} = (-1)^k k!\frac{t^k}{2\pi i}\int_{|\zeta-t|=\eta t} e^{-\zeta(L_{\nu,\phi}+C_2\nu^2)}\frac{d\zeta}{(\zeta-t)^{k+1}},$$

where the positive constant η is small enough, and the integral does not depend on η (the choice $\eta = \frac{1}{2}\sin\frac{\widetilde{\tau}}{2}$ insures that $\{\zeta : |\zeta - t| \le \eta t\}$ is contained in $\Sigma(\widetilde{\tau})$). From this and (2.15), we deduce that, for any $k \in \mathbb{N}$, there exists a positive constant $C_{(k)}$, depending on k, such that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\|[t(L_{\nu,\phi} + C_2\nu^2)]^k e^{-t(L_{\nu,\phi}+C_2\nu^2)}(f)\|_{L^2(w,\mathbb{R}^n)} \le C_{(k)}\|f\|_{L^2(w,\mathbb{R}^n)}. \tag{2.16}$$

To show the conclusion of Lemma 2.5 in the case $k \in \mathbb{N}$, we apply an induction argument. Assume that, for every $j \in \{0,\dots,k-1\}$, there exists a positive constant $C_{(j)}$, depending on j, such that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\|(tL_{\nu,\phi})^j e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} \lesssim e^{C_{(j)}\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)}. \tag{2.17}$$

Observe that for all $k \in \mathbb{N}$,

$$(L_{\nu,\phi} + C_2\nu^2)^k e^{-tL_{\nu,\phi}}(f) = \sum_{j=0}^{k-1}\binom{k}{j}(L_{\nu,\phi})^j (C_2\nu^2)^{k-j} e^{-tL_{\nu,\phi}}(f) + (L_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f),$$

where $\binom{k}{j}$ denotes the binomial coefficients. From this, (2.16), and (2.17), it follows that, for any $k \in \mathbb{N}$, there exists a positive constant $M_{(k)} > \max\{C_2, C_{(0)},\dots,C_{(k-1)}\}$, depending on k, such that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
&\|(L_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} \\
&\quad\lesssim \|(L_{\nu,\phi} + C_2\nu^2)^k e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} + \sum_{j=0}^{k-1}\|(L_{\nu,\phi})^j (C_2\nu^2)^{k-j} e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} \\
&\quad\lesssim \|(L_{\nu,\phi} + C_2\nu^2)^k e^{-t(L_{\nu,\phi}+C_2\nu^2)}(e^{C_2\nu^2 t}f)\|_{L^2(w,\mathbb{R}^n)} \\
&\qquad + \sum_{j=0}^{k-1}(C_2\nu^2)^{k-j}\|(L_{\nu,\phi})^j e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)} \\
&\quad\lesssim \frac{1}{t^k}\Big[e^{C_2\nu^2 t} + \sum_{j=0}^{k}(\nu^2 t)^{k-j} e^{C_{(j)}\nu^2 t}\Big]\|f\|_{L^2(w,\mathbb{R}^n)} \lesssim \frac{1}{t^k} e^{M_{(k)}\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)}.
\end{aligned}$$

Thus, (2.13) also holds true for k. This, together with (2.14), finishes the proof of Lemma 2.5. ∎

Since the semigroup $\{e^{-tL_w}\}_{t>0}$ satisfies the weighted Davies–Gaffney estimate (1.9) and e^{-zL_w} is holomorphic in $\Sigma(\pi/2 - \omega)$, where $\omega = \arctan(\Lambda/\lambda)$ (see [12, p. 293]), by an argument similar to the proof of [25, Proposition 3.1], we obtain the following proposition, the details being omitted.

Proposition 2.6 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator satisfying (1.2) and (1.3). Then, for every $k \in \mathbb{Z}_+$, the family of operators, $\{(tL_w)^k e^{-tL_w}\}_{t>0}$, satisfies the weighted Davies–Gaffney estimates; namely, there exist positive constants c and C such that, for all $t \in (0,\infty)$, closed subsets $E, F \subset \mathbb{R}^n$ and $f \in L^2(w,\mathbb{R}^n)$ with* supp $f \subset E$,

$$\|(tL_w)^k e^{-tL_w}\|_{L^2(w,F)} \le Ce^{-c\frac{[d(E,F)]^2}{t}}\|f\|_{L^2(w,E)}.$$

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We now turn to the weighted gradient estimates of $\{(tL_w)^k e^{-tL_w}\}_{t>0}$ with $k \in \mathbb{Z}_+$.

Proposition 2.7 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator satisfying (1.2) and (1.3). Then, for every $k \in \mathbb{Z}_+$, there exist positive constants C and $\widetilde{C}$ such that for all $t \in (0,\infty)$, closed sets $E, F \subset \mathbb{R}^n$, and $f \in L^2(w,\mathbb{R}^n)$ supported in E,*

$$\|\sqrt{t}\nabla([tL_w]^k e^{-tL_w}(f))\|_{L^2(w,F)} \le Ce^{-\widetilde{C}\frac{[d(E,F)]^2}{t}}\|f\|_{L^2(w,E)}.$$

Proof Let $k \in \mathbb{Z}_+$, $\nu \in \mathbb{R}_+$, and $\phi \in \mathcal{E}(\mathbb{R}^n)$. To prove Proposition 2.7, we first show that there exist positive constants M and M_0 such that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\|e^{\nu\phi}\sqrt{t}\nabla([tL_w]^k e^{-tL_w}(e^{-\nu\phi}f))\|_{L^2(w,\mathbb{R}^n)} \le Me^{M_0\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)}. \tag{2.18}$$

Indeed, from the fact that

$$e^{\nu\phi}\nabla([tL_w]^k e^{-tL_w}(e^{-\nu\phi}f)) = (\nabla - \nu\nabla\phi)(e^{\nu\phi}(tL_w)^k e^{-tL_w}(e^{-\nu\phi}f)),$$

it follows that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
&\|e^{\nu\phi}\sqrt{t}\nabla([tL_w]^k e^{-tL_w}(e^{-\nu\phi}f))\|_{L^2(w,\mathbb{R}^n)}\\
&\quad\le \|\sqrt{t}\nabla(e^{\nu\phi}[tL_w]^k e^{-tL_w}(e^{-\nu\phi}f))\|_{L^2(w,\mathbb{R}^n)}\\
&\qquad+ \|\nu\sqrt{t}e^{\nu\phi}(tL_w)^k e^{-tL_w}(e^{-\nu\phi}f)\nabla\phi\|_{L^2(w,\mathbb{R}^n)} =: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}$$

By the definition of ϕ, (2.3), (2.7), and Lemma 2.5, it is easy to see that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$ and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
\mathrm{J}_2 &\lesssim \nu\sqrt{t}\|(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f)\|_{L^2(w,\mathbb{R}^n)}\\
&\lesssim \nu\sqrt{t}e^{C_1\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)} \lesssim e^{(C_1+1)\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)},
\end{aligned} \tag{2.19}$$

where the positive constant C_1 is as in Lemma 2.5.

On the other hand, using (2.3), (2.7), and the degenerate elliptic condition (1.3), we see that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
(\mathrm{J}_1)^2 &\le \frac{t}{\lambda}\Re\big\{\mathfrak{a}((tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f), (tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f))\big\}\\
&\le \frac{t}{\lambda}\Re\big\{\mathfrak{a}((tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f), (tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f))\\
&\qquad\qquad - \mathfrak{a}_{\nu,\phi}((tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f), (tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f))\big\}\\
&\quad+ \frac{t}{\lambda}\Re\big\{\mathfrak{a}_{\nu,\phi}((tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f), (tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f))\big\} =: \mathrm{K}_1 + \mathrm{K}_2,
\end{aligned} \tag{2.20}$$

where the positive constant λ is as in (1.3).

By Lemmas 2.4 and 2.5, we see that for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0,\infty)$, and $f \in L^2(w,\mathbb{R}^n)$,

$$\begin{aligned}
\mathrm{K}_1 &\le \frac{t}{4\lambda}\Re\big\{\mathfrak{a}((tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f), (tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f))\big\}\\
&\quad+ C\frac{\nu^2 t}{\lambda}\|(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f)\|^2_{L^2(w,\mathbb{R}^n)}
\end{aligned}$$

$$\leq \frac{t}{4\lambda}\Re\big\{\mathfrak{a}\big((tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f),(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f)\big)\big\}$$
$$+ C\frac{\nu^2 t}{\lambda}e^{2C_1\nu^2 t}\|f\|^2_{L^2(w,\mathbb{R}^n)},$$

where the positive constants C and C_1 are, respectively, as in Lemmas 2.4 and 2.5. From this and (2.20), we further deduce that for all $\nu\in\mathbb{R}_+$, $\phi\in\mathcal{E}(\mathbb{R}^n)$, $t\in(0,\infty)$ and $f\in L^2(w,\mathbb{R}^n)$,

$$(2.21)\qquad (\mathrm{J}_1)^2 \leq \frac{4}{3\lambda}C\nu^2 te^{2C_1\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)} + \frac{4}{3}\mathrm{K}_2 \sim \nu^2 te^{2C_1\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)} + \mathrm{K}_2.$$

From (2.5), the Hölder inequality, and Lemma 2.5, we deduce that there exists a positive constant $\widetilde{C_1}$ such that, for all $\nu\in\mathbb{R}_+$, $\phi\in\mathcal{E}(\mathbb{R}^n)$, $t\in(0,\infty)$, and $f\in L^2(w,\mathbb{R}^n)$,

$$(2.22)\qquad \mathrm{K}_2 \lesssim t\big|\big(L_{\nu,\phi}(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f),(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}(f)\big)_{L^2(w,\mathbb{R}^n)}\big|$$
$$\lesssim \|(tL_{\nu,\phi})^{k+1}e^{-tL_{\nu,\phi}}\|_{L^2(w,\mathbb{R}^n)}\|(tL_{\nu,\phi})^k e^{-tL_{\nu,\phi}}\|_{L^2(w,\mathbb{R}^n)}$$
$$\lesssim e^{\widetilde{C_1}\nu^2 t}\|f\|^2_{L^2(w,\mathbb{R}^n)}.$$

Combining (2.21) and (2.22), there exists a constant $M_1>(\max\{2C_1,\widetilde{C_1}\})/2$ such that for all $\nu\in\mathbb{R}_+$, $\phi\in\mathcal{E}(\mathbb{R}^n)$, $t\in(0,\infty)$, and $f\in L^2(w,\mathbb{R}^n)$,

$$\mathrm{J}_1 \lesssim [\nu^2 te^{2C_1\nu^2 t}+e^{\widetilde{C_1}\nu^2 t}]^{1/2}\|f\|_{L^2(w,\mathbb{R}^n)} \lesssim e^{M_1\nu^2 t}\|f\|_{L^2(w,\mathbb{R}^n)}.$$

This, together with (2.19), implies that (2.18) holds true.

Take $\phi\in\mathcal{E}(\mathbb{R}^n)$ satisfying $\phi|_F\geq 0$ and $\phi|_E\leq -\frac{\widetilde{d}(E,F)}{1+\epsilon}$, where ϵ is some suitable positive constant (see [15, p. 151] for the existence of such a function). By this and (2.18), we find that for all $k\in\mathbb{Z}_+$, $t\in(0,\infty)$, closed sets $E,F\subset\mathbb{R}^n$, and $f\in L^2(w,E)$ supported in E,

$$\|\sqrt{t}\nabla((tL_w)^k e^{-tL_w}(f))\|_{L^2(w,F)}$$
$$= \|e^{-\nu\phi}e^{\nu\phi}\sqrt{t}\nabla((tL_w)^k e^{-tL_w}(e^{-\nu\phi}e^{\nu\phi}f))\|_{L^2(w,F)}$$
$$\leq \|e^{\nu\phi}\sqrt{t}\nabla((tL_w)^k e^{-tL_w}(e^{-\nu\phi}e^{\nu\phi}f))\|_{L^2(w,F)}$$
$$\lesssim e^{M_0\nu^2 t}\|e^{\nu\phi}f\|_{L^2(w,E)} \lesssim e^{M_0\nu^2 t}e^{-\nu\frac{\widetilde{d}(E,F)}{1+\epsilon}}\|f\|_{L^2(w,E)},$$

where the positive constant M_0 is as in (2.18). This, together with Lemma 2.3 and the choice that $\nu:=(\widetilde{d}(E,F))/(\widetilde{C_0}t)$ with $\widetilde{C_0}>(1+\epsilon)M_0$, implies that there exists a positive constant $\widetilde{C}$ such that, for all $k\in\mathbb{Z}_+$, $t\in(0,\infty)$, closed sets $E,F\subset\mathbb{R}^n$ and any $f\in L^2(w,\mathbb{R}^n)$ supported in E,

$$\|\sqrt{t}\nabla((tL_w)^k e^{-tL_w}(f))\|_{L^2(w,F)} \lesssim e^{-\frac{[\widetilde{d}(E,F)]^2}{t}\left(\frac{1}{1+\epsilon}-\frac{M_0}{\widetilde{C_0}}\right)\frac{1}{\widetilde{C_0}}}\|f\|_{L^2(w,E)}$$
$$\sim e^{-\widetilde{C}\frac{[d(E,F)]^2}{t}}\|f\|_{L^2(w,E)}.$$

This finishes the proof of Proposition 2.7. ∎

To show Proposition 1.5, we also need the following local weighted Sobolev embedding theorems (see [21, Theorem (1.2) and Property 4, p. 107], respectively).

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In what follows, for a subset $E \subset \mathbb{R}^n$, $C_c^\infty(E)$ denotes the set of all C^∞ functions with compact support in E.

