Canad. Math. Bull. Vol. **56** (2), 2013 pp. 337–343
http://dx.doi.org/10.4153/CMB-2011-183-9

Certain Properties of K_0-monoids Preserved by Tracial Approximation

Qingzhai Fan

Abstract. We show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$: (1) weak comparability, (2) strictly unperforated, (3) strictly cancellative.

1 Introduction

The Elliott program for the classification of amenable C^*-algebras might be said to have begun with the K-theoretical classification of AF-algebras in [2]. Since then, many classes of C^*-algebras have been found to be classified by the Elliott invariant. Among them, one important class is the class of simple unital AH-algebras. A very important axiomatic version of the classification of AH-algebras without dimension growth was given by H. Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property and showed that C^*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras.

Following the notion of Lin on the tracial approximation by interval algebras, G. A. Elliott and Z. Niu in [5] considered tracial approximation by more general C^*-algebras. Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebras in Ω, denoted by $TA\Omega$, is defined as follows. A simple unital C^*-algebra A is said to belong to the class $TA\Omega$, if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that
(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$,
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The question of the behavior of C^*-algebra properties under passage from a class Ω to the class $TA\Omega$ is interesting and sometimes important. In fact the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of the classification theorem in [5] by G. A. Elliott and Z. Niu.

Received by the editors December 31, 2010.
Published electronically September 22, 2011.
This article is supported by the National Natural Science Foundation of China (11101268, 11071159) and by the Science and Technology Program of Shanghai Maritime University, no. 20110052.
AMS subject classification: **46L05**, 46L80, 46L35.
Keywords: C^*-algebra, tracial approximation, K_0-monoid.

In this paper, we show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$:
(1) weak comparability,
(2) strictly unperforated,
(3) strictly cancellative.

2 Preliminaries and Definitions

Let a and b be two positive elements in a C^*-algebra A. We write $[a] \le [b]$ (*cf.* [11, Definition 3.5.2]), if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her}(a)$, v^*c, $cv \in A$, $vv^* = P_a$, where P_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her}(b)$. We write $[a] = [b]$ if $v^* \mathrm{Her}(a) v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \le [b]$, if there are n mutually orthogonal positive elements $b_1, b_2, \ldots, b_n \in \mathrm{Her}(b)$ such that $[a] \le [b_i]$, $i = 1, 2, \ldots, n$.

Let $0 < \sigma_1 < \sigma_2 \le 1$ be two positive numbers. Define

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & \text{if } t \ge \sigma_2 \\ \frac{t-\sigma_1}{\sigma_2-\sigma_1} & \text{if } \sigma_1 \le t \le \sigma_2 \\ 0 & \text{if } 0 < t \le \sigma_1 \end{cases}$$

Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras that can be tracially approximated by C^*-algebras in Ω is denoted by $TA\Omega$.

Definition 2.1 ([5]) A simple unital C^*-algebra A is said to belong to the class $TA\Omega$ if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \ge 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that
(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$,
(3) $[1 - p] \le [a]$.

Definition 2.2 ([7]) Let Ω be a class of unital C^*-algebras. A unital C^*-algebra A is said to have property (III) if, for any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a, and any integer $n > 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $B \in \Omega$ and $1_B = p$ such that
(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$, $\|pap\| \ge \|a\| - \varepsilon$,
(3) $n\big[f_{\sigma_1}^{\sigma_2}\big((1-p)a(1-p)\big)\big] \le [f_{\sigma_3}^{\sigma_4}(pap)]$.

Lemma 2.3 ([5]) *If the class Ω is closed under tensoring with matrix algebras, or closed under taking unital hereditary C^*-subalgebras, then $TA\Omega$ is closed under passing to matrix algebras or unital hereditary C^*-subalgebras.*

We say a C^*-algebra A has the SP-property if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Call projections $p, q \in M_\infty(A)$ equivalent, denoted $p \sim q$, when there is a partial isometry $v \in M_\infty(A)$ such that $p = v^*v$, $q = vv^*$. The equivalence classes are denoted $[\,\cdot\,]$ and the set of all these is

$$V(A) := \{[p] \mid p = p^* = p^2 \in M_\infty(A)\}.$$

Addition in $V(A)$ is defined by

$$[p] + [q] := [\mathrm{diag}(p, q)].$$

Then $V(A)$ becomes an abelian monoid, which we call $V(A)$ the K_0-monoid of A.

All abelian monoids have a natural pre-order, the algebraic ordering, defined as follows: if $x, y \in M$, we write $x \le y$ if there is a z in M such that $x+z = y$. In the case of $V(A)$, the algebraic ordering is given by Murray–von Neumann subequivalence, that is, $[p] \le [q]$ if and only if there is a projection $p' \le q$ such that $p \sim p'$. We also write, as is customary, $p \preceq q$ to mean that p is subequivalent to q.

