Integral equations for Lamé functions

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Introduction

1. In the theory of ordinary linear differential equations with three regular singularities and in the theory of their special and limiting cases, integral representations of the solutions are known to be very important. It seems that there is no corresponding simple integral representation of the solutions of ordinary linear differential equations with four regular singularities (Heun's equation) or of particular (*e.g.* Lamé's equation) or limiting (*e.g.* Mathieu's equation) cases of such equations. It has been suggested (Whittaker 1915 *c*) that the theorems corresponding in these latter cases to *integral representations* of the hypergeometric functions involve *integral equations* of the second kind. Such integral equations have been discovered for Mathieu functions (Whittaker 1912, *cf.* also Whittaker and Watson 1927 pp. 407-409 and 426) as well as for Lamé functions (Whittaker 1915 *a* and *b*, *cf.* also Whittaker and Watson 1927 pp. 564-567) and polynomial or "quasi-algebraic" solutions of Heun's equation (Lambe and Ward 1934). Ince (1921-22) investigated general integral equations connected with periodic solutions of linear differential equations.

In the present paper I restrict myself in the first instance to Lamé's differential equation. It appears that the theory of integral equations connected with periodic solutions of Lamé's equation is not as complete as the corresponding theory of integral representations of, say, Legendre functions. Professor Whittaker published (1915 *a* and *b*) two essentially different integral equations which are satisfied by Lamé polynomials, and recently Sharma (1937) added another, seemingly different, integral equation. The connections between these integral equations, however, are by no means obvious, and the question has not yet been dealt with whether the known integral equations exhaust all possible types of integral equations connected with Lamé polynomials. In fact it is very plausible that they do not. For denoting by $E_n^s(x)$ ($s = 0, 1, \ldots, 2n$) the $2n + 1$

linearly independent mutually orthogonal and normalised Lamé polynomials of degree n,

$$K(x,y) = \sum_{s=0}^{2n} \lambda_s^{-1} E_n^s(x) E_n^s(y), \tag{1.1}$$

with arbitrary (not vanishing) characteristic numbers λ_s, is a nucleus for Lamé polynomials. The most general nucleus thus depends on $2n+1$ arbitrary constants, whereas Whittaker's two integral equations depend on two and three arbitrary constants respectively. Sharma's integral equation contains only one (multiplicative) arbitrary constant.

It is comparatively simple to show that Sharma's integral equation is a particular case of the first of Whittaker's equations (1915 a). The connection between Whittaker's two integral equations is, however, far less obvious. After some unsuccessful attempts to establish some relation between these two integral equations, I realised that the most profitable way of doing this is to find the most general kernel (containing $2n+1$ parameters) all the characteristic functions of which are Lamé polynomials of a fixed degree n, and to see how Whittaker's two kernels fit into the general scheme.

2. In the following section I first establish the most general nucleus, $K(x, y)$, all characteristic functions of which are Lamé polynomials of a fixed degree n [though not necessarily all Lamé polynomials of degree n will be eigen-functions of the nucleus $K(x, y)$]. The result is very simple indeed: *Interpreting x and y as ellipsoidal coordinates on an ellipsoid, transform the latter by means of an affine transformation into the unit sphere. Then any surface harmonic of degree n on the unit sphere is a nucleus $K(x, y)$.* It is easily seen that, there being $2n+1$ linearly independent surface harmonics, this nucleus depends on $2n+1$ arbitrary constants and hence it is the most general nucleus. Whittaker's two nuclei correspond to the two extreme cases of zonal and sectorial harmonics respectively. There are some special nuclei, however, which are understood to originate from tesseral surface harmonics of order one and two.

The general theorem could be inferred from some developments of Heine. For the sake of completeness, however, and because it is so brief, I give the complete proof[1].

[1] [Added in proof]. Since this paper was written, Professor E. T. Whittaker has pointed out that my proof of Heine's results is identical with that of Professor Copson (these *Proceedings* (2), 1 (1927-28), 62-64). Since this paper was submitted I have also come across a thesis by E. O. Stanaitis (*Mém. Fac. Sci. Univ. Lithuanie* 13 (1939), fasc. 1) which contains some results of my paper.

There are also transcendental periodic Lamé functions, integral equations for which have been investigated by Ince (1940 *a*). I discuss briefly the application of the general nuclei to these transcendental Lamé functions.