Theorem 2.8 ([21]) *For any given $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, there exist positive constants c and δ such that for all balls $B \equiv B(x_B, r_B)$ of $\mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, all $u \in C_c^\infty(B)$, and all numbers $k_0 \in \mathbb{R}_+$ satisfying $1 \le k_0 \le \frac{n}{n-1} + \delta$,*

(2.23)
$$\Big[\frac{1}{w(B)}\int_B |u(x)|^{k_0 p} w(x)\,dx\Big]^{1/(k_0 p)} \le c r_B \Big[\frac{1}{w(B)}\int_B |\nabla u(x)|^p w(x)\,dx\Big]^{1/p}.$$

Theorem 2.9 ([21]) *Let $w \in QC(\mathbb{R}^n)$. Then there exist positive constants c and $k_0 \in (1,\infty)$ such that for all balls $B \equiv B(x_B, r_B)$ of $\mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and all $u \in C_c^\infty(B)$,*

$$\Big[\frac{1}{w(B)}\int_B |u(x)|^{2k_0} w(x)\,dx\Big]^{1/(2k_0)} \le c r_B \Big[\frac{1}{w(B)}\int_B |\nabla u(x)|^2 w(x)\,dx\Big]^{1/2}. \tag{2.24}$$

We are now in a position to prove Proposition 1.5.

Proof of Proposition 1.5 We first prove that for all $l \in \mathbb{Z}_+$,

$$(tL_w)^l e^{-tL_w} \in \mathcal{O}_w(L^2 - L^{2k_0}),$$

where the positive number $k_0 \in (1,\infty)$ satisfies (2.23) with $p = 2$ and (2.24) (when $w \in A_2(\mathbb{R}^n)$, we choose $k_0 \equiv \frac{n}{n-1}$).

Given any ball $B \equiv B(x_B, r_B)$ of $\mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, we define $H_0^1(w,B)$ to be the closure of $C_c^\infty(B)$ with respect to the norm

$$\|f\|_{H_0^1(w,B)} := \Big\{\int_B [|f(x)|^2 + |\nabla f(x)|^2] w(x)\,dx\Big\}^{1/2}.$$

Take $\phi \in C_c^\infty(2B)$ such that $|\nabla\phi(x)| \lesssim 1/r_B$, $\operatorname{supp}\phi \subset 2B$, $\phi \equiv 1$ on B, and for all $x \in \mathbb{R}^n$, $0 \le \phi(x) \le 1$. Then it is easy to show that for all $l \in \mathbb{Z}_+$ and $f \in L^2_{\mathrm{loc}}(w,\mathbb{R}^n)$,

$$\phi[(tL_w)^l e^{-tL_w}(\chi_B f)] \in H_0^1(w, 2B).$$

Since $C_c^\infty(2B)$ is dense in $H_0^1(w,2B)$, by the choice of ϕ, Lemma 2.2, Theorems 2.8 and 2.9, Propositions 2.6 and 2.7 and a density argument, we know that for all $l \in \mathbb{Z}_+$, $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, $t \in (0,\infty)$ and $f \in L^2_{\mathrm{loc}}(w,\mathbb{R}^n)$,

$$\begin{aligned}
&\Big[\frac{1}{w(B)}\int_B |(tL_w)^l e^{-tL_w}(\chi_B f)(x)|^{2k_0} w(x)\,dx\Big]^{1/(2k_0)}\\
&\quad\lesssim \Big[\frac{1}{w(2B)}\int_{2B} |\phi(x)(tL_w)^l e^{-tL_w}(\chi_B f)(x)|^{2k_0} w(x)\,dx\Big]^{1/(2k_0)}\\
&\quad\lesssim r_B\Big[\frac{1}{w(2B)}\int_{2B} |\nabla(\phi[(tL_w)^l e^{-tL_w}(\chi_B f)])(x)|^2 w(x)\,dx\Big]^{1/2}\\
&\quad\lesssim \Big[\frac{1}{w(2B)}\int_{2B} |(tL_w)^l e^{-tL_w}(\chi_B f)(x)|^2 w(x)\,dx\Big]^{1/2}\\
&\qquad + \frac{r_B}{\sqrt{t}}\Big[\frac{1}{w(2B)}\int_{2B} |\sqrt{t}\nabla((tL_w)^l e^{-tL_w}(\chi_B f))(x)|^2 w(x)\,dx\Big]^{1/2}
\end{aligned}$$

$$\lesssim \Big(1+\frac{r_B}{\sqrt{t}}\Big)\Big[\frac{1}{w(B)}\int_B |f(x)|^2 w(x)\,dx\Big]^{1/2}$$
$$\lesssim \Upsilon\Big(\frac{r_B}{\sqrt{t}}\Big)\Big[\frac{1}{w(B)}\int_B |f(x)|^2 w(x)\,dx\Big]^{1/2},$$

where Υ is as in (1.14). This shows that (1.11) holds true in the case where $q = 2k_0$ and $p = 2$.

Next, we prove (1.12) in the case where $q = 2k_0$ and $p = 2$. For all $j \in \mathbb{N}$ and $j \geq 3$, let $S_j(B) := (2^{j+1}B) \setminus (2^{j-2}B)$. Take $\eta_j \in C_c^\infty(S_j(B))$ satisfying that, for all $x \in \mathbb{R}^n$, $0 \leq \eta_j(x) \leq 1$, $|\nabla \eta_j(x)| \lesssim \frac{1}{2^j r_B}$, and $\eta_j \equiv 1$ on $U_j(B)$. By the fact that $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$ and Lemma 2.1(ii), we know that there exists some $r \in (1,\infty)$ such that $w \in RH_r(\mathbb{R}^n)$. From the choice of η_j, Lemma 2.2, Theorems 2.8, and 2.9, Propositions 2.6 and 2.7, and a density argument, it follows that there exists a positive constant c such that for all $l \in \mathbb{Z}_+$, $j \in \mathbb{N} \cap [3,\infty)$, $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, $t \in (0,\infty)$, and $f \in L^2_{\mathrm{loc}}(w, \mathbb{R}^n)$,

$$\Big[\frac{1}{w(2^jB)}\int_{U_j(B)} |(tL_w)^l e^{-tL_w}(\chi_B f)(x)|^{2k_0} w(x)\,dx\Big]^{1/(2k_0)}$$
$$\lesssim \Big[\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B} |\eta_j(x)(tL_w)^l e^{-tL_w}(\chi_B f)(x)|^{2k_0} w(x)\,dx\Big]^{1/(2k_0)}$$
$$\lesssim \Big[\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B} |(tL_w)^l e^{-tL_w}(\chi_B f)(x)|^2 w(x)\,dx\Big]^{1/2}$$
$$+\frac{2^j r_B}{\sqrt{t}}\Big[\frac{1}{w(2^{j+1}B)}\int_{2^{j+1}B} |\sqrt{t}\nabla((tL_w)^l e^{-tL_w}(\chi_B f))(x)|^2 w(x)\,dx\Big]^{1/2}$$
$$\lesssim 2^{jn\frac{r-1}{2r}}\Big(1+\frac{2^j r_B}{\sqrt{t}}\Big) e^{-c\frac{(2^j r_B)^2}{t}}\Big[\frac{1}{w(B)}\int_B |f(x)|^2 w(x)\,dx\Big]^{1/2}$$
$$\lesssim 2^{j\theta_1}\Upsilon\Big(\frac{2^j r_B}{\sqrt{t}}\Big) e^{-c\frac{(2^j r_B)^2}{t}}\Big[\frac{1}{w(B)}\int_B |f(x)|^2 w(x)\,dx\Big]^{1/2},$$

where $\theta_1 \equiv \frac{r-1}{2r} n$ and Υ is as in (1.14). This implies that (1.12) in the case where $q = 2k_0$ and $p = 2$ holds true.

Similarly, (1.13) in the case where $q = 2k_0$ and $p = 2$ also holds true.

Thus, we conclude that there exists a number $k_0 \in (1,\infty)$ such that for all $l \in \mathbb{Z}_+$,

$$(tL_w)^l e^{-tL_w} \in \mathcal{O}_w(L^2 - L^{2k_0}).$$

The remainder of the proof of Proposition 1.5 follows from the duality and the composition rule of the weighted off-diagonal estimates on balls (see [1, Comments (6), Theorem 2.3(b)]), the details being omitted. This finishes the proof of Proposition 1.5. ∎

Remark 2.10 Recall that in [4] Bui et al. establish an abstract theory of Hardy spaces on the space $(\mathcal{X}, d, \mu)$ of homogenous type, associated with operators satisfying the bounded H_∞ functional calculus and the off-diagonal estimates on balls. Proposition 1.5 shows that L_w satisfies the off-diagonal estimates on balls when

$$(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, w(x)\,dx).$$

Moreover, by [12, pp. 291–294], we know that L_w has a bounded H_∞ functional calculus in $L^2(w,\mathbb{R}^n)$. Therefore, L_w satisfies the assumptions of the operators in [4].

3 The Maximal Function Characterization of $H^p_{L_w}(\mathbb{R}^n)$

In this section, we give the proof of Theorem 1.3. We begin by introducing some notions and recalling some needed results from [4, 27, 32, 35].

Definition 3.1 Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, $p \in (0,1]$, $M \in \mathbb{N}$, and $\epsilon \in (0,\infty)$. A function $m \in L^2(w,\mathbb{R}^n)$ is called a $(p,2,M,\epsilon)_{L_w}$*-molecule* if $m \in R(L_w^M)$ (the range of L_w^M) and there exists a ball $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, such that for every $k \in \{0,1,\dots,M\}$ and $j \in \mathbb{Z}_+$, it holds true that

$$\|(r_B^{-2}L_w^{-1})^k(m)\|_{L^2(w,U_j(B))} \le 2^{-j\epsilon}[w(2^jB)]^{1/2}[w(B)]^{-1/p},$$

where $U_j(B)$ is as in (1.15).

Remark 3.2 We point out that by the weighted Poincaré inequality (see [21, p. 95 and p. 110]), L_w is injective from $D(L_w) \subset L^2(w,\mathbb{R}^n)$ to $L^2(w,\mathbb{R}^n)$, where $D(L_w)$ denotes the domain of L_w. Hence, L_w^{-1} makes sense.

Definition 3.3 Let $p \in (0,1]$ and f be a measurable function on $\mathbb{R}^n$. The formula $f = \sum_{j=1}^\infty \lambda_j m_j$ is called a *molecular* $(p,2,M,\epsilon)_{L_w}$*-representation* of f if $\{\lambda_j\}_{j=1}^\infty \in l^p$, each m_j is a $(p,2,M,\epsilon)_{L_w}$-molecule and the summation converges in $L^2(w,\mathbb{R}^n)$. Let

$$\mathbb{H}^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n) := \{f \in L^2(w,\mathbb{R}^n) : f \text{ has a molecular } (p,2,M,\epsilon)_{L_w}\text{-representation}\}.$$

The *molecular Hardy space* $H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)$ is defined as the completion of $\mathbb{H}^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)$ with respect to the *quasi-norm*

$$\|f\|_{\mathbb{H}^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)} := \inf\Big\{\Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{1/p} : f = \sum_{j=1}^\infty \lambda_j m_j\Big\},$$

where the infimum is taken over all the molecular $(p,2,M,\epsilon)_{L_w}$-representations of f as above.

Since L_w satisfies the assumptions of the operators in [4] (see Remark 2.10), we have the following theorem, which is just a special case of [4, Theorem 4.8].

Theorem 3.4 ([4]) *Let $w \in A_q(\mathbb{R}^n)$ with $q \in [1,\infty)$ and $p \in (0,1]$. Assume that $M \in \mathbb{N}$ with*

$$M > \frac{nq}{2}\Big[\frac{q}{p} + \frac{p}{nq(2-p)} - \frac{1}{nq}\Big] \quad \textit{and} \quad \varepsilon \in \Big(\frac{nq^2}{p}, \infty\Big).$$

Then $H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n) = H^p_{L_w}(w,\mathbb{R}^n)$ with equivalent quasi-norms.

Let us now introduce an auxiliary square operator $\widetilde{\mathcal{S}}_{L_w}^{(\beta)}$, which, when $w(x) \equiv 1$, is just [27, (6.3)]. Let $\beta \in (0, \infty)$. For any $f \in L^2(w, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\widetilde{\mathcal{S}}_{L_w}^{(\beta)}(f)(x) := \Big\{ \iint_{\Gamma_\beta(x)} |t \nabla e^{-t^2 L_w}(f)(y)|^2 w(y) \frac{dy}{w(B(x,t))} \frac{dt}{t} \Big\}^{1/2}, \tag{3.1}$$

where Γ_β is as in (1.7) with α replaced by β. We denote $\widetilde{\mathcal{S}}_{L_w}^{(1)}(f)$ simply by $\widetilde{\mathcal{S}}_{L_w}(f)$.

In the following subsections, we will prove the following theorems using this auxiliary operator.

Theorem 3.5 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$. Then, for all $p \in (0, \infty)$, there exists a positive constant $C := C_{(n,p)}$, depending on n and p, such that for all $f \in L^2(w, \mathbb{R}^n)$,*

$$\|\mathcal{S}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \le C \|\widetilde{\mathcal{S}}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

Theorem 3.6 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$. Then, for all $p \in (0, 1]$, there exists a positive constant $C := C_{(n,p)}$, depending on n and p, such that for all $f \in L^2(w, \mathbb{R}^n)$,*

$$\|\widetilde{\mathcal{S}}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \le C \|\mathcal{N}_h(f)\|_{L^p(w,\mathbb{R}^n)}. \tag{3.2}$$

Recall that $QC(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$ (see [21, p. 107]).

