

Representations of metabelian groups satisfying the minimal condition for normal subgroups

Howard L. Silcock

A question of John S. Wilson concerning indecomposable representations of metabelian groups satisfying the minimal condition for normal subgroups is answered negatively, by means of an example. It is shown that such representations need not be irreducible, even when the group being represented is an extension of an elementary abelian p -group by a quasicyclic q -group of the type first described by V.S. Čarin, and the characteristic of the field is a prime distinct from both p and q . This implies that certain techniques used in the study of metabelian groups satisfying the minimal condition for normal subgroups are not available for the corresponding class of soluble groups of derived length 3.

1.

In this paper we answer a question raised by Wilson in his talk [12] at the Mini-Conference on Group Theory at the Australian National University in November 1975.

Wilson's question arises in the context of a more general question, whether a soluble group that satisfies $\text{min-}n$ (the minimal condition for normal subgroups) is necessarily countable. The latter question remains open [see note added in proof], though McDougall [6] has dealt with an important case by proving that all metabelian groups satisfying $\text{min-}n$ are

Received 16 December 1975.

countable (see also [9]). By a result of Wilson [11], it is sufficient for the countability question to consider soluble groups satisfying $\text{min-}n$ that are also $\underline{\mathbb{F}}$ -perfect: that is, have no proper subgroups of finite index. Further, it is easy to see that every group satisfying $\text{min-}n$ is a subdirect product of a finite number of monolithic groups satisfying $\text{min-}n$. (A group is *monolithic* if its non-trivial normal subgroups have non-trivial intersection.) Therefore, when it is convenient, we may restrict our attention further by considering only monolithic groups.

Detailed information about the structure of metabelian groups satisfying $\text{min-}n$ has been obtained by Hartley and McDougall [2]. Indeed these authors have completely classified the $\underline{\mathbb{F}}$ -perfect metabelian groups satisfying $\text{min-}n$. Their results imply, in particular, that a metabelian group which satisfies $\text{min-}n$ and is in addition both $\underline{\mathbb{F}}$ -perfect and monolithic is very like one of the "standard" examples of metabelian groups satisfying $\text{min-}n$ constructed by Čarin [1]. For each pair of distinct primes p and q , Čarin's construction gives a group satisfying $\text{min-}n$ which is an extension of an infinite elementary abelian p -group by an abelian group of type q^∞ . Following Wilson, we denote this group by $C(p, q^\infty)$.

Now suppose G is a soluble group of derived length 3 which satisfies $\text{min-}n$ and is both $\underline{\mathbb{F}}$ -perfect and monolithic, and write $A = G''$. Then G/A is an $\underline{\mathbb{F}}$ -perfect metabelian group satisfying $\text{min-}n$. In attempting to prove that G is countable it is natural, in view of the remarks made above, to consider first the special case where G/A is actually isomorphic to $C(p, q^\infty)$, for some p and q . Now A is periodic (see Theorem 5.25 of [8]) and its non-trivial G -invariant subgroups intersect non-trivially because G is monolithic. Therefore A has only one primary component, and so is an r -group, for some prime r . Further, each of the subgroups

$$A[r^n] = \left\{ a \in A : a^{r^n} = 1 \right\},$$

for $n = 0, 1, 2, \dots$, is G -invariant, and G will be countable if and only if all the factors $A[r^n]/A[r^{n-1}]$ are countable. Thus we may further suppose that A has exponent r .

After making the simplifications just indicated, we may view A as a

representation module for $C(p, q^\infty)$ over the field of r elements Z_r . The problem of describing G thus leads naturally to the problem of describing representations of $C(p, q^\infty)$ over Z_r . This is the subject of Wilson's question, which is part of Problem 4 of his talk [see note added in proof]:

"Study irreducible and indecomposable modules over the fields of r elements for Čarin's group $C(p, q^\infty)$. If r is distinct from p and q , are monolithic modules necessarily irreducible?"

In the analogous situation for metabelian groups, all the monolithic modules which occur are irreducible, by a result of Kovács and Newman [4]. Nevertheless we shall show, by means of an example based on Neumann's example in [4], that Wilson's question has a negative answer. In fact, we shall prove:

THEOREM. *Let p and q be distinct primes. If r is a prime such that $r - 1$ is divisible by p , then the Čarin group $C(p, q^\infty)$ has a monolithic representation module over Z_r that is artinian but not irreducible.*

2.