Ellipsoidal surface harmonics can be expressed in terms of products of Lamé polynomials. Integral equations for these products are again most easily expressed by means of an affine transformation of the ellipsoid into the unit sphere. *Ellipsoidal surface harmonics are the eigen-functions of an arbitrary nucleus depending only on the spherical distance on the unit sphere.*

The last part of the paper contains the corresponding results on Mathieu functions, obtained by a limiting process.

Lamé polynomials

3. The "Jacobian" form of Lamé's differential equation (sometimes called Hermite's equation) reads

$$\frac{d^2 E}{dx^2} + \{h - n(n+1) k^2 \operatorname{sn}^2 x\} E = 0, \tag{3.1}$$

where n is supposed to be a non-negative integer. There are $2n+1$ particular values of h corresponding to solutions of (3.1) which are polynomials of degree n in $\operatorname{sn} x$, $\operatorname{cn} x$ and $\operatorname{dn} x$. These solutions will be called Lamé polynomials and denoted by $E_n^s(x)$ $(s = 0, 1, \ldots, 2n)$. We may take them in such an order that $E_n^s(x)$ has exactly s zeros in the interval $0 \leqq x \leqq 2K$; n will be kept fixed throughout, unless the contrary is stated.

The nucleus of an integral equation

$$E_n^s(x) = \lambda_s \int_{-2K}^{+2K} K(x, y)\, E_n^s(y)\, dy, \qquad (s = 0, 1, \ldots, 2n), \tag{3.2}$$

in which some of the λ_s may be infinite, and which is supposed to have no solutions except Lamé polynomials (and, possibly, linear combinations of Lamé polynomials) is bound to be a mod $4K$ periodic function of x as well as of y, satisfying the partial differential equation [Ince 1921-22 (2), (2′)]

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial y^2} - n(n+1) k^2 (\operatorname{sn}^2 x - \operatorname{sn}^2 y) K = 0. \tag{3.3}$$

Now let us introduce spherical polar coordinates θ, ϕ, by means of the substitution

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(3.4) $\cos\theta = k \operatorname{sn} x \operatorname{sn} y$, $\sin\theta\cos\phi = \frac{1}{k'}\operatorname{dn} x \operatorname{dn} y$, $\sin\theta\sin\phi = i\frac{k}{k'}\operatorname{cn} x \operatorname{cn} y$.

(3.3) changes by this substitution into (Heine 1878 p. 354 ; Hobson 1931 pp. 456, 457)

$$\frac{\partial^2 K}{\partial\theta^2} + \cot\theta\frac{\partial K}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 K}{\partial\phi^2} + n(n+1)K = 0, \tag{3.5}$$

the equation of spherical surface harmonics of degree n. Hence there are integral equations of the form (3.2) with nuclei $K(x, y) = Y_n(\theta, \phi)$ where Y_n is any surface harmonic of degree n. It is well to note that

$$Y_n(\theta, \phi) = \sum_{r=0}^{n} P_n^r(\cos\theta)\{a_r e^{ir\phi} + b_r e^{-ir\phi}\} \tag{3.6}$$

is a polynomial of degree n in $\cos\theta$, $\sin\theta\cos\phi$ and $\sin\theta\sin\phi$ and hence a polynomial of degree n in $\operatorname{sn} x \operatorname{sn} y$, $\operatorname{cn} x \operatorname{cn} y$ and $\operatorname{dn} x \operatorname{dn} y$. On the other hand Y_n contains $2n+1$ arbitrary constants, namely $a_0 + b_0, a_1, \ldots, a_n, b_1, \ldots, b_n$. Hence it is the most general nucleus the complete system of eigen-functions of which consists of (some or all) Lamé polynomials of degree n.

Instead of $Y_n(\theta, \phi)$ we may equally well take $Y_n(\theta', \phi')$ where θ', ϕ' are polar coordinates on the unit sphere with an arbitrary pole, so that

$$\begin{aligned} \cos\theta' &= \cos\gamma\cos\theta + \sin\gamma\sin\theta\cos(\phi - a), \\ \sin\theta'\cos(\phi' - a') &= \sin\gamma\cos\theta - \cos\gamma\sin\theta\cos(\phi - a) \\ \sin\theta'\sin(\phi' - a') &= -\sin\theta\sin(\phi - a) \end{aligned} \tag{3.7}$$

where a, a', γ are three arbitrary angles. Putting

$$\cos\gamma = k \operatorname{sn} p \operatorname{sn} q, \quad \sin\gamma\cos a = \frac{1}{k'}\operatorname{dn} p \operatorname{dn} q, \quad \sin\gamma\sin a = i\frac{k}{k'}\operatorname{cn} p \operatorname{cn} q,$$