Theorem 3.7 *Suppose $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$. Let $q \in [2, \infty)$ be such that $w \in A_q(\mathbb{R}^n)$. Then for all $p \in (0, 1]$, $M \in \mathbb{N}$ satisfying $M > \frac{qn}{2p}(1 - \frac{p}{2})$ and $\varepsilon \in (\frac{nq}{p}, \infty)$, it holds true that $H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n) \subset H^p_{L_w,\mathcal{N}_h}(w, \mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for all $f \in H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)$,*

$$\|\mathcal{N}_h(f)\|_{L^p(w,\mathbb{R}^n)} \le C \|f\|_{H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)}.$$

Remark 3.8 In Theorem 3.7, if $w \in A_2(\mathbb{R}^n)$, then, by Lemma 2.1(i), we know that, for all $q \in [2, \infty)$, $w \in A_q(\mathbb{R}^n)$.

If $w \in QC(\mathbb{R}^n)$, then $w \in RH_{n/(n-2)}(\mathbb{R}^n)$. Indeed, if $n = 2$, this is obviously true. Now, we assume $n > 2$. Then for any quasi-conformal mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$, let

$$L_f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

From the definition of quasi-conformal mappings, we deduce that for almost every $x \in \mathbb{R}^n$,

$$[L_f(x)]^n \sim |f'(x)|, \tag{3.3}$$

where $f'(x)$ is as in (2.1).

By the Gehring lemma (see [24, Lemma 4]), we know that for all balls B in $\mathbb{R}^n$,

$$\Big(\frac{1}{|B|} \int_B [L_f(x)]^n \, dx \Big)^{\frac{1}{n}} \lesssim \frac{1}{|B|} \int_B L_f(x) \, dx,$$

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which, together with (3.3) and the Hölder inequality, implies that if $w \in QC(\mathbb{R}^n)$, then

$$\begin{aligned}\Big(\frac{1}{|B|}\int_B [w(x)]^{\frac{n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} &= \Big[\frac{1}{|B|}\int_B |f'(x)|\,dx\Big]^{\frac{n-2}{n}} \sim \Big(\frac{1}{|B|}\int_B [L_f(x)]^n\,dx\Big)^{\frac{n-2}{n}}\\ &\lesssim \Big(\frac{1}{|B|}\int_B [L_f(x)]\,dx\Big)^{n-2} \sim \Big(\frac{1}{|B|}\int_B [w(x)]^{\frac{1}{n-2}}\,dx\Big)^{n-2}\\ &\lesssim \frac{1}{|B|}\int_B w(x)\,dx,\end{aligned}$$

namely, $w \in RH_{n/(n-2)}(\mathbb{R}^n)$. By this and Lemma 2.1(ii), we see that $w \in A_q(\mathbb{R}^n)$ for some $q \in [1, \infty)$.

Our main Theorem 1.3 then follows directly from Theorems 3.5–3.7 as follows.

Proof of Theorem 1.3 Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and $p \in (0, 1]$. For any $g \in L^2(w, \mathbb{R}^n)$, by Theorems 3.5 and 3.6, we see that

$$\|\mathcal{S}_{L_w}(g)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\mathcal{N}_h(g)\|_{L^p(w,\mathbb{R}^n)}.$$

Then it follows from a density argument that for all $f \in H^p_{L_w,\mathcal{N}_h}(w, \mathbb{R}^n)$,

$$\|\mathcal{S}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\mathcal{N}_h(f)\|_{L^p(w,\mathbb{R}^n)},$$

which further implies that

$$H^p_{L_w,\mathcal{N}_h}(w, \mathbb{R}^n) \subset H^p_{L_w}(w, \mathbb{R}^n). \tag{3.4}$$

Next, we prove the converse of (3.4), namely, $H^p_{L_w}(w, \mathbb{R}^n) \subset H^p_{L_w,\mathcal{N}_h}(w, \mathbb{R}^n)$. By Remark 3.8, we know that, for $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, there exists some $q \in [2, \infty)$, such that $w \in A_q(\mathbb{R}^n)$. Let

$$M > \max\Big\{\frac{qn}{2p}\Big(1 - \frac{p}{2}\Big),\quad \frac{nq}{2}\Big[\frac{q}{p} + \frac{p}{nq(2-p)} - \frac{1}{nq}\Big]\Big\} \quad\text{and}\quad \varepsilon \in \Big(\frac{nq^2}{p}, \infty\Big).$$

From Theorems 3.4 and 3.7, we deduce that, for all $f \in H^p_{L_w}(w, \mathbb{R}^n)$,

$$\|\mathcal{N}_h(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|f\|_{H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)} \sim \|f\|_{H^p_{L_w}(w,\mathbb{R}^n)},$$

which implies $H^p_{L_w}(w, \mathbb{R}^n) \subset H^p_{L_w,\mathcal{N}_h}(w, \mathbb{R}^n)$. This, together with (3.4), shows that $H^p_{L_w}(w, \mathbb{R}^n)$ and $H^p_{L_w,\mathcal{N}_h}(w, \mathbb{R}^n)$ coincide with equivalent quasi-norms, which completes the proof of Theorem 1.3. ■

In subsections 3.1, 3.2, and 3.3, we prove Theorems 3.5, 3.6, and 3.7, respectively, and hence complete the proof of Theorem 1.3.

3.1 Proof of Theorem 3.5

For $\alpha \in (0, \infty)$ and a closed set F of $\mathbb{R}^n$, we set $\mathcal{R}_\alpha(F) := \bigcup_{x\in F} \Gamma_\alpha(x)$, where $\Gamma_\alpha(x)$ for all $x \in F$ is as in (1.7). For simplicity, we often write $\mathcal{R}(F)$ instead of $\mathcal{R}_1(F)$.

Let $F \subset \mathbb{R}^n$ be a closed set and $O := F^{\complement}$. For any fixed $\gamma \in (0,1)$, the *set F^*_γ of points with global γ-density with respect to F* is defined by

$$F^*_\gamma := \Big\{ x \in \mathbb{R}^n : \frac{w(B(x,r) \cap F)}{w(B(x,r))} \geq \gamma \text{ for all } r \in (0,\infty) \Big\}. \tag{3.5}$$

It is easy to see that $F^*_\gamma \subset F$ and

$$(F^*_\gamma)^{\complement} = \{x \in \mathbb{R}^n : M_w(\chi_O)(x) > 1-\gamma\}, \tag{3.6}$$

where M_w denotes the *central weighted Hardy-Littlewood maximal operator*; namely, for any $f \in L^1_{\text{loc}}(w,\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_w(f)(x) := \sup_{r\in(0,\infty)} \frac{1}{w(B(x,r))} \int_{B(x,r)} |f(y)| w(y)\, dy.$$

Lemma 3.9 is just an analogue of [32, Lemma 6.2], which was proved by borrowing some ideas from the proof of [11, Proposition 4], the details being omitted.

Lemma 3.9 *For any $\alpha \in (0,\infty)$, measurable function f on $\mathbb{R}^{n+1}_+$, and $x \in \mathbb{R}^n$, let*

$$A_\alpha(f)(x) := \Big[\iint_{\Gamma_\alpha(x)} |f(y,t)|^2 w(y) \frac{dy}{w(B(x,\alpha t))} \frac{dt}{t} \Big]^{1/2}.$$

Then for $p \in (0,\infty)$ and $\alpha, \beta \in (0,\infty)$, there exists a positive constant $C := C_{(n,\alpha,\beta,p)}$, depending on n, α, β, and p, such that, for all measurable functions f on $\mathbb{R}^{n+1}_+$,

$$C^{-1} \|A_\beta(f)\|_{L^p(w,\mathbb{R}^n)} \leq \|A_\alpha(f)\|_{L^p(w,\mathbb{R}^n)} \leq C \|A_\beta(f)\|_{L^p(w,\mathbb{R}^n)}.$$

Finally, we have the following weighted elliptic Caccioppoli inequality for solutions to the degenerate parabolic system.

Lemma 3.10 *Let $w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ and let L_w be the degenerate elliptic operator satisfying (1.2) and (1.3). Assume, in the distributional sense, that $\partial_t u = -2tL_w u$ in $B(x_0,2r) \times [t_0 - 2cr, t_0 + 2cr]$, where $x_0 \in \mathbb{R}^n$, $r, c \in (0,\infty)$ and $3cr < t_0 < \infty$. Then there exists a positive constant $C := C_{(n,\lambda,\Lambda,c)}$, depending on n, λ, Λ, and c, but independent of x_0, t_0, and r, such that*

$$\int_{t_0-cr}^{t_0+cr} \int_{B(x_0,r)} t|\nabla u(x,t)|^2 w(x)\, dx\, dt \leq \frac{C}{r^2} \int_{t_0-2cr}^{t_0+2cr} \int_{B(x_0,2r)} t|u(x,t)|^2 w(x)\, dx\, dt. \tag{3.7}$$

The proof of Lemma 3.10 is an analogue of the corresponding Caccioppoli inequality in the case where $w \equiv 1$ (see, for example, [30, Lemma 3.3]), choosing a suitable cut-off function, the details being omitted.

Proof of Theorem 3.5 For all $\alpha \in (0,\infty)$, $0 < \epsilon < R < \infty$ and $x \in \mathbb{R}^n$, we define the truncated cone $\Gamma_{\varepsilon,R,\alpha}(x)$ by

$$\Gamma_{\varepsilon,R,\alpha}(x) := \{(y,t) \in \mathbb{R}^n \times (\varepsilon,R) : |x-y| < \alpha t\}.$$

Take a function $\eta \in C^\infty_c(\Gamma_{\varepsilon/2,2R,3/2}(x))$ satisfying $\eta \equiv 1$ on $\Gamma_{\varepsilon,R,1}(x)$, $0 \leq \eta \leq 1$, and, for all $(y,t) \in \Gamma_{\varepsilon/2,2R,3/2}(x)$, $|\nabla_{y,t}\eta(y,t)| \lesssim 1/t$, where the implicit constant is independent of y and t. Then, by the definition of L_w (see (1.5)), the degenerate elliptic condition (1.2), and the Hölder inequality, we conclude that

(3.8)

$$
\begin{aligned}
&\Big\{\iint_{\Gamma_{\varepsilon,R,1}(x)} |t^2L_we^{-t^2L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/2}\\
&\quad\le \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} t^2L_we^{-t^2L_w}(f)(y)\\
&\qquad\qquad\times \overline{t^2L_we^{-t^2L_w}(f)(y)}\eta(y,t)w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/2}\\
&\quad= \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} [tA(y)\nabla e^{-t^2L_w}(f)(y)\cdot \overline{t\nabla[t^2L_we^{-t^2L_w}(f)](y)}\eta(y,t)\\
&\qquad + t^2A(y)\nabla e^{-t^2L_w}(f)(y)\cdot\nabla\eta(y,t)\overline{t^2L_we^{-t^2L_w}(f)(y)}]\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/2}\\
&\quad\lesssim \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t\nabla e^{-t^2L_w}(f)(y)|\\
&\qquad\qquad\times |t\nabla[t^2L_we^{-t^2L_w}(f)](y)|w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/2}\\
&\qquad + \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t\nabla e^{-t^2L_w}(f)(y)||t^2L_we^{-t^2L_w}(f)(y)|w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/2}\\
&\quad\lesssim \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t\nabla e^{-t^2L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/4}\\
&\qquad \times \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t\nabla[t^2L_we^{-t^2L_w}(f)](y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/4}\\
&\qquad + \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t\nabla e^{-t^2L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/4}\\
&\qquad \times \Big\{\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t^2L_we^{-t^2L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\Big\}^{1/4}.
\end{aligned}
$$

To control the above integrals, we first decompose $\Gamma_{\varepsilon/4,3R,2}(x)$ into a family of Whitney balls, $\{B((y_k,t_k),r_k)\}_{k=0}^{\infty}$, such that $\bigcup_{k=0}^{\infty} B((y_k,t_k),r_k) = \Gamma_{\varepsilon/4,3R,2}(x)$,

$$c_1r_k \le \operatorname{dist}(B((y_k,t_k),r_k),(\Gamma_{\varepsilon/4,3R,2}(x))^{\complement}) \le c_2r_k,$$

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and for all $z \in \Gamma_{\varepsilon/4,3R,2}(x)$, $\sum_{k=0}^{\infty} \chi_{B((y_k,t_k),3r_k)}(z) \le N_0$, where $(y_k, t_k) \in \mathbb{R}^n \times (0,\infty)$, $3 < c_1 < c_2 < \infty$, and $N_0 \in \mathbb{N}$ are fixed constants independent of $\Gamma_{\varepsilon/4,3R,2}(x)$. Consider a subsequence of $\{B((y_k,t_k),r_k)\}_{k=0}^{\infty}$ (without loss of generality, we may use the same notation as the original sequence) such that

$$\Gamma_{\varepsilon/2,2R,3/2}(x) \subset \bigcup_{k=0}^{\infty} B((y_k,t_k),r_k) \quad \text{and} \quad \operatorname{dist}(B((y_k,t_k),r_k),\{t=0\}) \sim r_k.$$

Then by Lemma 3.10, we know that

$$\begin{aligned}
&\iint_{\Gamma_{\varepsilon/2,2R,3/2}(x)} |t\nabla[t^2 L_w e^{-t^2 L_w}(f)](y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\\
&\quad\le \sum_{k=0}^{\infty} \iint_{B((y_k,t_k),r_k)} |t\nabla[t^2 L_w e^{-t^2 L_w}(f)](y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\\
&\quad\le \sum_{k=0}^{\infty} \int_{t_k-r_k}^{t_k+r_k} \int_{B(y_k,r_k)} |t\nabla[t^2 L_w e^{-t^2 L_w}(f)](y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\\
&\quad\lesssim \sum_{k=0}^{\infty} \int_{t_k-2r_k}^{t_k+2r_k} \int_{B(y_k,2r_k)} |t^2 L_w e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\\
&\quad\lesssim \sum_{k=0}^{\infty} \iint_{B((y_k,t_k),3r_k)} |t^2 L_w e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\\
&\quad\lesssim \iint_{\Gamma_{\varepsilon/4,3R,2}(x)} |t^2 L_w e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}.
\end{aligned}$$

This, together with (3.8) and the Young inequality and via letting $\varepsilon \to 0$ and $R \to \infty$, shows that for any $\widetilde{\varepsilon} \in (0,\infty)$,

$$\begin{aligned}
&\left\{\iint_{\Gamma(x)} |t^2 L_w e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\right\}^{1/2}\\
&\quad\lesssim \left\{\iint_{\Gamma_{3/2}(x)} |t\nabla e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\right\}^{1/4}\\
&\qquad\times \left\{\iint_{\Gamma_2(x)} |t^2 L_w e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\right\}^{1/4}\\
&\quad\lesssim \widetilde{\varepsilon}\left\{\iint_{\Gamma_2(x)} |t^2 L_w e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\right\}^{1/2}\\
&\qquad+ \frac{1}{\widetilde{\varepsilon}}\left\{\iint_{\Gamma_{3/2}(x)} |t\nabla e^{-t^2 L_w}(f)(y)|^2 w(y)\,\frac{dy}{w(B(x,t))}\,\frac{dt}{t}\right\}^{1/2},
\end{aligned}$$

which, combined with Lemma 3.9 and a suitable choice of $\widetilde{\varepsilon}$, implies that for all $f \in L^2(w,\mathbb{R}^n)$,

$$\|\mathcal{S}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\widetilde{\mathcal{S}}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

This finishes the proof of Theorem 3.5. ∎

3.2 Proof of Theorem 3.6

Before showing Theorem 3.6, let us first introduce the *non-tangential maximal function* of β-angle, $\beta \in (0,\infty)$, by setting, for all $f \in L^2(w,\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{N}_h^{(\beta)}(f)(x) := \sup_{(y,t)\in\Gamma_\beta(x)} \Big[\frac{1}{w(B(y,\beta t))}\int_{B(y,\beta t)} |e^{-t^2 L_w}(f)(z)|^2 w(z)\,dz\Big]^{1/2}.$$

The following lemma is an analogue of [32, Lemma 6.2], the details being omitted.

