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A NOTE ON 2-DISTRIBUTORS

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Abstract

We construct an associative tensor product for 2-distributors by means of cartesian coends and we prove the usual adjointness property for a suitable Hom.

Let $\mathscr{A}$, $\mathscr{B}$ be 2-categories.

A 2-*distributor* Bozapalides (1975), or *pro*–2–*functor* Gray (1969), from $\mathscr{A}$ to $\mathscr{B}$ is a 2–functor of the form

$$\phi : \mathscr{B}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{Cat};$$

we write $\phi : \mathscr{A} \to \mathscr{B}$.

A 1–*cell* from $\phi : \mathscr{A} \to \mathscr{B}$ to $\psi : \mathscr{A} \to \mathscr{B}$ is a quasi-natural transformation of 2–functors Gray (1974)

$$m : \phi \to \psi : \mathscr{B}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{Cat}.$$

A 2–*cell* from $m : \phi \to \psi$ to $m' : \phi \to \psi$ is a modification of quasi-natural transformations Gray (1974)

$$a : m \Rightarrow m' : \phi \to \psi : \mathscr{B}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{Cat}.$$

We denote by $\mathrm{Dist}(\mathscr{A}, \mathscr{B})$ the 2–category so defined, i.e. — adopting Gray's notation —

$$\mathrm{Dist}(\mathscr{A}, \mathscr{B}) = \mathrm{Fun}(\mathscr{B}^{\mathrm{op}} \times \mathscr{A}, \mathrm{Cat}).$$

The theory of cartesian (co)ends developed in Bozapalides (1975) gives us a means of composing 2-distributors.

More precisely, if

$$\mathscr{A} \xrightarrow{\phi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

is a pair of 2-distributors, then their composition

$$\psi \otimes \phi : \mathscr{A} \to \mathscr{C}$$

is defined by the formula

$$\text{(i)} \qquad (\psi \otimes \phi)(C, A) = \text{cart} - \int^{B} \psi(C, B) \times \phi(B, A)$$

where the right member of (i) denotes the cartesian coend of the 2-functor

$$\psi(C, -) \times \phi(-, A) \colon \mathscr{B}^{\text{op}} \times \mathscr{B} \to \text{Cat}.$$

It should be observed that in the one-dimensional case the above tensor product coincides with the tensor product of ordinary distributors Bènabou (1973).

For $\Omega \colon \mathscr{C} \to \mathscr{D}$ we have

$$\begin{aligned}
(\Omega \otimes \psi) \otimes \phi(D, A) &= \text{cart} - \int^{B} (\Omega \otimes \psi)(D, B) \times \phi(B, A) \\
&= \text{cart} - \int^{B} \left(\text{cart} - \int^{C} \Omega(D, C) \times \psi(C, B)) \times \phi(B, A)\right) \\
\text{(1)} \qquad &\simeq \text{cart} - \int^{B} \text{cart} - \int^{C} (\Omega(D, C) \times \psi(C, B) \times \phi(B, A)) \\
\text{(2)} \qquad &= \text{cart} - \int^{C} \left(\Omega(D, C) \times \text{cart} - \int^{B} \psi(C, B) \times \phi(B, A)\right) \\
&= \text{cart} - \int^{C} \Omega(D, C) \times (\psi \otimes \phi)(C, A) \\
&= \Omega \otimes (\psi \otimes \phi)(C, A),
\end{aligned}$$

where (1) and (2) are combinations of Fubini's formula and the fact that the product of two categories is a cartesian coend in Cat [see Bozapalides (1975)].

So this tensor product is associative.

Moreover it has a closure property in the following sense: For every diagram of 2-distributors

$$\mathscr{B} \xleftarrow{\phi} \mathscr{A} \xrightarrow{\psi} \mathscr{C}$$

there is a 2-distributor

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$$[\phi, \psi]: \mathscr{B} \to \mathscr{C}$$

such that

$$\text{Dist}(\mathscr{A}, \mathscr{C})(\theta \otimes \phi, \psi) \simeq \text{Dist}(\mathscr{B}, \mathscr{C})(\theta, [\phi, \psi])$$

naturally in $\theta: \mathscr{B} \to \mathscr{C}$.