In our proof of the theorem we shall need to refer to a property of the Čarin group $C(p, q^\infty)$ that does not seem to have been noted in the literature. To explain this property we first introduce some notation.

We assume that distinct primes p and q have been chosen once and for all, and we let F be the field associated with $C(p, q^\infty)$. Thus F is the field obtained from Z_p (the field with p elements) by adjoining a primitive q^n -th root of unity, ω_n say, for each positive integer n . This field is the union of a tower of finite subfields

$$F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots,$$

where $F_n = Z_p(\omega_n)$, for each n . We write Ω for the subgroup of the multiplicative group of F generated by $\omega_1, \omega_2, \dots$; and we set

$\Omega_n = \langle \omega_n \rangle$, for $n = 1, 2, 3, \dots$.

LEMMA 1. *There is a positive integer d such that, for each integer*

$n \geq d$, the elements of any transversal to Ω_n in Ω form a basis for F as a vector space over F_n .

Proof. Let n be a positive integer and let Γ be a transversal to Ω_n in Ω . Since F is generated as a field by Ω , it is clear that the elements of Γ span F as a vector space over F_n . As a set is linearly independent if and only if every finite subset is linearly independent, it is sufficient to prove that, if n is sufficiently large, the set $\Gamma \cap \Omega_m$ is linearly independent over F_n for every $m > n$.

Now $\Gamma \cap \Omega_m$ is a transversal to Ω_n in Ω_m , so it contains $|\Omega_m/\Omega_n| = q^{m-n}$ elements. These elements span F_m as a vector space over F_n , so to show they are linearly independent it will be enough to prove that $|F_m : F_n| = q^{m-n}$.

By its definition, F_m contains a primitive q^m -th root of unity and is the smallest field of characteristic p with this property. So if $F_m = \text{GF}\left(\begin{smallmatrix} f \\ p \end{smallmatrix}^m\right)$, then the order of the multiplicative group of F_m , namely $p^{\frac{f}{m}} - 1$, must be divisible by q^m ; and f_m must be the least positive integer with this property. Thus f_m is the order of p modulo q^m , in the sense of number theory. Similarly f_n is the order of p modulo q^n . As $f_m = |F_m : \mathbb{Z}_p|$ and $f_n = |F_n : \mathbb{Z}_p|$, we have

$$|F_m : F_n| = f_m/f_n.$$

The result is therefore a consequence of the following simple number-theoretic lemma.

LEMMA 2. Let p and q be distinct primes, and let f_n be the order of p modulo q^n , for $n = 1, 2, 3, \dots$. Let d be the positive integer determined by the following conditions:

(i) if q is odd, or if $q = 2$ and $p \equiv 1 \pmod{4}$, then q^d is the largest power of q dividing $p^{f_1} - 1$;

(ii) if $q = 2$ and $p \equiv 3 \pmod{4}$, then 2^d is the largest power of 2 dividing $p^2 - 1$.

Then, for all integers m, n with $m > n \geq d$, we have

$$f_m / f_n = q^{m-n}.$$

We omit the proof of this lemma, as the result may be deduced easily from Theorem 4-6 of [5] (p. 52).

We now consider the significance of Lemma 1 for the structure of $C(p, q^\infty)$. By definition, $C(p, q^\infty)$ is a split extension of the additive group of F by the multiplicative group Ω defined above. To avoid the confusion caused by the simultaneous use of additive and multiplicative notations, we shall think of the elements of $C(p, q^\infty)$ as ordered pairs

$$\left(\omega_i^k, \alpha \right),$$

where i, k are positive integers, and $\alpha \in F$. These pairs multiply according to the rule

$$(1) \quad \left(\omega_i^k, \alpha \right) \left(\omega_j^l, \beta \right) = \left(\omega_i^k \omega_j^l, \alpha \omega_j^l + \beta \right).$$

Let us now write $b(\alpha) = (1, \alpha)$ and $c_i = (\omega_i, 0)$, for all $\alpha \in F$ and all $i > 0$. Then, from (1), we have

$$(2) \quad b(\alpha)b(\beta) = b(\alpha + \beta),$$

$$(3) \quad c_i^{-1} b(\alpha) c_i = b(\omega_i \alpha),$$

for all $\alpha, \beta \in F$ and all $i > 0$. The elements $b(\alpha)$, for $\alpha \in F$, form a normal subgroup, B say, of $C(p, q^\infty)$, and the mapping $b(\alpha) \mapsto \alpha$ is an isomorphism from B onto the additive group of F . The subgroup C generated by c_1, c_2, \dots is a complement to B , and the mapping $c_i \mapsto \omega_i$ evidently extends to an isomorphism between C and Ω .