Lemma 3.11 *Let $0<\gamma<\beta<\infty$ and $p \in (0,1]$. Then there exists a positive constant $C := C_{(n,\gamma,\beta)}$, depending on n, γ, and β, such that for all $f \in L^2(w,\mathbb{R}^n)$,*

$$C^{-1}\|\mathcal{N}_h^{(\gamma)}(f)\|_{L^p(w,\mathbb{R}^n)} \le \|\mathcal{N}_h^{(\beta)}(f)\|_{L^p(w,\mathbb{R}^n)} \le C\|\mathcal{N}_h^{(\gamma)}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

Proof of Theorem 3.6 By Lemma 3.9, we see that

$$\|\widetilde{\mathcal{S}}_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)\|_{L^p(w,\mathbb{R}^n)}, \tag{3.9}$$

for all $p \in (0,1]$ and $f \in L^2(w,\mathbb{R}^n)$. Therefore, to finish the proof of Theorem 3.6, it suffices to prove (3.2) with $\widetilde{\mathcal{S}}_{L_w}$ replaced by $\widetilde{\mathcal{S}}_{L_w}^{(1/2)}$.

For $0<\varepsilon\le R<\infty$, $\beta \in (0,\infty)$, $f \in L^2(w,\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, let

$$\widetilde{\mathcal{S}}_{L_w}^{(\varepsilon,R,\beta)}(f)(x) := \Big[\iint_{\Gamma_{\varepsilon,R,\beta}(x)} |t\nabla e^{-t^2 L_w}(f)(y)|^2 \frac{w(y)\,dy}{w(B(x,\beta t))}\frac{dt}{t}\Big]^{1/2}.$$

For any $\sigma \in (0,\infty)$, let

$$E := \{x \in \mathbb{R}^n : \mathcal{N}_h^{(\beta)}(f)(x) \le \sigma\}, \tag{3.10}$$

where β is a fixed positive constant to be determined later and $\mathcal{E}^* := E^*_{1/2}$ is the set of points having the global 1/2-density with respect to E (see (3.5)). Let $B^* := (\mathcal{E}^*)^{\complement}$, $R_{\varepsilon,R,\beta}(\mathcal{E}^*) := \bigcup_{x\in\mathcal{E}^*}\Gamma_{\varepsilon,R,\beta}(x)$ and let $u(y,t) := e^{-t^2L_w}(f)(y)$, $t \in (0,\infty)$, $y \in \mathbb{R}^n$. By [12, Proposition 3.7], it is easy to see that u is a weak solution of the parabolic equation $2t\,\mathrm{div}(A\nabla u) = w\partial_t u$. By the definition of $\widetilde{\mathcal{S}}_{L_w}^{(2\varepsilon,R,1/2)}$ and the Fubini theorem, we know that

$$\int_{\mathcal{E}^*}[\widetilde{\mathcal{S}}_{L_w}^{(2\varepsilon,R,1/2)}(f)(x)]^2 w(x)\,dx \lesssim \iint_{R_{\varepsilon,2R,1}(\mathcal{E}^*)} t|\nabla u(y,t)|^2 w(y)\,dy\,dt. \tag{3.11}$$

Let $G := R_{\varepsilon,2R,1}(\mathcal{E}^*)$ and $G_1 := R_{\varepsilon/2,4R,2}(\mathcal{E}^*)$. Take a real-valued function $\eta \in C_c^\infty(G_1)$ satisfying $\eta \equiv 1$ on G, $0 \le \eta \le 1$ and, for all $(y,t) \in G_1$, $|\nabla_{y,t}\eta(y,t)| \lesssim 1/t$. By (1.3), the definition of L_w, integration by parts, (1.2), and the Hölder inequality, we conclude that

(3.12)

$$
\begin{aligned}
&\iint_G t|\nabla u(y,t)|^2 w(y)\,dy\,dt\\
&\leq \frac{1}{\lambda}\Re\Big\{\iint_{G_1} tA(y)\nabla u(y,t)\cdot\overline{\nabla u(y,t)}\eta(y,t)\,dy\,dt\Big\}\\
&= \frac{1}{\lambda}\Re\Big\{\iint_{G_1}[tA(y)\nabla u(y,t)\cdot\overline{\nabla(\eta u)(y,t)}\\
&\qquad\qquad - tA(y)\nabla u(y,t)\cdot\nabla\eta(y,t)\overline{u(y,t)}]\,dy\,dt\Big\}\\
&= \frac{1}{\lambda}\Re\Big\{\iint_{G_1}\big[tL_w u(y,t)\overline{(\eta u)(y,t)}w(y)\\
&\qquad\qquad - tA(y)\nabla u(y,t)\cdot\nabla\eta(y,t)\overline{u(y,t)}\big]\,dy\,dt\Big\}\\
&= \frac{1}{\lambda}\Re\Big\{\iint_{G_1}\Big[-\frac{1}{4}\partial_t(|u(y,t)|^2)\eta(y,t)w(y)\\
&\qquad\qquad - tA(y)\nabla u(y,t)\cdot\nabla\eta(y,t)\overline{u(y,t)}\Big]\,dy\,dt\Big\}\\
&\lesssim \iint_{G_1}|u(y,t)|^2|\partial_t\eta(y,t)|w(y)\,dy\,dt\\
&\quad + \iint_{G_1} t|A(y)\nabla u(y,t)\cdot\nabla\eta(y,t)u(y,t)|\,dy\,dt\\
&\lesssim \iint_{G_1\smallsetminus G}|u(y,t)|^2 w(y)\,dy\,\frac{dt}{t}\\
&\quad + \Big[\iint_{G_1\smallsetminus G} t|\nabla u(y,t)|^2 w(y)\,dy\,dt\Big]^{1/2}\Big[\iint_{G_1\smallsetminus G}|u(y,t)|^2 w(y)\,dy\,\frac{dt}{t}\Big]^{1/2}.
\end{aligned}
$$

For $\varepsilon\in(0,\infty)$, consider the following three regions:

$$B^{\varepsilon}(\mathcal{E}^*) := \{(x,t)\in\mathbb{R}^n\times(\varepsilon/2,\varepsilon) : \operatorname{dist}(x,\mathcal{E}^*) < 2t\}, \tag{3.13}$$

$$B^{R}(\mathcal{E}^*) := \{(x,t)\in\mathbb{R}^n\times(2R,4R) : \operatorname{dist}(x,\mathcal{E}^*) < 2t\}, \tag{3.14}$$

$$\widetilde{B}(\mathcal{E}^*) := \{(x,t)\in B^*\times(\varepsilon,2R) : t < \operatorname{dist}(x,\mathcal{E}^*) < 2t\}, \tag{3.15}$$

and observe that

$$(G_1\smallsetminus G)\subset(B^{\varepsilon}(\mathcal{E}^*)\cup B^{R}(\mathcal{E}^*)\cup\widetilde{B}(\mathcal{E}^*)).$$

Next, we consider integrals in (3.12) corresponding, respectively, to the regions in (3.13) through (3.15).

For each $\varepsilon\in(0,\infty)$, let

$$\mathrm{I}^{(\varepsilon)} := \iint_{B^{\varepsilon}(\mathcal{E}^*)}|u(y,t)|^2 w(y)\,dy\,\frac{dt}{t}.$$

For every $(y,t)\in B^{\varepsilon}(\mathcal{E}^*)$, there exists some $y^*\in\mathcal{E}^*$ such that $|y-y^*|<2t$. From the definition of $\mathcal{E}^*$, it follows that $w(E\cap B(y^*,2t))\geq\frac{1}{2}w(B(y^*,2t))$. By the fact that $B(y^*,2t)\subset B(y,4t)$ and Lemma 2.2, we see that

$$w\big(E\cap B(y,4t)\big)\geq w\big(E\cap B(y^*,2t)\big)\gtrsim w\big(B(y^*,2t)\big)\gtrsim w\big(B(y,4t)\big).$$

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By this, Lemma 2.2, and the Fubini theorem, we have

$$\text{(3.16)}\quad \mathrm{I}^{(\varepsilon)} \lesssim \iint_{B^{\varepsilon}(\mathcal{E}^*)} \int_{E\cap B(y,4t)} |u(y,t)|^2 w(z)\,dz\,w(y)\,\frac{dy}{w(B(y,4t))}\,\frac{dt}{t}$$
$$\lesssim \int_{\varepsilon/2}^{\varepsilon}\int_E \Big[\frac{1}{w(B(z,4t))}\int_{B(z,4t)} |u(y,t)|^2 w(y)\,dy\Big] w(z)\,dz\,\frac{dt}{t}$$
$$\lesssim \int_{\varepsilon/2}^{\varepsilon}\int_E \big[\mathcal{N}_h^{(\beta)}(f)(z)\big]^2 w(z)\,dz\,\frac{dt}{t} \lesssim \int_E \big[\mathcal{N}_h^{(\beta)}(f)(z)\big]^2 w(z)\,dz,$$

for all $\beta \geq 4$.

For each $\varepsilon \in (0,\infty)$, let

$$\mathrm{II}^{(\varepsilon)} := \iint_{B^{\varepsilon}(\mathcal{E}^*)} t|\nabla u(y,t)|^2 w(y)\,dy\,dt.$$

By an argument similar to that used in the estimate for $\mathrm{I}^{(\varepsilon)}$, we conclude that

$$\text{(3.17)}\quad \mathrm{II}^{(\varepsilon)} \lesssim \int_{\varepsilon/2}^{\varepsilon}\int_E \Big[\frac{1}{w(B(z,4t))}\int_{B(z,4t)} t|\nabla u(y,t)|^2 w(y)\,dy\Big] w(z)\,dz\,dt$$
$$\lesssim \int_E \int_{\varepsilon/2}^{\varepsilon}\int_{B(z,4\varepsilon)} t|\nabla u(y,t)|^2 w(y)\,dy\,dt\,\frac{w(z)\,dz}{w(B(z,4\varepsilon))}.$$

From the definition of $u(y,t) = e^{-t^2 L_w} f(y)$, together with the Caccioppoli inequality (3.7), we deduce that

$$\int_{\varepsilon/2}^{\varepsilon}\int_{B(z,4\varepsilon)} t|\nabla u(y,t)|^2 w(y)\,dy\,dt \lesssim \frac{1}{\varepsilon^2}\int_{\varepsilon/4}^{5\varepsilon/4}\int_{B(z,8\varepsilon)} t|u(y,t)|^2 w(y)\,dy\,dt.$$

Combining this, Lemma 2.2, and (3.17), we find that

$$\text{(3.18)}\quad \mathrm{II}^{(\varepsilon)} \lesssim \int_E \frac{1}{\varepsilon^2}\int_{\varepsilon/4}^{5\varepsilon/4}\int_{B(z,8\varepsilon)} t|u(y,t)|^2\,dy\,dt\,\frac{w(z)\,dz}{w(B(z,4\varepsilon))}$$
$$\lesssim \int_E \int_{\varepsilon/4}^{5\varepsilon/4} \frac{1}{w(B(z,32t))}\int_{B(z,32t)} |u(y,t)|^2 w(y)\,dy\,\frac{dt}{t}\,w(z)\,dz$$
$$\lesssim \int_{\varepsilon/4}^{5\varepsilon/4}\int_E \big[\mathcal{N}_h^{(\beta)}(f)(z)\big]^2 w(z)\,dz\frac{dt}{t} \lesssim \int_E \big[\mathcal{N}_h^{(\beta)}(f)(z)\big]^2 w(z)\,dz,$$

for all $\beta \geq 32$. By the same argument as above, we have

$$\text{(3.19)}\qquad \iint_{B^{R}(\mathcal{E}^*)} |u(y,t)|^2 w(y)\,dy\,\frac{dt}{t} \lesssim \int_E [\mathcal{N}_h^{(\beta)}(f)(z)]^2 w(z)\,dz$$

and

$$\text{(3.20)}\qquad \iint_{B^{R}(\mathcal{E}^*)} t|\nabla u(y,t)|^2 w(y)\,dy\,dt \lesssim \int_E [\mathcal{N}_h^{(\beta)}(f)(z)]^2 w(z)\,dz$$

for all $\beta \geq 32$.