If $x, y \in M$, we will write $x \le^* y$ if there is a nonzero element z in M, such that $x + z = y$.

Let us recall that an element u in a monoid M is an order unit provided $u \neq 0$ and, for any x in M, there is $n \in \mathbb{N}$ such that $x \le nu$.

We say that a monoid M is conical if $x + y = 0$ only when $x = y = 0$. Note that, for any C^*-algebra A, the projection monoid $V(A)$ is conical.

A monoid M with order unit u was called weak comparability if for every nonzero element x in M such that $x \le u$, there exists $k \in \mathbb{N}$ such that, if $y \in M$ and $ky \le u$, then also $y \le x$.

Recall also that a monoid M is called strictly unperforated if, whenever $nx+z = ny$ for $x, y, z \in M$ with $z \neq 0$ and $n \in \mathbb{N}$, there is a nonzero element w such that $x + w = y$.

We say that a monoid M is strictly cancellative if for $a, b, c \in M$ such that $a + c \le^* b + c$ implies $a \le^* b$.

Theorem 2.4 ([7]) *Let Ω be a class of unital C^*-algebras such that Ω is closed under taking unital hereditary C^*-subalgebras and closed taking finite direct sums. Let A be a simple unital C^*-algebra. Then the following are equivalent:*

(1) *$A \in TA\Omega$,*

(2) *A has property* (III).

Theorem 2.5 ([7]) *Let Ω be a class of unital C^*-algebras with the SP-property. Then any simple unital C^*-algebra $A \in TA\Omega$ has the SP-property.*

3 The Main Results

Theorem 3.1 *Let Ω be a class of unital C^*-algebras with SP-property. Suppose that Ω is closed under tensoring with matrix algebras and for any $B \in \Omega$ the K_0-monoid $V(B)$ is weak comparability. Then the K_0-monoid $V(A)$ is weak comparability for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof We need to show that $[q] \leq [p]$ when $[p] \leq [1]$ and $k[q] \leq [1]$ for some $k \in \mathbb{N}$ where $p, q \in M_j(A)$ for some integer j.

By Lemma 2.3 we may assume that $p, q \in \mathrm{proj}(A)$. By Theorem 2.5, A has the property SP: there exist nonzero projections p', p'' such that $[p] = [p'] + [p'']$, $[p'] = [p'']$. We have $[p'] \leq [1]$.

Since $A \in TA\Omega$, by Theorem 2.4, for $F = \{p, p', p'', q\}$, any $\varepsilon > 0$, there exist a projection $s \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = s$ such that

(1) $\|xs - sx\| < \varepsilon$ for all $x \in F$,

(2) $sxs \in_\varepsilon B$ for all $x \in F$,

(3) $[1 - s] \leq [p'']$.

By (1) and (2) there exist projections $p_1', q_1 \in B$ and $p_2, q_2 \in (1-s)A(1-s)$, such that

$$[p'] = [p_1'] + [p_2'], \quad [q] = [q_1] + [q_2]$$

and $[p_1'] \leq [s]$, $[p_2'] \leq [1 - s]$, $k[q_1] \leq [s]$ and $k[q_2] \leq [1 - s]$.

Since $V(B)$ is weak comparability, we have

$$[q_1] \leq [p_1'], \quad k[q_2] \leq [1 - s] \leq [p''].$$

We have

$$[q_2] \leq [1 - s] \leq [p''].$$

Therefore

$$[q] = [q_1] + [q_2] \leq [p_1'] + [q_2] \leq [p_1'] + [p''] = [p]. \qquad \blacksquare$$

Theorem 3.2 *Let Ω be a class of unital C^*-algebras with SP-property. Suppose that Ω is closed under tensoring with matrix algebras and for any $B \in \Omega$ the K_0-monoid $V(B)$ is strictly unperforated. Then the K_0-monoid $V(A)$ is strictly unperforated for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof We need to show there exists a nonzero element w such that $x + w = y$ whenever $nx + z = ny$ for $x, y, z \in V(A)$ with $z \neq 0$ and $n \in \mathbb{N}$. We may assume that $x = [p], z = [r], y = [q]$ where $p, q, r \in M_j(A)$ with $r \neq 0$ for some large integer $j \in \mathbb{N}$.