Further, B is a minimal normal subgroup of $BC = C(p, q^\infty)$ (see, for example, §5.2 of [8]).

The following property of BC turns out to be crucial for the construction to be described later.

LEMMA 3. *There is a homomorphism π from B onto a finite direct factor B_0 of B such that, for every $b \in B$, the set*

$$\{c \in C : b^c \pi \neq 1\}$$

is finite.

Proof. Choose a positive integer d satisfying the conditions of Lemma 1, and let Γ be a transversal to Ω_d in Ω containing the identity element. Then Lemma 1 shows that Γ is a basis for F as a vector space over F_d , so F is the direct sum of the subspaces γF_d , as γ ranges over Γ .

Now the mapping $\alpha \mapsto b(\alpha)$ is an isomorphism from the additive group of F onto B , so this direct sum decomposition of F determines a direct product decomposition of B , in which the summand γF_d of F corresponds to the factor

$$\{b(\gamma\alpha) : \alpha \in F_d\}$$

of B . Since Γ contains the identity element, the subgroup

$$B_0 = \{b(\alpha) : \alpha \in F_d\}$$

occurs among these direct factors. Furthermore, if we set $t_\gamma = (\gamma, 0)$ for each $\gamma \in \Gamma$, then using equation (3) above we see that

$${}^t_{B_0}\gamma = \{b(\gamma\alpha) : \alpha \in F_d\}.$$

Consequently we can express B as a direct product

$$(4) \quad B = \text{dr}_{\gamma \in \Gamma} {}^t_{B_0}\gamma.$$

Also, as the generators $c_i = (\omega_i, 0)$ of C transform B according to the mapping $b(\alpha) \mapsto b(\omega_i \alpha)$, the normalizer $N_C(B_0)$ is the finite subgroup $C_0 = \langle c_d \rangle$ of C . But C_0 is the image of Ω_d under the

isomorphism $\omega \mapsto (\omega, 0)$ between C and Ω , and this isomorphism maps Γ onto the set

$$T = \{t_\gamma : \gamma \in \Gamma\}.$$

Therefore T is a transversal to C_0 in C .

Now define π to be the canonical projection of B onto B_0 associated with the direct decomposition (4). We claim that π satisfies the conditions of the lemma. To see this, let $b \in B$. Then the projection of b in the direct factor B_0^t is trivial for all except a finite number of elements $t \in T$. Write T_0 for the finite subset of exceptional elements. Then, for any $c \in C$, the direct factors in which b^c has non-trivial projection are precisely the factors B_0^{tc} , for t ranging over T_0 . In particular, the projection $b^c \pi$ of b^c in B_0 is non-trivial if and only if $B_0^{tc} = B_0$, for some $t \in T_0$. This will be the case if and only if $tc \in N_C(B_0) = C_0$, for some $t \in T_0$. However, both T_0 and C_0 are finite, so there are only finitely many $c \in C$ such that $c \in t^{-1}C_0$ for some $t \in T_0$. Therefore $b^c \pi = 1$ for all but a finite number of $c \in C$, and the result follows.

3.

We now come to the construction required to prove the theorem. We start by defining the *unrestricted treble product*, a variant of the treble product defined in [3].