To control the integral over $\widetilde{B}(\mathcal{E}^*)$, we first decompose $B^* := (\mathcal{E}^*)^{\complement}$ into a family of Whitney balls, $\{B(x_k,r_k)\}_{k=0}^{\infty}$, such that $B^* = \bigcup_{k=0}^{\infty} B(x_k,r_k)$,

$$c_1\,\mathrm{dist}(x_k,\mathcal{E}^*) \leq r_k \leq c_2\,\mathrm{dist}(x_k,\mathcal{E}^*)$$

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and every point $x \in B^*$ belongs to at most c_3 balls. Here $0 < c_1 < c_2 < 1$ and $c_3 \in \mathbb{N}$ are some fixed constants, independent of B^* (see, for example, [36, Theorem 3]). Then by the definition of $\widetilde{B}(\mathcal{E}^*)$ and Lemma 2.2, we see that

(3.21)
$$\begin{aligned}\widetilde{\mathrm{I}} &:= \iint_{\widetilde{B}(\mathcal{E}^*)} |u(y,t)|^2 w(y)\,dy\,\frac{dt}{t}\\ &\le \sum_{k=0}^{\infty} \int_{\frac12(\frac{1}{c_2}-1)r_k}^{(1+\frac{1}{c_1})r_k} \int_{B(x_k,r_k)} |u(y,t)|^2 w(y)\,dy\,\frac{dt}{t}\\ &\lesssim \sum_{k=0}^{\infty} \int_{(\frac{1}{c_2}-1)\frac{r_k}{2}}^{(1+\frac{1}{c_1})r_k} w(B(x_k,r_k))\Big[\frac{1}{w(B(x_k,\frac{2c_2}{1-c_2}t))}\int_{B(x_k,\frac{2c_2}{1-c_2}t)} |u(y,t)|^2 w(y)\,dy\Big]\frac{dt}{t}.\end{aligned}$$

From the fact that $\mathcal{E}^* \subset E$, it follows that $\operatorname{dist}(x_k, E) \le \operatorname{dist}(x_k, \mathcal{E}^*) \le \frac{2c_2}{(1-c_2)c_1} t$. Hence, we have

$$\frac{1}{w(B(x_k,\frac{2c_2}{1-c_2}t))}\int_{B(x_k,\frac{2c_2}{1-c_2}t)} |u(y,t)|^2 w(y)\,dy \lesssim [\sup_{z\in E} \mathcal{N}_h^{(\beta)}(f)(z)]^2,$$

for all $\beta \ge \frac{2c_2}{(1-c_2)c_1}$. By this and (3.21), we see that

(3.22) $$\widetilde{\mathrm{I}} \lesssim \sum_{k=0}^{\infty} w\big(B(x_k,r_k)\big)\Big[\sup_{z\in E}\mathcal{N}_h^{(\beta)}(f)(z)\Big]^2 \lesssim w(B^*)\Big[\sup_{z\in E}\mathcal{N}_h^{(\beta)}(f)(z)\Big]^2,$$

for all $\beta \ge \frac{2c_2}{(1-c_2)c_1}$.

Similar to (3.21) and (3.22), by using Lemma 3.10 to control the gradient of u, we conclude that there exist positive constants C and $\widetilde{C} := \widetilde{C}_{(c_1,c_2)}$, depending on c_1 and c_2, such that

(3.23) $$\widetilde{\mathrm{II}} := \iint_{\widetilde{B}(\mathcal{E}^*)} t|\nabla u(y,t)|^2 w(y)\,dy\,dt \le Cw(B^*)[\sup_{z\in E}\mathcal{N}_h^{(\beta)}(f)(z)]^2,$$

for all $\beta \ge \widetilde{C}$.

Now, by choosing

$$\beta := \max\Big\{32, \frac{2c_2}{(1-c_2)c_1}, \widetilde{C}\Big\}$$

in (3.10), and via (3.22) and (3.23), we conclude that

$$\widetilde{\mathrm{I}} \lesssim \sigma^2 w(B^*) \quad \text{and} \quad \widetilde{\mathrm{II}} \lesssim \sigma^2 w(B^*).$$

By this, (3.11), (3.12), (3.16), (3.18), (3.19), and (3.20), we further find that

$$\int_{\mathcal{E}^*} [\widetilde{\mathcal{S}}_{L_w}^{(2\varepsilon,R,1/2)}(f)(x)]^2 w(x)\,dx \lesssim \sigma^2 w(B^*) + \int_E [\mathcal{N}_h^{(\beta)}(f)(z)]^2 w(z)\,dz.$$

Passing to the limit as $\varepsilon \to 0$ and $R \to \infty$, we see that

(3.24) $$\int_{\mathcal{E}^*} [\widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)(x)]^2 w(x)\,dx \lesssim \sigma^2 w(B^*) + \int_E [\mathcal{N}_h^{(\beta)}(f)(z)]^2 w(z)\,dz.$$

Let $\lambda_{\mathcal{N}_h^{(\beta)}(f)}$ be the distribution function of $\mathcal{N}_h^{(\beta)}(f)$ with respect to w; namely, for any $a \in (0,\infty)$,

$$\lambda_{\mathcal{N}_h^{(\beta)}(f)}(a) := w\big(\big\{x \in \mathbb{R}^n : \mathcal{N}_h^{(\beta)}(f)(x) > a\big\}\big).$$

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Recall that $\mathcal{N}_h^{(\beta)}(f) \le \sigma$ on E (see (3.10)). From the definition of B^*, (3.6) and the boundedness of M_w from $L^1(w,\mathbb{R}^n)$ to the weak-$L^1(w,\mathbb{R}^n)$, it follows that

$$w(B^*) = w\big(\{x \in \mathbb{R}^n : M_w(\chi_{E^\complement})(x) > 1/2\}\big) \lesssim w(E^\complement) \sim \lambda_{\mathcal{N}_h^{(\beta)}(f)}(\sigma).$$

By this and (3.24) we have

$$\begin{aligned}\lambda_{\widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)}(\sigma) &\le w\big(\{x \in \mathcal{E}^* : \widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)(x) > \sigma\}\big) + w(B^*)\\ &\lesssim \frac{1}{\sigma^2}\int_{\mathcal{E}^*}[\widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)(x)]^2 w(x)\,dx + w(B^*)\\ &\lesssim \frac{1}{\sigma^2}\int_0^\sigma t\lambda_{\mathcal{N}_h^{(\beta)}(f)}(t)\,dt + \lambda_{\mathcal{N}_h^{(\beta)}(f)}(\sigma).\end{aligned}$$

From this and Lemma 3.9 we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^n}[\widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)(x)]^p w(x)\,dx\\ &= \int_0^\infty u^{p-1}\lambda_{\widetilde{\mathcal{S}}_{L_w}^{(1/2)}(f)}(u)\,du\\ &\lesssim \int_0^\infty u^{p-1}\frac{1}{u^2}\int_0^u t\lambda_{\mathcal{N}_h^{(\beta)}(f)}(t)\,dt + \int_0^\infty u^{p-1}\lambda_{\mathcal{N}_h^{(\beta)}(f)}(u)\,du\\ &\lesssim \int_0^\infty t\lambda_{\mathcal{N}_h^{(\beta)}(f)}(t)\int_t^\infty u^{p-3}\,du\,dt + \int_{\mathbb{R}^n}[\mathcal{N}_h^{(\beta)}(f)(x)]^p w(x)\,dx\\ &\lesssim \int_{\mathbb{R}^n}[\mathcal{N}_h^{(\beta)}(f)(x)]^p w(x)\,dx \lesssim \int_{\mathbb{R}^n}[\mathcal{N}_h f(x)]^p w(x)\,dx,\end{aligned}$$

which together with (3.9) completes the proof of Theorem 3.6. ∎

3.3 Proof of Theorem 3.7

The following lemma is a special case of [4, Corollary 4.7].

Lemma 3.12 ([4]) *Let $w \in A_q(\mathbb{R}^n)$ with $q \in [1,\infty)$, $p \in (0,1]$, $\varepsilon \in (0,\infty)$, and $M \in \mathbb{N}$ satisfy $M > C_{(p,q,n)}$, where $C_{(p,q,n)}$ is a positive constant depending on p, q, and n. Suppose that T is a linear (resp. non-negative sublinear) operator that maps $L^2(w,\mathbb{R}^n)$ continuously into weak-$L^2(w,\mathbb{R}^n)$. If there exists a positive constant C such that for any $(p,2,M,\epsilon)_{L_w}$-molecule m associated with the ball B,*

$$\int_{\mathbb{R}^n}|T(m)(x)|^p w(x)\,dx \le C,$$

then T can extend to be a bounded linear (resp. sublinear) operator from $H_{L_w,\mathrm{mol}}^{p,2,M}(\mathbb{R}^n)$ to $L^p(w,\mathbb{R}^n)$.

Recall that an operator T is said to be *non-negative*, if $T(f) \ge 0$ for all non-negative functions f in the domain of T. Theorem 3.7 then follows from establishing the boundedness of $\mathcal{N}_h$ on all $(p,2,M,\epsilon)_{L_w}$-molecules.

Proof of Theorem 3.7 For $M \in \mathbb{N}$, we first introduce the *radial maximal functions*, $\mathcal{R}_h$ and $\mathcal{R}_h^{(M)}$, respectively, by setting, for all $f \in L^2(w, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{R}_h(f)(x) := \sup_{t\in(0,\infty)} \Big[\frac{1}{w(B(x,t))} \int_{B(x,t)} |e^{-t^2 L_w}(f)(y)|^2 w(y)\,dy \Big]^{1/2}$$

and

$$\mathcal{R}_h^{(M)}(f)(x) := \sup_{t\in(0,\infty)} \Big[\frac{1}{w(B(x,t))} \int_{B(x,t)} |(t^2 L_w)^M e^{-t^2 L_w}(f)(y)|^2 w(y)\,dy \Big]^{1/2}.$$

Both of the operators above are bounded on $L^2(w, \mathbb{R}^n)$. Indeed, by Proposition 1.5, we know that there exists some $p \in (1,2)$ such that $e^{-tL_w} \in \mathcal{O}_w(L^p\text{-}L^2)$. From this and the boundedness of M_w in $L^{2/p}(w, \mathbb{R}^n)$, it follows that for all $f \in L^2(w, \mathbb{R}^n)$,

$$\begin{aligned}
&\|\mathcal{R}_h(f)\|^2_{L^2(w,\mathbb{R}^n)}\\
&\quad\lesssim \int_{\mathbb{R}^n} \Big\{ \sup_{t\in(0,\infty)} \sum_{j=0}^{\infty} \Big[\frac{1}{w(B(x,t))}\\
&\qquad\qquad \times \int_{B(x,t)} |e^{-t^2 L_w}(\chi_{U_j(B(x,t))} f)(y)|^2 w(y)\,dy \Big]^{1/2} \Big\}^2 w(x)\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n} \Big\{ \sup_{t\in(0,\infty)} \sum_{j=3}^{\infty} 2^{j\theta_1} \Big[\Upsilon\Big(\frac{2^j t}{t}\Big)\Big]^{\theta_2} e^{-c\frac{4^j t^2}{t^2}}\\
&\qquad\qquad \times \Big[\frac{1}{w(2^j B(x,t))} \int_{2^j B(x,t)} |f(y)|^p w(y)\,dy \Big]^{1/p}\\
&\qquad\quad + \sup_{t\in(0,\infty)} \Big[\frac{1}{w(B(x,4t))} \int_{B(x,4t)} |f(y)|^p w(y)\,dy \Big]^{1/p} \Big\}^2 w(x)\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n} \Big\{ \sum_{j=2}^{\infty} 2^{j(\theta_1+\theta_2)} e^{-c4^j} [M_w(|f|^p)(x)]^{1/p} + [M_w(|f|^p)(x)]^{1/p} \Big\}^2 w(x)\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n} [M_w(|f|^p)(x)]^{2/p} w(x)\,dx \lesssim \int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx,
\end{aligned}$$

where θ_1, θ_2, Υ, and c are as in Definition 1.4 with $q = 2$ and $\{U_j(B(x,t))\}_{j\in\mathbb{Z}_+}$ are as in (1.15) with B replaced by $B(x,t)$. By a similar argument as above, we also obtain the boundedness of $\mathcal{R}_h^{(M)}$ in $L^2(w, \mathbb{R}^n)$.

Observe that by the definitions of $\mathcal{R}_h(f)$ and $\mathcal{N}_h^{(1/2)}(f)$, together with Lemma 2.2, we conclude that for all $f \in L^2(w, \mathbb{R}^n)$, $\mathcal{N}_h^{(1/2)}(f) \lesssim \mathcal{R}_h(f)$. From this and Lemma 3.11 we further deduce that for all $f \in L^2(w, \mathbb{R}^n)$,

$$\|\mathcal{N}_h(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\mathcal{N}_h^{(1/2)}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\mathcal{R}_h(f)\|_{L^p(w,\mathbb{R}^n)}.$$

By this and Lemma 3.12, to prove the desired conclusion of Theorem 3.7, it suffices to prove that for all $(p, 2, M, \epsilon)_{L_w}$-molecules m associated with the ball $B \equiv B(x_B, r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,

$$\|\mathcal{R}_h(m)\|_{L^p(w,\mathbb{R}^n)} \lesssim 1.$$

To this end, by the Hölder inequality, we write

$$
\begin{aligned}
&\int_{\mathbb{R}^n}[\mathcal{R}_h(m)(x)]^p w(x)\,dx\\
&\quad\le \sum_{j=0}^{\infty}\int_{U_j(B)}[\mathcal{R}_h(m)(x)]^p w(x)\,dx\\
&\quad\le \sum_{j=0}^{\infty}[w(U_j(B))]^{1-\frac{p}{2}}\Big\{\int_{U_j(B)}[\mathcal{R}_h(m)(x)]^2 w(x)\,dx\Big\}^{\frac{p}{2}}\\
&\quad\le \sum_{j=0}^{10}[w(U_j(B))]^{1-\frac{p}{2}}\|\mathcal{R}_h(m)\|^p_{L^2(w,U_j(B))}\\
&\qquad+\sum_{j=11}^{\infty}[w(U_j(B))]^{1-\frac{p}{2}}\|\mathcal{R}_h(m)\|^p_{L^2(w,U_j(B))} =: \mathrm{I}+\mathrm{II},
\end{aligned}
$$

where $U_j(B)$ is as in (1.15).