By Lemma 2.3 we may assume that $p, q, r \in \mathrm{proj}(A)$, we have $n[p] + [r] = n[q]$ with $r \neq 0$. Since $A \in TA\Omega$, by Theorem 2.4, for $F = \{p, q, r\}$, any $\varepsilon > 0$, there exist a projection $s \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = s$ such that:

(1) $\|xs - sx\| < \varepsilon$ for all $x \in F$,

(2) $sxs \in_\varepsilon B$ for all $x \in F$,

(3) $(n + 1)[1 - s] \leq [r]$.

By (3) there exist mutually equivalent and mutually orthogonal projections $r_1, r_2, \ldots, r_{n+1}$ such that

$$r_i \leq 1 - s$$

for all $1 \leq i \leq n + 1$, and

$$(n + 1)[r_1] = [r_1 + r_2 \cdots + r_{n+1}] \leq [r].$$

https://doi.org/10.4153/CMB-2011-183-9 Published online by Cambridge University Press

By Theorem 2.5, A has the *SP* property such that there exists a projection $t \leq 1 - p$ such that $[t] \leq [r_1]$. We have

$$n[p] + n[t] \leq^* n[q],$$

i.e., $n[p + t] \leq^* n[q]$.

Since $A \in TA\Omega$, for $G = \{p, q, t\}$, any $\varepsilon > 0$, and any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, there exist a projection $m \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega$, $1_C = m$ such that

(1′) $\|xm - mx\| < \varepsilon$ for all $x \in G$,

(2′) $mxm \in_\varepsilon C$ for all $x \in G$,

(3′) $\big[f_{\sigma_1}^{\sigma_2}\big((1-m)t(1-m)\big)\big] \leq [f_{\sigma_3}^{\sigma_4}(mtm)]$.

By (1′) and (2′) there exist projections

$$p_1, q_1, t_1 \in C \quad \text{and} \quad p_2, q_2, t_2 \in (1-m)A(1-m)$$

such that

$$[p] = [p_1] + [p_2], \quad [q] = [q_1] + [q_2],$$
$$n([p_1] + [t_1]) \leq^* n[q_1], \quad n([p_2] + [t_2]) \leq^* n[q_2].$$

By (3′) we have $[t_2] \leq [t_1]$.

Since $V(C)$ has the strictly cancellative property, we have $([p_1] + [t_1]) \leq^* [q_1]$.

For $H = \{p_2, q_2, r_2, t_2\}$, any $\varepsilon > 0$, by Theorem 2.4, there exist a projection $e \in A$ and a C^*-subalgebra $E \subseteq A$ with $E \in \Omega$, $1_E = e$ such that

(1″) $\|xe - ex\| < \varepsilon$ for all $x \in H$,

(2″) $exe \in_\varepsilon E$ for all $x \in H$,

(3″) $[1 - e] \leq [t_2]$.

By (1″) and (2″), there exist projections $p_3, q_3 \in E$ and $p_4, q_4 \in (1-e)A(1-e)$ such that

$$[p_2] = [p_3] + [p_4],$$
$$[q_2] = [q_3] + [q_4],$$
$$n([p_3] + [t_3]) \leq^* n[q_3], \quad \text{and}$$
$$n([q_4] + [t_4]) \leq^* n[q_4].$$

Since $V(E)$ has the strictly cancellative property, we have $[p_3] + [t_3] \leq^* [q_3]$.

By (3″) we have $[p_4] \leq [1 - e] \leq [t_2]$. We have

$$\begin{aligned}[p] &= [p_1] + [p_2] = [p_1] + [p_3] + [p_4] \leq [p_1] + [p_3] + [t_2] \\ &\leq [p_1] + [q_3] + [t_1] \leq^* [q_1] + [q_3] \leq^* [q].\end{aligned}$$

Therefore there exists an element w such that $[p] + w = [q]$. ■

https://doi.org/10.4153/CMB-2011-183-9 Published online by Cambridge University Press

Theorem 3.3 *Let Ω be a class of unital C^*-algebras with SP-property. Suppose that Ω is closed under tensoring with matrix algebras and for any $B \in \Omega$ the K_0-monoid $V(B)$ is strictly cancellative. Then the K_0-monoid $V(A)$ is strictly cancellative for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof We need to show that $a+c \leq^* b+c$ imply that $a \leq^* b$. There exist projections $p, q, r \in M_k(A)$ for sufficiently large integer k such that $[p] = a$, $[r] = c$, $[q] = b$. By Lemma 2.3, we may assume that $p, q, r \in \operatorname{proj}(A)$.

Since $A \in TA\Omega$, for $G = \{p, q, r\}$, any $\varepsilon > 0$, and any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, there exist a projection $m \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega$, $1_C = m$ such that

$(1')$ $\|xm - mx\| < \varepsilon$ for all $x \in G$,

$(2')$ $mxm \in {}_{\varepsilon}C$ for all $x \in G$.