$[\phi, \psi]$ is defined by

$$\text{(ii)} \qquad [\phi, \psi](C, B) = \text{cart} - \int_A \psi(C, A)^{\phi(B,A)}$$

where $M^N = \text{Cat}(M, N)$ and $\text{cart} - \int_A$ this time denotes the cartesian end of the 2-functor

$$\psi(C, -)^{\phi(B,-)}: \mathscr{A}^{\text{op}} \times \mathscr{A} \to \text{Cat}.$$

In fact, we have

$$\text{Dist}(\mathscr{A}, \mathscr{C})(\theta \otimes \phi, \psi) = \text{Fund}(\mathscr{C}^{\text{op}} \times \mathscr{A}, \text{Cat})(\theta \otimes \phi, \psi) \simeq$$

$$\text{(1)} \qquad \simeq \text{cart} - \int_{(C,A)} \psi(C, A)^{(\theta \otimes \phi)(C,A)}$$

$$\text{(2)} \qquad \simeq \text{cart} - \int_{(C,A)} \psi(C, A)^{\text{cart} - \int^B \theta(C,B) \times \phi(B,A)}$$

$$\text{(3)} \qquad \simeq \text{cart} - \int_{(C,A)} \text{cart} - \int_B \psi(C, A)^{\theta(C,B) \times \phi(B,A)}$$

$$\text{(4)} \qquad \simeq \text{cart} - \int_{(C,B)} \text{cart} - \int_A (\psi(C, A)^{\phi(B,A)})^{\theta(C,B)}$$

$$\text{(5)} \qquad \simeq \text{cart} - \int_{(C,B)} \left(\text{cart} - \int_A \psi(C, A)^{\phi(B,A)}\right)^{\theta(C,B)}$$

$$\text{(6)} \qquad \simeq \text{cart} - \int_{(C,B)} [\phi, \psi](C, B)^{\theta(C,B)}$$

$$\text{(7)} \qquad \simeq \text{Fun}(\mathscr{C}^{\text{op}} \times \mathscr{B}, \text{Cat})(\theta, [\phi, \psi])$$

$$= \text{Dist}(\mathscr{B}, \mathscr{C})(\theta, [\phi, \psi]),$$

where:

(1) and (7) result from Bozapalides (1975 example b);

(2) we use the relation (i);

(3) the representable 2-functors

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$$\psi(C, A)^{(-)}: \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}$$

commute with cart $-\int_B$;
(4) is the Fubini formula

$$\int_{(C,A)} \int_B \simeq \int_{(C,B,A)} \simeq \int_{(C,B)} \int_A,$$

and the classical adjunction in Cat,

$$(M^N)^K \simeq M^{N\times K};$$

(5) the representable 2-functors

$$(-)^{\theta(C,B)}: \mathrm{Cat} \to \mathrm{Cat}$$

commute with the cart-$\int_A$;
(6) we use (ii).

If we have the diagram

$$\mathscr{A} \xrightarrow{\phi} \mathscr{B} \xleftarrow{\psi} \mathscr{C}$$

then the 2-distributor $]\phi, \psi[: \mathscr{C} \to \mathscr{A}$, defined by

$$]\phi, \psi[(A, C) = \mathrm{cart} - \int_B \psi(B, C)^{\phi(B,A)}$$

is such that

$$\mathrm{Dist}(\mathscr{C}, \mathscr{A})(\theta,]\phi, \psi[) \simeq \mathrm{Dist}(\mathscr{C}, \mathscr{B})(\phi \otimes \theta, \psi)$$

for every $\theta: \mathscr{C} \to \mathscr{A}$.

Remarks. 1) In the situations

$$\mathbf{1} \xrightarrow{\phi} \mathscr{B} \xleftarrow{\psi} \mathbf{1}, \ \mathbf{1} \xleftarrow{\phi} \mathscr{A} \xrightarrow{\psi} \mathbf{1},$$

$$]\phi, \psi[= \mathrm{Fun}(\mathscr{B}^{\mathrm{op}}, \mathrm{Cat})(\phi, \psi)$$

$$[\phi, \psi] = \mathrm{Fun}(\mathscr{A}, \mathrm{Cat})(\phi, \psi)$$

2) In the situations

$$\mathscr{A} \xrightarrow{\phi} \mathscr{B} \xleftarrow{\mathscr{B}(-,-)} \mathscr{B}, \ \mathscr{A} \xleftarrow{\mathscr{A}(-,-)} \mathscr{A} \xrightarrow{\phi} \mathscr{B}$$

$]\phi, \mathscr{B}(-,-)[$ and $[\phi, \mathscr{A}(-,-)]$ are denoted by $\check{D}\phi$ and $\hat{D}\phi$ respectively, and we call them the *duals* of ϕ.

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