Since $\mathcal{R}_h$ is bounded on $L^2(w,\mathbb{R}^n)$, from the definition of m, it follows that $\mathrm{I}\lesssim 1$.

To estimate the term II, we fix some constant $a\in(0,1)$ such that $M>\frac{qn}{2ap}(1-\frac{p}{2})$, which is possible, since $M>\frac{qn}{2p}(1-\frac{p}{2})$. Then for every $j\ge 11$ and $x\in U_j(B)$, write

(3.25)

$$
\begin{aligned}
\mathcal{R}_h(m)(x)\le{}& \sup_{t\in(0,2^{aj-2}r_B]}\Big[\frac{1}{w(B(x,t))}\int_{B(x,t)}|e^{-t^2L_w}(m)(y)|^2 w(y)\,dy\Big]^{1/2}\\
&+\sup_{t\in(2^{aj-2}r_B,\infty)}\Big[\frac{1}{w(B(x,t))}\int_{B(x,t)}|e^{-t^2L_w}(m)(y)|^2 w(y)\,dy\Big]^{1/2}\\
=:{}& \mathrm{II}_{1,j}+\mathrm{II}_{2,j}.
\end{aligned}
$$

To handle $\mathrm{II}_{1,j}$, let $S_j(B):=(2^{j+3}B)\setminus(2^{j-3}B)$,

$$
R_j(B):=(2^{j+5}B)\setminus(2^{j-5}B)\quad\text{and}\quad E_j(B):=[R_j(B)]^{\complement}.
$$

Write $m=m\chi_{R_j(B)}+m\chi_{E_j(B)}$. Since $t\le 2^{aj-2}r_B$, it follows that for any $x\in U_j(B)$,

$$
B(x,t)\subset S_j(B)\quad\text{and}\quad \operatorname{dist}(S_j(B),E_j(B))\sim[2^{j+5}-2^{j+3}]r_B\sim 2^j r_B.
$$

By Lemma 2.2, we see that for any $x\in U_j(B)$ and $t\in(0,2^{aj-2}r_B]$,

$$
w(B(x_B,2^jr_B))\sim w(B(x,2^jr_B))\lesssim w(B(x,t))\Big(\frac{2^jr_B}{t}\Big)^{qn}.
$$

From this and (1.9) we deduce that for every $j\ge 11$,

$$
\begin{aligned}
&\Big\|\sup_{t\in(0,2^{aj-2}r_B]}\Big[\frac{1}{w(B(\cdot,t))}\int_{B(\cdot,t)}|e^{-t^2L_w}(m\chi_{E_j(B)})(y)|^2 w(y)\,dy\Big]^{1/2}\Big\|_{L^2(w,U_j(B))}\\
&\quad\le\Big\|\sup_{t\in(0,2^{aj-2}r_B]}\frac{1}{[w(B(\cdot,t))]^{1/2}}\\
&\qquad\qquad\times\Big[\int_{S_j(B)}|e^{-t^2L_w}(m\chi_{E_j(B)})(y)|^2 w(y)\,dy\Big]^{1/2}\Big\|_{L^2(w,U_j(B))}\\
&\quad\lesssim\Big\|\sup_{t\in(0,2^{aj-2}r_B]}\frac{1}{[w(B(\cdot,t))]^{1/2}}e^{-c\frac{[\operatorname{dist}(S_j(B),E_j(B))]^2}{t^2}}\|m\|_{L^2(w,E_j(B))}\Big\|_{L^2(w,U_j(B))}
\end{aligned}
$$

$$\lesssim \Big\| \sup_{t\in(0,2^{aj-2}r_B]} \frac{1}{[w(B(\cdot,t))]^{1/2}} \Big(\frac{t}{2^j r_B}\Big)^N \Big\|_{L^2(w,U_j(B))} \|m\|_{L^2(w,\mathbb{R}^n)}$$
$$\lesssim \Big\| \sup_{t\in(0,2^{aj-2}r_B]} \frac{1}{[w(B(x_B,2^j r_B))]^{1/2}} 2^{(\frac{qn}{2}-N)j} \Big(\frac{t}{r_B}\Big)^{N-\frac{qn}{2}} \Big\|_{L^2(w,U_j(B))} \|m\|_{L^2(w,\mathbb{R}^n)}$$
$$\lesssim 2^{(\frac{qn}{2}-N)j} \Big(\frac{2^{aj} r_B}{r_B}\Big)^{N-\frac{qn}{2}} \|m\|_{L^2(w,\mathbb{R}^n)} \lesssim 2^{(1-a)(qn/2-N)j} \|m\|_{L^2(w,\mathbb{R}^n)},$$

where the positive constant N is greater than $\frac{qn(2-a)}{2p(1-a)}$. Thus, by this and the definition of m, we further conclude that

$$\begin{aligned}(3.26)\quad &\sum_{j=11}^{\infty} [w(U_j(B))]^{1-\frac{p}{2}} \Big\| \sup_{t\in(0,2^{aj-2}r_B]} \Big[\frac{1}{w(B(\cdot,t))} \\ &\qquad\times \int_{B(\cdot,t)} |e^{-t^2 L_w}(m\chi_{E_j(B)})(y)|^2 w(y)\,dy\Big]^{1/2} \Big\|^p_{L^2(w,U_j(B))} \\ &\lesssim \sum_{j=11}^{\infty} 2^{p(1-a)(\frac{qn}{2}-N)j} 2^{(1-\frac{p}{2})jqn} [w(B)]^{1-p/2} \|m\|^p_{L^2(w,\mathbb{R}^n)} \lesssim 1.\end{aligned}$$

As for the estimate of $m\chi_{R_j(B)}$, from the $L^2(w,\mathbb{R}^n)$-boundedness of $\mathcal{R}_h$, the definition of m and the fact that $\varepsilon\in(\frac{nq}{p},\infty)$, it follows that

$$\begin{aligned}(3.27)\quad &\sum_{j=0}^{\infty} [w(U_j(B))]^{1-\frac{p}{2}} \|\mathcal{R}_h(m\chi_{R_j(B)})\|^p_{L^2(w,U_j(B))} \\ &\lesssim \sum_{j=0}^{\infty} [w(U_j(B))]^{1-\frac{p}{2}} \|m\|^p_{L^2(w,R_j(B))} \lesssim \sum_{j=0}^{\infty} 2^{-jp\varepsilon} 2^{jnq} \lesssim 1.\end{aligned}$$

Combining (3.26) and (3.27), we find that

$$(3.28)\qquad \sum_{j=11}^{\infty} [w(U_j(B))]^{1-\frac{p}{2}} \|\mathrm{II}_{1,j}\|^p_{L^2(w,U_j(B))} \lesssim 1.$$

Now we consider the term $\mathrm{II}_{2,j}$. For every $j\ge 11$ and $x\in U_j(B)$, we have

$$\begin{aligned}\mathrm{II}_{2,j} &= \sup_{t\in(2^{aj-2}r_B,\infty)} \Big[\frac{1}{w(B(x,t))} \\ &\qquad\times \int_{B(x,t)} |t^{2M} L_w^{2M} e^{-t^2 L_w}(t^{-2M} L_w^{-M}(m))(y)|^2 w(y)\,dy\Big]^{1/2} \\ &\lesssim 2^{-2aMj} \sup_{t\in(2^{aj-2}r_B,\infty)} \Big[\frac{1}{w(B(x,t))} \\ &\qquad\times \int_{B(x,t)} \Big| t^{2M} L_w^{2M} e^{-t^2 L_w}(r_B^{-2M} L_w^{-M}(m))(y)\Big|^2 w(y)\,dy\Big]^{1/2} \\ &\lesssim 2^{-2aMj} \mathcal{R}_h^{(M)}\big(r_B^{-2M} L_w^{-M}(m)\big)(x),\end{aligned}$$

which, together with the boundedness of $\mathcal{R}_h^{(M)}$ in $L^2(w,\mathbb{R}^n)$ and the definition of m, further implies that

$$\sum_{j=11}^{\infty}\big[w(U_j(B))\big]^{1-\frac{p}{2}}\|\mathrm{II}_{2,j}\|_{L^2(w,U_j(B))}^p \tag{3.29}$$
$$\lesssim \sum_{j=11}^{\infty}2^{-2apMj}[w(2^jB)]^{1-\frac{p}{2}}\|\mathcal{R}_h^{(M)}(r_B^{-2M}L_w^{-M}(m))\|_{L^2(w,\mathbb{R}^n)}^p$$
$$\lesssim \sum_{j=11}^{\infty}2^{-2apMj}[w(2^jB)]^{1-\frac{p}{2}}\|r_B^{-2M}L_w^{-M}(m)\|_{L^2(w,\mathbb{R}^n)}^p$$
$$\lesssim \sum_{j=11}^{\infty}2^{-[2apM-(1-\frac{p}{2})qn]j}\lesssim 1,$$

where $M>\frac{qn}{2ap}(1-\frac{p}{2})$.

By combining (3.25), (3.28), and (3.29), we have $\mathrm{II}\lesssim 1$. This further implies that $\|\mathcal{R}_h(m)\|_{L^p(w,\mathbb{R}^n)}\lesssim 1$, which completes the proof of Theorem 3.7. ■

4 Boundedness of Riesz Transforms

In this section, we give the proof of Theorem 1.6. Before going into the details, we present some technical propositions.

Observe that when $w\in A_2(\mathbb{R}^n)$, $\nabla L_w^{-1/2}$ is bounded from $L^2(w,\mathbb{R}^n)$ to itself (see [13, Theorem 1.1]) and $\sqrt{t}\nabla e^{-tL_w}$ satisfies the weighted Davies–Gaffney estimate (see Proposition 2.7). Proposition 4.1 is a special case of [5, Lemma 4.4] with $(X,d,\mu):=(\mathbb{R}^n,|\cdot|,w(x)\,dx)$ and $DL^{-1/2}:=\nabla L_w^{-1/2}$.

Proposition 4.1 *For every $M\in\mathbb{N}$, there exists a positive constant $C_{(M)}$, depending on M, such that for all $t\in(0,\infty)$, closed subsets E,F of $\mathbb{R}^n$ with $\operatorname{dist}(E,F)>0$ and $f\in L^2(w,\mathbb{R}^n)$ supported in E,*

$$\|\nabla L_w^{-1/2}(I-e^{-tL_w})^M(f)\|_{L^2(w,F)}\le C_{(M)}\Big(\frac{t}{[d(E,F)]^2}\Big)^M\|f\|_{L^2(w,E)},$$
$$\|\nabla L_w^{-1/2}(tL_we^{-tL_w})^M(f)\|_{L^2(w,F)}\le C_{(M)}\Big(\frac{t}{[d(E,F)]^2}\Big)^M\|f\|_{L^2(w,E)}.$$

We also need the following technical lemma.

Proposition 4.2 *Let $M\in\mathbb{N}$ and let E,F be closed subsets of $\mathbb{R}^n$. If $d(E,F)>0$, then there exists a positive constant $C_{(M)}$, depending on M, but independent of E and F, such that for all $t\in(0,\infty)$ and $f\in L^2(w,\mathbb{R}^n)$ supported in F,*

$$\|L_w^{-1/2}(I-e^{-tL_w})^M(f)\|_{L^2(w,E)}\le C_{(M)}\sqrt{t}\Big(\frac{t}{[d(E,F)]^2}\Big)^{M-\frac12}\|f\|_{L^2(w,F)},$$
$$\|L_w^{-1/2}(tL_we^{-tL_w})^M(f)\|_{L^2(w,E)}\le C_{(M)}\sqrt{t}\Big(\frac{t}{[d(E,F)]^2}\Big)^{M-\frac12}\|f\|_{L^2(w,F)}.$$

If $d(E,F)=0$, then there exists a positive constant $C_{(M)}$, depending on M, but independent of E and F, such that for all $t\in(0,\infty)$ and $f\in L^2(w,\mathbb{R}^n)$ supported in F,

$$\|L_w^{-1/2}(I-e^{-tL_w})^M(f)\|_{L^2(w,E)}\le C_{(M)}\sqrt{t}\|f\|_{L^2(w,F)},$$
$$\|L_w^{-1/2}(tL_we^{-tL_w})^M(f)\|_{L^2(w,E)}\le C_{(M)}\sqrt{t}\|f\|_{L^2(w,F)}.$$

Proof Notice that for every $k\in\mathbb{Z}_+$, $\{(tL_w)^ke^{-tL_w}\}_{t>0}$ satisfy the weighted Davies–Gaffney estimates (see Proposition 2.6), namely, there exists a positive constant C such that for all $t\in(0,\infty)$, closed subsets E,F of $\mathbb{R}^n$ and $f\in L^2(w,\mathbb{R}^n)$ supported in F,

$$\|(tL_w)^ke^{-tL_w}(f)\|_{L^2(w,E)}\lesssim e^{-C\frac{[d(E,F)]^2}{t}}\|f\|_{L^2(w,F)}. \tag{4.1}$$

The remainder of the proof of this proposition is completely analogous to that of [26, Lemma 2.2], replacing the Davies–Gaffney estimates used therein for the gradient of semigroup by (4.1) above, the details being omitted. This finishes the proof of Proposition 4.2. ∎

In what follows, let $\mathcal{S}(\mathbb{R}^n)$ denote the *space of all Schwartz functions* and let $\mathcal{S}'(\mathbb{R}^n)$ be the *space of all Schwartz distributions*.