By $(1')$ and $(2')$ there exist projections

$$p_1, q_1, r_1 \in C \quad \text{and} \quad p_2, q_2, r_2 \in (1-m)A(1-m)$$

such that

$$[p] = [p_1] + [p_2], \quad [q] = [q_1] + [q_2], \quad [r] = [r_1] + [r_2],$$
$$[p_1] + [r_1] \leq^* [q_1] + [r_1], \quad [p_2] + [r_2] \leq^* [q_2] + [r_2].$$

Since $V(C)$ has the strictly cancellative property, we have

$$[p_1] \leq^* [q_1].$$

There exist projection $e_1 \neq 0$ such that $[p_1] + [e_1] = [q_1]$. We may assume that $e_1 \in \operatorname{proj}(A)$.

For $H = \{p_2, q_2, r_2, e_1\}$, any $\varepsilon > 0$, there exist a projection $n \in A$ and a C^*-subalgebra $E \subseteq A$ with $E \in \Omega$, $1_E = n$ such that

$(1'')$ $\|xn - nx\| < \varepsilon$ for all $x \in H$,

$(2'')$ $nxn \in {}_{\varepsilon}E$ for all $x \in H$,

$(3'')$ $[1 - n] \leq [e_1]$.

By $(1'')$ and $(2'')$ there exist projections

$$p_3, q_3, r_3, e_3 \in E \quad \text{and} \quad p_4, q_4, r_4, e_4 \in (1-n)A(1-n)$$

such that

$$[p_2] = [p_3] + [p_4], \quad [q_2] = [q_3] + [q_4], \quad [r_2] = [r_3] + [r_4],$$
$$[p_3] + [r_3] \leq^* [q_3] + [r_3], \quad [p_4] + [r_4] \leq^* [q_4] + [r_4].$$

Since $V(E)$ has the strictly cancellative property, we have $[p_3] \leq^* [q_3]$.

By $(3'')$ we have $[p_4] \leq [1 - n] \leq [e_1]$. Therefore

$$[p] = [p_1] + [p_2] = [p_1] + [p_3] + [p_4] \leq [p_1] + [p_3] + [e_1]$$
$$\leq [q_1] + [p_3] \leq^* [q_1] + [q_3] + [q_4] = [q]. \qquad \blacksquare$$

https://doi.org/10.4153/CMB-2011-183-9 Published online by Cambridge University Press

References

[1] B. Blackadar, *K-theory for operator algebra.* Springer-Verlag, New York, 1986.
[2] G. A. Elliott, *On the classification of the inductive limits of sequences of semisimple finite dimensional algebras*. J. Algebra **38**(1976), 29–44. http://dx.doi.org/10.1016/0021-8693(76)90242-8
[3] ———, *On the classification of C^*-algebras of real rank zero.* J. Reine Angew. Math. **443**(1993), 179–219. http://dx.doi.org/10.1515/crll.1993.443.179
[4] G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero, II.* Ann. Math. **144**(1996), 497–610. http://dx.doi.org/10.2307/2118565
[5] G. A. Elliott and Z. Niu, *On tracial approximation.* J. Funct. Anal. **25**(2008), 396–440. http://dx.doi.org/10.1016/j.jfa.2007.08.005
[6] Q. Fan and X. Fang, *Stable rank one and real rank zero for crossed product by automorphisms with the tracial Rokhlin property.* Chinese Ann. Math. (Series B) **30**(2009), 179–186. http://dx.doi.org/10.1007/s11401-007-0563-7
[7] Q. Fan, *Certain properties preserved by tracial approximation*. Preprint.
[8] ———, *Classification of certain simple C^*-algebras.* J. Ramanujan Math. Soc. **26**(2011), 1–7.
[9] Q. Fan and X. Fang, *Non-simple tracial approximation.* Houston J. Math., to appear.
[10] H. Lin, *The tracial topological rank of C^*-algebras*. Proc. London Math. Soc. **83**(2001), 199–234. http://dx.doi.org/10.1112/plms/83.1.199
[11] ———, *An introduction to the classification of amenable C^*-algebras*. World Sci. Publ., New Jersey–London–Singapore–Hong Kong, 2001.
[12] E. Ortega, F. Perera, and M. Rordam, *The Corona Factorization Property and refinement monoids.* arxiv:0904.0541v1.
[13] F. Wehrung, *Injective positively ordered monoids I*. J. Pure Appl. Algebra **83**(1992), 43–82. http://dx.doi.org/10.1016/0022-4049(92)90104-N

Department of Mathematics, Shanghai Maritime University, 1550 Haigang Ave. in New Harbor City, Shanghai 201306, China
e-mail: qzfan@shmtu.edu.cn