Let A, B, C be three groups and let

$$\sigma : B \rightarrow \text{aut } A,$$

$$\tau : C \rightarrow \text{aut } B$$

be homomorphisms (where we write $\text{aut } X$ for the automorphism group of a group X). Using the homomorphism τ , we first form the semi-direct product $H = BC$. Now write P for the set of all functions from C into A . Under the usual pointwise operations, P is a group. We make H act

as a group of operators on P as follows. If $f \in P$ and $h = bc_1 \in H$, where $b \in B$ and $c_1 \in C$, then we define f^h by

$$(5) \quad f^h(c) = f\left(cc_1^{-1}\right)^{b^{c^{-1}}\sigma},$$

for all $c \in C$. It is a routine matter to check that the mapping $f \mapsto f^h$ is an automorphism of P for each $h \in H$, and that the mapping which takes h to the automorphism $f \mapsto f^h$ is a homomorphism from H into $\text{aut } P$. We define the unrestricted treble product of A, B, C (with associated homomorphisms σ, τ) to be the semi-direct product $K = PH$, formed according to the action just described.

If Q denotes the subgroup of P consisting of all functions from C into A with finite support, then Q is easily seen to be invariant under the action of H . The subgroup $T = QH$ of K is just the treble product

$$\text{tr}(A, B, C; \sigma, \tau)$$

defined in [3]. As usual, we may regard A as a subgroup of K by identifying each element $a \in A$ with the function $\hat{a} : C \rightarrow A$ defined by

$$\begin{aligned} \hat{a}(c) &= 1, \text{ if } c \neq 1, \\ \hat{a}(1) &= a. \end{aligned}$$

When this identification is made, it follows from (5) that A is invariant under the action of B , and that AB is isomorphic to the semi-direct product of A and B formed using σ .

In the terminology of Neumann [7], K is an unrestricted twisted wreath product of A and H , while T is the corresponding restricted twisted wreath product.

We now consider the subgroup $M = \langle Q, D \rangle$ of P obtained by adjoining to Q the subgroup D of all constant functions from C into A . If C is infinite, the elements of D have infinite support, so that $D \cap Q = 1$ and hence $M = D \times Q$. In general, M need not be invariant under the action of H . However, we now show that if the homomorphisms σ and τ are subject to certain conditions M will be invariant.

LEMMA. 4. *If σ and τ are chosen so that, for each $b \in B$, the*

set $\{c \in C : b^c \sigma \neq 1\}$ is finite, then M is invariant under the action of H . Furthermore, we have

$$[M, H] \leq Q$$

in this case.

Proof. As we know that Q is H -invariant, it will be sufficient to prove that the conditions on σ and τ imply that $[D, H] \leq Q$.

Let $f \in D$; then f is a constant function, say $f(c) = a$, for all $c \in C$. If $h \in H$, then we can write $h = bc_1$ for suitable $b \in B$ and $c_1 \in C$, and then

$$f^h(c) = f\left(cc_1^{-1}\right)b^{c^{-1}\sigma} = a^{b^{c^{-1}\sigma}},$$

for all $c \in C$. By hypothesis, $b^{c^{-1}\sigma} = 1$ for all but a finite number of c , so f and f^h agree except on a finite subset of C . Therefore $[f, h] = f^{-1}f^h$ has finite support. Hence $[D, H] \leq Q$, and the result follows.

Now let p, q, r be primes satisfying the conditions of the theorem. We specialize the construction just described, taking B and C to have the meanings ascribed to them in §2, and choosing $\tau : C \rightarrow \text{aut } B$ so that the associated semi-direct product $H = BC$ is the Čarin group $C(p, q^\infty)$. For the group A we take a cyclic group of order r . Then $\text{aut } A$ has order $r - 1$, which by hypothesis is divisible by p ; hence $\text{aut } A$ has a subgroup of order p .

Now let $\pi : B \rightarrow B_0$ be the homomorphism described in Lemma 3. As B_0 is a non-trivial p -group, we can choose a non-trivial homomorphism $\sigma_0 : B_0 \rightarrow \text{aut } A$. We take σ to be the homomorphism from B into $\text{aut } A$ obtained by composing π and σ_0 .

With this choice of σ and τ , the treble product

$$T = \text{tr}(A, B, C; \sigma, \tau)$$

is of the type constructed in §4 of [10]. Therefore, by Lemma 6 of [10],

the normal subgroups of T are well-ordered by inclusion; and, as A has prime order, its normal closure Q is contained in each non-trivial normal subgroup of T .

We now observe that the conditions of Lemma 4 are satisfied. For by our choice of σ we have $\ker \pi \leq \ker \sigma$; hence the set

$\{c \in C : b^c \sigma \neq 1\}$ is contained in $\{c \in C : b^c \pi \neq 1\}$. But the latter is finite by Lemma 3. Hence Lemma 4 is applicable, and we conclude that $M = Q \times D$ is invariant under the action of H .