Let $\psi\in\mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\psi(x)\,dx=1$ and $\psi_t(x):=t^{-n}\psi(\frac{x}{t})$ for all $x\in\mathbb{R}^n$ and $t\in(0,\infty)$. For all $f\in\mathcal{S}'(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, the *non-tangential maximal function* $\psi_\triangledown^*(f)(x)$ is defined by setting

$$\psi_\triangledown^*(f)(x):=\sup_{\substack{|x-y|<t\\ t\in(0,\infty)}}|(\psi_t*f)(y)|.$$

Then for $p\in(0,1]$ and $w\in A_\infty(\mathbb{R}^n)$, $f\in\mathcal{S}'(\mathbb{R}^n)$ is said to belong to the *weighted Hardy space* $H_w^p(\mathbb{R}^n)$, if $\psi_\triangledown^*(f)\in L^p(w,\mathbb{R}^n)$; moreover, define

$$\|f\|_{H_w^p(\mathbb{R}^n)}:=\|\psi_\triangledown^*(f)\|_{L^p(w,\mathbb{R}^n)}.$$

An important fact is that every element in the Hardy space $H_w^p(\mathbb{R}^n)$ admits an atomic decomposition. Let us first recall the definition of $(p,q,s)_w$-atoms as follows. Recall that $\lfloor s\rfloor$ for any $s\in\mathbb{R}$ denotes the *maximal integer not more than* s.

Definition 4.3 ([23]) Let $p\in(0,1]$, $q\in[1,\infty)$ with $q>p$ and $w\in A_q(\mathbb{R}^n)$. Assume that $s\in\mathbb{Z}_+$ satisfies $s\ge\lfloor n(q_w/p-1)\rfloor$, where

$$q_w:=\inf\{q\in[1,\infty):w\in A_q(\mathbb{R}^n)\}.$$

A function a is called a $(p,q,s)_w$*-atom* associated with the ball B if

(i) supp $a\subset B$;
(ii) $\|a\|_{L^q(w,\mathbb{R}^n)}\le[w(B)]^{1/q-1/p}$;
(iii) for all $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\le s$, $\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$.

Definition 4.4 Let p, q, s, and w be as in Definition 4.3. The *atomic weighted Hardy space* $H_w^{p,q,s}(\mathbb{R}^n)$ is defined by setting

$$H_w^{p,q,s}(\mathbb{R}^n):=\Big\{f\in\mathcal{S}'(\mathbb{R}^n):f=\sum_{j=0}^\infty\lambda_ja_j\text{ in }\mathcal{S}'(\mathbb{R}^n)\Big\},$$

where $\{a_j\}_{j=0}^{\infty}$ is a sequence of $(p,q,s)_w$-atoms and $\{\lambda_j\}_{j=0}^{\infty} \subset \mathbb{C}$ satisfies $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. The *quasi-norm* of f is defined by setting

$$\|f\|_{H_w^{p,q,s}(\mathbb{R}^n)} := \inf\Big\{ \Big(\sum_{j=0}^{\infty} |\lambda_j|^p \Big)^{1/p} \Big\},$$

where the infimum is taken over all possible decompositions of f as above.

The following atomic characterization of $H_w^p(\mathbb{R}^n)$ can be found in [23].

Lemma 4.5 ([23]) *Let p, q, s, and w be as in Definition 4.3. Then the spaces $H_w^p(\mathbb{R}^n)$ and $H_w^{p,q,s}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Definition 4.6 Let $p \in (0,1]$, $w \in A_2(\mathbb{R}^n)$ and $\varepsilon \in (0,\infty)$. A function $m \in L^2(w,\mathbb{R}^n)$ is called a $(p,2,\varepsilon)_w$-*molecule* associated with the ball B if

(i) for every $j \in \mathbb{Z}_+$, $\|m\|_{L^2(w,U_j(B))} \le 2^{-j\varepsilon}[w(2^jB)]^{1/2-1/p}$, where $U_j(B)$ is as in (1.15);

(ii) $\int_{\mathbb{R}^n} m(x)\,dx = 0$.

Proposition 4.7 *Let*

$$p \in \Big(\frac{n}{n+1}, 1\Big], \quad w \in A_{q_0}(\mathbb{R}^n),$$

with $q_0 \in [1, \frac{p(n+1)}{n})$ and $\varepsilon \in (2n+2,\infty)$. Then there exists a positive constant C such that for all $(p,2,\varepsilon)_w$-molecules m, it holds true that

$$m = \sum_{j=0}^{\infty} \lambda_j \alpha_j \text{ in } L^2(w,\mathbb{R}^n),$$

where $\{\lambda_j\}_{j=0}^{\infty} \subset \mathbb{C}$ and $\{\alpha_j\}_{j=0}^{\infty}$ is a family of $(p,2,0)_w$-atoms up to a harmless constant multiple, and $\|m\|_{H_w^{p,2,0}(\mathbb{R}^n)} \le C$.

Proof Let m be a $(p,2,\varepsilon)_w$-molecule associated with a ball B. To prove Proposition 4.7, we borrow some ideas from [7] (see also [3, 32]).

For each $j \in \mathbb{Z}_+$, let $\beta_j := \int_{U_j(B)} m(y)\,dy$ and $\chi_j := \frac{1}{|U_j(B)|}\chi_{U_j(B)}$. Then for each $x \in \mathbb{R}^n$, we define

$$M_j(x) := m(x)\chi_{U_j(B)}(x) - \beta_j\chi_j(x)$$

and $N_j := \sum_{k=j}^{\infty} \beta_k$. Since $\int_{\mathbb{R}^n} m(x)\,dx = 0$, we write

$$m = \sum_{j=0}^{\infty} M_j + \sum_{j=0}^{\infty} N_{j+1}(\chi_{j+1} - \chi_j) =: \sum_{j=0}^{\infty} M_j + \sum_{j=0}^{\infty} P_j, \tag{4.2}$$

where the summations converge for almost every $x \in \mathbb{R}^n$.

For each $j \in \mathbb{Z}_+$, it is easy to see that $\int_{\mathbb{R}^n} M_j(x)\,dx = 0$ and $\operatorname{supp} M_j \subset 2^jB$. Moreover, by the fact $w \in A_2(\mathbb{R}^n)$, the Hölder inequality, and the definition of m, we find that

$$\|M_j\|_{L^2(w,\mathbb{R}^n)}$$
$$\le \|m\|_{L^2(w,U_j(B))} + \frac{|\beta_j|}{|U_j(B)|}[w(U_j(B))]^{1/2}$$

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$$\begin{aligned}&\lesssim \|m\|_{L^2(w,U_j(B))}\\&\quad+\frac{[w(2^jB)]^{1/2}}{|2^jB|}\Big[\int_{2^jB}[w(x)]^{-1}\,dx\Big]^{1/2}\Big[\int_{U_j(B)}|m(x)|^2w(x)\,dx\Big]^{1/2}\\&\lesssim \|m\|_{L^2(w,U_j(B))}\lesssim 2^{-j(\varepsilon-2n/p)}[w(2^jB)]^{1/2-1/p}.\end{aligned}$$

By this together with the fact that $q_0\in[1,\frac{p(n+1)}{n})$ implies $\lfloor n(\frac{q_0}{p}-1)\rfloor=0$, we see that $2^{j(\varepsilon-2n/p)}M_j$ is a $(p,2,0)_w$-atom associated with the ball 2^jB, up to a harmless constant multiple.

On the other hand, for each $j\in\mathbb{Z}_+$, we see that $\int_{\mathbb{R}^n}P_j(x)\,dx=0$, $\operatorname{supp}P_j\subset 2^{j+1}B$ and

$$\|P_j\|_{L^2(w,\mathbb{R}^n)}\le|N_{j+1}|\Big\{\frac{[w(U_{j+1}(B))]^{1/2}}{|U_{j+1}(B)|}+\frac{[w(U_j(B))]^{1/2}}{|U_j(B)|}\Big\}. \tag{4.3}$$

Since $w\in A_2(\mathbb{R}^n)$ and $\varepsilon\in(2n+2,\infty)$, by Lemma 2.1(ii), we know that there exists some $r\in(1,\infty)$ such that $w\in RH_r(\mathbb{R}^n)$. Moreover, by this, the Hölder inequality, Lemma 2.2 and the definition of m, we have

$$\begin{aligned}|N_{j+1}|&\le\sum_{k=j}^{\infty}\int_{U_k(B)}|m(x)|\,dx\\&\le\sum_{k=j}^{\infty}\Big\{\int_{U_k(B)}[w(x)]^{-1}\,dx\Big\}^{1/2}\Big[\int_{U_k(B)}|m(x)|^2w(x)\,dx\Big]^{1/2}\\&\lesssim\sum_{k=j}^{\infty}\frac{|2^kB|}{[w(2^kB)]^{1/2}}\|m\|_{L^2(w,U_k(B))}\\&\lesssim\frac{|2^jB|}{[w(2^jB)]^{1/2}}[w(2^jB)]^{1/2-1/p}2^{-j(\varepsilon-2n/p)}\sum_{k=j}^{\infty}2^{-(k-j)[\varepsilon-\frac{n}{2}(3+\frac{1}{r})]}\\&\lesssim 2^{-j(\varepsilon-2n/p)}|2^jB|[w(2^jB)]^{-1/p},\end{aligned}$$

which, together with (4.3) and Lemma 2.2, implies that

$$\begin{aligned}\|P_j\|_{L^2(w,\mathbb{R}^n)}&\lesssim 2^{-j(\varepsilon-2n/p)}|2^jB|[w(2^jB)]^{-1/p}\Big\{\frac{[w(U_{j+1}(B))]^{1/2}}{|U_{j+1}(B)|}+\frac{[w(U_j(B))]^{1/2}}{|U_j(B)|}\Big\}\\&\lesssim 2^{-j(\varepsilon-2n/p)}[w(2^{j+1}B)]^{1/2-1/p}.\end{aligned}$$

Hence, $2^{j(\varepsilon-2n/p)}P_j$ is a $(p,2,0)_w$-atom associated with the ball $2^{j+1}B$, up to a harmless constant multiple. By (4.2), we have

$$\|m\|_{H^{p,2,0}_w(\mathbb{R}^n)}\lesssim\Big(\sum_{j=0}^{\infty}2^{-pj\varepsilon}\Big)^{1/p}\lesssim 1,$$

which completes the proof of Proposition 4.7. ∎

Using Proposition 4.7, we now prove Theorem 1.6.

Proof of Theorem 1.6 Suppose that m is a $(p,2,M,\epsilon)_{L_w}$-molecule associated with a ball $B\equiv B(x_B,r_B)$ with $x_B\in\mathbb{R}^n$ and $r_B\in(0,\infty)$, and $\varepsilon\in(2n,\infty)$. We first show that $\nabla L_w^{-1/2}(m)$ is a $(p,2,\varepsilon)_w$-molecule associated with B.

By the boundedness of $\nabla L_w^{-1/2}$ in $L^2(w,\mathbb{R}^n)$ (see [13, Theorem 1.1]), together with Definition 3.1 and Lemma 2.2, we conclude that for $j \in \{0,1,\dots,10\}$,

$$\|\nabla L_w^{-1/2}(m)\|_{L^2(w,U_j(B))} \le \|\nabla L_w^{-1/2}(m)\|_{L^2(w,\mathbb{R}^n)} \lesssim \|m\|_{L^2(w,\mathbb{R}^n)} \lesssim [w(B)]^{1/2-1/p} \lesssim 2^{-j\varepsilon}[w(2^jB)]^{1/2-1/p}.$$

For $j \ge 11$, let $W_j(B) := (2^{j+3}B) \setminus (2^{j-3}B)$ and $E_j(B) := [W_j(B)]^{\complement}$. Write

$$\begin{aligned}\|\nabla L_w^{-1/2}(m)\|_{L^2(w,U_j(B))} &\le \|\nabla L_w^{-1/2}(I-e^{-r_B^2L_w})^M(m)\|_{L^2(w,U_j(B))}\\ &\quad + \|\nabla L_w^{-1/2}(I-[I-e^{-r_B^2L_w}]^M)(m)\|_{L^2(w,U_j(B))}\\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

From Proposition 4.1 and the boundedness of $\nabla L_w^{-1/2}$ in $L^2(w,\mathbb{R}^n)$, together with Definition 3.1 and Lemma 2.2, it follows that

$$\begin{aligned}\mathrm{I}_1 &\le \|\nabla L_w^{-1/2}(I-e^{-r_B^2L_w})^M(m\chi_{W_j(B)})\|_{L^2(w,U_j(B))}\\ &\quad + \|\nabla L_w^{-1/2}(I-e^{-r_B^2L_w})^M(m\chi_{E_j(B)})\|_{L^2(w,U_j(B))}\\ &\lesssim \|m\|_{L^2(w,W_j(B))} + \Big(\frac{r_B^2}{2^{2j}r_B^2}\Big)^M \|m\|_{L^2(w,E_j(B))}\\ &\lesssim 2^{-j\varepsilon}[w(2^jB)]^{1/2-1/p} + 2^{-2jM}[w(B)]^{1/2-1/p}\\ &\lesssim \Big\{2^{-j\varepsilon} + 2^{-2j[M-n(1/p-1/2)]}\Big\}[w(2^jB)]^{1/2-1/p} \lesssim 2^{-j\varepsilon}[w(2^jB)]^{1/2-1/p},\end{aligned}$$

where $0 < \varepsilon/2 \le M - n(1/p-1/2)$.

Similar to the estimate for I_1, by Proposition 4.1, the boundedness of $\nabla L_w^{-1/2}$ in $L^2(w,\mathbb{R}^n)$, Definition 3.1, and Lemma 2.2, we see that

$$\begin{aligned}\mathrm{I}_2 &\lesssim \sup_{1\le k\le M}\Big\|\nabla L_w^{-1/2}\Big[\Big(\frac{kr_B^2L_w}{M}\Big)e^{-\frac{kr_B^2L_w}{M}}\Big]^M\Big(\chi_{W_j(B)}(r_B^2L_w)^{-M}(m)\Big)\Big\|_{L^2(w,U_j(B))}\\ &\quad + \sup_{1\le k\le M}\Big\|\nabla L_w^{-1/2}\Big[\Big(\frac{kr_B^2L_w}{M}\Big)e^{-\frac{kr_B^2L_w}{M}}\Big]^M\Big(\chi_{E_j(B)}(r_B^2L_w)^{-M}(m)\Big)\Big\|_{L^2(w,U_j(B))}\\ &\lesssim \|(r_B^2L_w)^{-M}(m)\|_{L^2(w,W_j(B))} + \Big(\frac{r_B^2}{2^{2j}r_B^2}\Big)^M\big\|(r_B^2L_w)^{-M}(m)\big\|_{L^2(w,E_j(B))}\\ &\lesssim 2^{-j\varepsilon}[w(2^jB)]^{1/2-1/p}.\end{aligned}$$

Since $w \in A_2(\mathbb{R}^n)$ and $\varepsilon \in (n,\infty)$, combining the above estimates for I_1 and I_2, and using the Hölder inequality we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla L_w^{-1/2}(m)(x)|\,dx &= \sum_{j=0}^\infty \int_{U_j(B)}|\nabla L_w^{-1/2}(m)(x)|\,dx\\ &\le \sum_{j=0}^\infty\Big[\int_{2^jB}\frac{1}{w(x)}\,dx\Big]^{1/2}\|\nabla L_w^{-1/2}(m)\|_{L^2(w,U_j(B))}\\ &\lesssim \sum_{j=0}^\infty |2^jB|[w(2^jB)]^{-1/2}2^{-j\varepsilon}[w(2^jB)]^{1/2-1/p}\end{aligned}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-j(\varepsilon-n)}|B|[w(B)]^{-1/p} \lesssim |B|[w(B)]^{-1/p},$$

which further implies that $\nabla L_w^{-1/2}(m) \in L^1(\mathbb{R}^n)$.

For $j \in \{0,1,\dots,10\}$, using the facts that $L_w^{-1/2}(m) = \int_0^\infty e^{-s^2 L_w}(m)\,ds$ and $(tL_w)^k e^{-tL_w}$ is bounded on $L^2(w,\mathbb{R}^n)$ for every $k \in \mathbb{Z}_+$, together with Definition 3.1, we see that

$$\begin{aligned}\|L_w^{-1/2}(m)\|_{L^2(w,U_j(B))} &\le \int_0^\infty \|e^{-s^2 L_w}(m)\|_{L^2(w,U_j(B))}\,ds\\ &\le \Big\{\int_0^{r_B} + \int_{r_B}^\infty\Big\}\|e^{-s^2 L_w}(m)\|_{L^2(w,U_j(B))}\,ds\\ &\lesssim r_B\|m\|_{L^2(w,\mathbb{R}^n)} + \int_{r_B}^\infty s^{-2}\|s^2 L_w e^{-s^2 L_w}(L_w^{-1}m)\|_{L^2(w,\mathbb{R}^n)}\,ds\\ &\lesssim r_B\|m\|_{L^2(w,\mathbb{R}^n)} + r_B^{-1}\|L_w^{-1}(m)\|_{L^2(w,\mathbb{R}^n)} \lesssim r_B[w(B)]^{1/2-1/p}.\end{aligned}$$

From this, the fact that $w \in A_2(\mathbb{R}^n)$, and the Hölder inequality, we deduce that

$$\begin{aligned}\|L_w^{-1/2}(m)\|_{L^1(U_j(B))} &\le \Big[\int_{2^jB}\frac{1}{w(x)}\,dx\Big]^{1/2}\|L_w^{-1/2}(m)\|_{L^2(w,U_j(B))}\\ &\lesssim r_B\frac{|2^jB|}{[w(2^jB)]^{1/2}}[w(B)]^{1/2-1/p} \lesssim r_B|B|[w(B)]^{-1/p}.\end{aligned}\tag{4.4}$$

For $j \ge 11$, let $W_j(B) = (2^{j+3}B)\setminus(2^{j-3}B)$ and $E_j(B) = [W_j(B)]^\complement$. By the Hölder inequality, we have

$$\begin{aligned}&\|L_w^{-1/2}(m)\|_{L^1(U_j(B))}\\ &\quad\le \Big[\int_{2^jB}\frac{1}{w(x)}\,dx\Big]^{1/2}\|L_w^{-1/2}(I-e^{-r_B^2 L_w})^M(m)\|_{L^2(w,U_j(B))}\\ &\qquad + \Big[\int_{2^jB}\frac{1}{w(x)}\,dx\Big]^{1/2}\|L_w^{-1/2}[I-(I-e^{-r_B^2 L_w})^M](m)\|_{L^2(w,U_j(B))} =: \mathrm{J}_1 + \mathrm{J}_2.\end{aligned}$$

By Lemma 2.2, we see that there exists some $r \in (1,\infty)$ such that $w \in RH_r(\mathbb{R}^n)$. This, together with Proposition 4.2 and Definition 3.1, implies that

$$\begin{aligned}\mathrm{J}_1 &\le \Big[\int_{2^jB}\frac{1}{w(x)}\,dx\Big]^{1/2}\Big\{\|L_w^{-1/2}(I-e^{-r_B^2 L_w})^M(\chi_{W_j(B)}m)\|_{L^2(w,U_j(B))}\\ &\qquad + \|L_w^{-1/2}(I-e^{-r_B^2 L_w})^M(\chi_{E_j(B)}m)\|_{L^2(w,U_j(B))}\Big\}\\ &\lesssim \frac{|2^jB|}{[w(2^jB)]^{1/2}}\Big\{r_B\|m\|_{L^2(w,W_j(B))} + r_B\Big(\frac{r_B^2}{2^{2j}r_B^2}\Big)^{M-\frac12}\|m\|_{L^2(w,E_j(B))}\Big\}\\ &\lesssim \Big\{2^{-j[\varepsilon-n(\frac{3r+1}{2r})]} + 2^{-j[2M-1-n(\frac{r+1}{2r})]}\Big\}r_B|B|[w(B)]^{-1/p}\end{aligned}$$

and

$$\begin{aligned}\mathrm{J}_2 &\lesssim \Big[\int_{2^jB}\frac{1}{w(x)}\,dx\Big]^{1/2}\sup_{1\le k\le M}\Big\|L_w^{-1/2}\Big[\Big(\frac{kr_B^2L_w}{M}\Big)e^{-\frac{kr_B^2L_w}{M}}\Big]^M\\ &\qquad\times[(\chi_{W_j(B)}+\chi_{E_j(B)})(r_B^2L_w)^{-M}(m)]\Big\|_{L^2(w,U_j(B))}\end{aligned}$$

$$\lesssim \frac{|2^jB|}{[w(2^jB)]^{1/2}}\Big\{ r_B\|(r_B^2L_w)^{-M}(m)\|_{L^2(w,W_j(B))} + r_B\Big(\frac{r_B^2}{2^{2j}r_B^2}\Big)^{M-\frac12}\|(r_B^2L_w)^{-M}(m)\|_{L^2(w,E_j(B))}\Big\}$$
$$\lesssim \Big\{2^{-j[\varepsilon-n(\frac{3r+1}{2r})]}+2^{-j[2M-1-n(\frac{r+1}{2r})]}\Big\} r_B|B|[w(B)]^{-1/p},$$

which, together with (4.4) and $\varepsilon\in(2n,\infty)$, further implies that $L_w^{-1/2}(m)\in L^1(\mathbb{R}^n)$.

Next, we prove that $\int_{\mathbb{R}^n}\nabla L_w^{-1/2}(m)(x)\,dx = 0$. From [34, Theorem 8.1], it follows that $D(L_w^{1/2}) = D(\mathfrak{a})$, where $D(\mathfrak{a})\subset H_0^1(w,\mathbb{R}^n)$ is the domain of the sesquilinear form (1.1) associated with L_w, which implies that $R(L_w^{-1/2})\subset H_0^1(w,\mathbb{R}^n)$, where $R(L_w^{-1/2})$ denotes the range of $L_w^{-1/2}$.

We now choose $\{\phi_j\}_{j=1}^\infty\subset C_c^\infty(\mathbb{R}^n)$ such that

(a) $\sum_{j=1}^\infty\phi_j(x)=1$ for almost everywhere $x\in\mathbb{R}^n$;
(b) for each $j\in\mathbb{Z}_+$, there exists a ball $B_j\subset\mathbb{R}^n$ such that $\operatorname{supp}\phi_j\subset 2B_j$, $\phi_j=1$ on B_j and $0\le\phi_j\le 1$;
(c) there exists a positive constant C_ϕ such that for all $j\in\mathbb{N}$ and $x\in\mathbb{R}^n$, $|\nabla\phi_j(x)|\le C_\phi$;
(d) there exists $N_\phi\in\mathbb{N}$ such that $\sum_{k=1}^\infty\chi_{2B_j}\le N_\phi$.

For all $j\in\mathbb{N}$, let $\eta_j\in C_c(\mathbb{R}^n)$ such that $\eta_j=1$ on $2B_j$ and $\operatorname{supp}\eta_j\subset 4B_j$. Since $R(L_w^{-1/2})\subset H^1(w,\mathbb{R}^n)$ and $\nabla L_w^{-1/2}(m)\in L^1(\mathbb{R}^n)$, from the properties of $\{\phi_j\}_j$, the facts that $L_w^{-1/2}(m)$, $\nabla L_w^{-1/2}(m)\in L^1(\mathbb{R}^n)$, and integration by parts, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n}\nabla L_w^{-1/2}(m)(x)\,dx &= \int_{\mathbb{R}^n}\nabla\Big(\Big[\sum_{j=1}^\infty\phi_j\Big]L_w^{-1/2}(m)\Big)(x)\,dx\\ &=\sum_{j=1}^\infty\int_{\mathbb{R}^n}\nabla(\phi_jL_w^{-1/2}(m))(x)\,dx\\ &=\sum_{j=1}^\infty\int_{\mathbb{R}^n}\eta_j(x)\nabla(\phi_jL_w^{-1/2}(m))(x)\,dx\\ &=-\sum_{j=1}^\infty\int_{\mathbb{R}^n}\nabla\eta_j(x)\phi_j(x)L_w^{-1/2}(m)(x)\,dx=0.\end{aligned}$$

By the above arguments, we see that $\nabla L_w^{-1/2}(m)$ is a $(p,2,\varepsilon)_w$-molecule, associated with B, up to a positive constant multiple.

Now, suppose that $f\in\mathbb{H}^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)$. By the definition of $\mathbb{H}^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)$, there exist a family $\{m_j\}_{j=1}^\infty$ of $(p,2,M,\epsilon)_{L_w}$-molecules and numbers $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ such that

$$\|f\|_{H^{p,2,M}_{L_w,\mathrm{mol}}(\mathbb{R}^n)}\sim\Big(\sum_{j=1}^\infty|\lambda_j|^p\Big)^{1/p}.$$

For each $(p,2,M,\epsilon)_{L_w}$-molecule m_j, by the above arguments, we see that $\nabla L_w^{-1/2}(m_j)$ is a $(p,2,\varepsilon)_w$-molecule up to a positive constant multiple. Moreover, by Proposition 4.7, we know that there exist $\{\Lambda_{j,k}\}_{k=1}^\infty\subset\mathbb{C}$ and a family $\{\alpha_k\}_{k=1}^\infty$ of $(p,2,0)_w$-atoms

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with a harmless constant multiple such that

$$\nabla L_w^{-1/2}(m_j) = \sum_{k=1}^{\infty} \Lambda_{j,k}\alpha_k \text{ in } L^2(w, \mathbb{R}^n)$$

and

$$\|\nabla L_w^{-1/2}(m_j)\|_{H_w^{p,2,0}(\mathbb{R}^n)} \le \Big(\sum_{k=1}^{\infty} |\Lambda_{j,k}|^p\Big)^{1/p} \le C,$$

where C is a positive constant independent of j. By the boundedness of $\nabla L_w^{-1/2}$ in $L^2(w, \mathbb{R}^n)$, we know that

$$\nabla L_w^{-1/2}(f) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \lambda_j \Lambda_{j,k}\alpha_k$$

in $L^2(w, \mathbb{R}^n)$. Hence, from the definition of $H_w^{p,2,0}(\mathbb{R}^n)$, we deduce that

$$\begin{aligned}\|\nabla L_w^{-1/2}(f)\|_{H_w^{p,2,0}(\mathbb{R}^n)} &\le \Big[\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\lambda_j|^p |\Lambda_{j,k}|^p\Big]^{1/p} \\ &\lesssim \Big[\sum_{j=1}^{\infty} |\lambda_j|^p\Big]^{1/p} \sim \|f\|_{H_{L_w,\mathrm{mol}}^{p,2,M}}(\mathbb{R}^n).\end{aligned}$$

Then by a standard argument we see that $\nabla L_w^{-1/2}$ extends to a bounded linear operator from $H_{L_w,\mathrm{mol}}^{p,2,M}(\mathbb{R}^n)$ to $H_w^{p,2,0}(\mathbb{R}^n)$. This, together with Lemma 4.5, finishes the proof of Theorem 1.6. ■

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