Now M is isomorphic to a direct power of A , so it is an elementary abelian r -group. Hence M may be viewed in a natural way as a representation module for H over Z_r . From this point of view, the H -invariant subgroups of M are just the submodules of M ; and, as Q is a minimal normal subgroup in $T = QH$, it follows that Q is an irreducible submodule of M . Further, M/Q has order r and is centralized by H , so it is a one-dimensional factor-module of M . Thus M has composition length 2, and so is certainly artinian. To complete the proof of the theorem, we need one more fact concerning M .

LEMMA 5. Q has no H -admissible complement in M .

Proof. Suppose on the contrary that Q has an admissible complement L . Then L is operator-isomorphic to M/Q , so it is a cyclic group of order r and is centralized by H .

Suppose f is a generator of L . Then for some $c \in C$ we have $f(c) \neq 1$. Write $f(c) = a$. Then, as f commutes with c , we have

$$f(c) = f^c(c) = f(1),$$

so that $f(1) = a$. As A has prime order, we have $A = \langle a \rangle$. Now choose $b \in B$ such that $b \notin \ker \sigma$. Then $a^{b\sigma} \neq a$. However, f commutes with b , so

$$a = f(1) = f^b(1) = f(1)^b = a^b.$$

This contradiction completes the proof.

From this lemma it follows at once that Q is contained in every non-trivial submodule of M . Consequently M is a monolithic representation module for H ; further M is artinian and has a proper submodule Q .

Hence the theorem is established.

Note added in proof, 5 March 1976. After this paper was accepted for publication the author learnt that Dr B. Hartley has independently obtained much more general results on the problems discussed here, in a forthcoming paper. In particular he has shown that soluble groups satisfying $\text{min-}n$ can be uncountable.

The problem of Wilson's quoted in §1 was taken from a draft version of [12]: however, the author understands that this version is now to be modified to take account of Hartley's results, and that the published version of [12] may no longer include this problem in the form stated here.

References

- [1] В.С. Чарин [V.S. Čarin], "Замечание об условии минимальности для подгрупп" [A remark on the minimal condition for subgroups], *Dokl. Akad. Nauk SSSR (N.S.)* 66 (1949), 575-576.
- [2] B. Hartley and D. McDougall, "Injective modules and soluble groups satisfying the minimal condition for normal subgroups", *Bull. Austral. Math. Soc.* 4 (1971), 113-135.
- [3] H. Heineken and J.S. Wilson, "Locally soluble groups with $\text{min-}n$ ", *J. Austral. Math. Soc.* 17 (1974), 305-318.
- [4] L.G. Kovács and M.F. Newman, "Direct complementation in groups with operators", *Arch. Math. (Basel)* 13 (1962), 427-433.
- [5] William Judson LeVeque, *Topics in number theory*, Vol. I (Addison-Wesley, Reading, Massachusetts, 1956).
- [6] David McDougall, "Soluble groups with the minimum condition for normal subgroups", *Math. Z.* 118 (1970), 157-167.
- [7] B.H. Neumann, "Twisted wreath products of groups", *Arch. Math. (Basel)* 14 (1963), 1-6.
- [8] Derek J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 1 (Ergebnisse der Mathematik und ihrer Grenzgebiete, 62. Springer-Verlag, Berlin, Heidelberg, New York, 1972).

- [9] Howard L. Silcock, "Metanilpotent groups satisfying the minimal condition for normal subgroups", *Math. Z.* 135 (1974), 165-173.
- [10] Howard L. Silcock, "On the construction of soluble groups satisfying the minimal condition for normal subgroups", *Bull. Austral. Math. Soc.* 12 (1975), 231-257.
- [11] John S. Wilson, "Some properties of groups inherited by normal subgroups of finite index", *Math. Z.* 114 (1970), 19-21.
- [12] John S. Wilson, "Locally soluble groups satisfying the minimal condition for normal subgroups", *Proc. Mini-Conference Group Theory*, Australian National University, Canberra, 1975 (Lecture Notes in Mathematics, to appear).

Department of Pure Mathematics,
University of Adelaide,
Adelaide,
South Australia.