we have

$$\begin{aligned} \cos\theta' &= k^2 \operatorname{sn} p \operatorname{sn} q \operatorname{sn} x \operatorname{sn} y + \frac{1}{k'^2}\operatorname{dn} p \operatorname{dn} q \operatorname{dn} x \operatorname{dn} y - \frac{k^2}{k'^2}\operatorname{cn} p \operatorname{cn} q \operatorname{cn} x \operatorname{cn} y, \\ \sin\theta' \,.\, e^{\pm i(\phi' - a')} &= k\sin\gamma \operatorname{sn} x \operatorname{sn} y - \frac{1}{k'}(\cos\gamma\cos a \mp i\sin a)\operatorname{dn} x \operatorname{dn} y \\ &\quad - i\frac{k}{k'}(\cos\gamma\sin a \pm i\cos a)\operatorname{cn} x \operatorname{cn} y. \end{aligned} \tag{3.8}$$

The most general integral equation of the form (3.2) reads

$$E_n^s(x) = \lambda_s \int_{-2K}^{2K} Y_n(\theta', \phi') E_n^s(y)\, dy. \tag{3.9}$$

Instead of from $-2K$ to $2K$, the integration may be extended over any common period of $E_n^s(y)$ and $Y_n(\theta', \phi')$ (*qua* function of y).

4. The nucleus of the integral equation reduces to an elementary function when Y_n is a sectorial harmonic

$$(4.1) \qquad Y_n = \{\sin\theta' \,.\, e^{\pm i(\phi' - a')}\}^n.$$

By (3.8), this reduces essentially to Whittaker's nucleus (1915 *a*)

$$(4.2) \qquad (\operatorname{dn} x \operatorname{dn} y + k \cosh\eta \operatorname{cn} x \operatorname{cn} y + k\,k' \sinh\eta \operatorname{sn} x \operatorname{sn} y)^n$$

where

$$(4.3) \qquad \sinh\eta = \frac{-\sin\gamma}{\cos\gamma\cos a \mp i\sin a}, \quad \cosh\eta = i\,\frac{\cos\gamma\sin a \pm i\cos a}{\cos\gamma\cos a \mp i\sin a}.$$

The other important particular case arises when Y_n is a zonal harmonic

$$(4.4) \qquad Y_n = P_n(\cos\theta').$$

By (3.8), in this case we have Whittaker's second nucleus (1915 *b*)

$$(4.5) \qquad P_n\left(k^2 \operatorname{sn} p \operatorname{sn} q \operatorname{sn} x \operatorname{sn} y + \frac{1}{k'^2}\operatorname{dn} p \operatorname{dn} q \operatorname{dn} x \operatorname{dn} y - \frac{k^2}{k'^2}\operatorname{cn} p \operatorname{cn} q \operatorname{cn} x \operatorname{cn} y\right).$$

Several simpler nuclei [Whittaker and Watson 1927 § 23.6; Lambe and Ward 1934, equation (4.15)] are particular cases of (4.5).

Nuclei of the type given by Whittaker and Watson (1927 § 23.61) and Lambe and Ward [1934, equations (4.16)-(4.18)] originate from tesseral harmonics of lowest order

$$Y_n = P_n^1(\cos\theta') \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (\phi' - a') \text{ and } Y_n = P_n^2(\cos\theta')\sin 2(\phi' - a').$$

Taking $a = \gamma = 0$, we obtain the nuclei

$$(4.6) \qquad \operatorname{cn} x \operatorname{cn} y\, P_n'(k \operatorname{sn} x \operatorname{sn} y), \qquad \operatorname{dn} x \operatorname{dn} y\, P_n'(k \operatorname{sn} x \operatorname{sn} y),$$
$$\operatorname{cn} x \operatorname{dn} x \operatorname{cn} y \operatorname{dn} y\, P_n''(k \operatorname{sn} x \operatorname{sn} y).$$

Similarly a great number of other special nuclei may be deduced.

5. The main result of section 3 immediately follows from Heine's theorem that any $Y_n(\theta', \phi')$ is expressible as a (finite) series of products of Lamé functions (Heine 1878, p. 376 Schema A) which in our notations may be written

$$(5.1) \qquad Y_n(\theta', \phi') = \sum_{s=0}^{2n} g_s\, E_n^s(x)\, E_n^s(y).$$

From this (3.9) immediately follows by the orthogonality property of Lamé functions. I preferred, however, to prove (3.9) independently not only because both the orthogonal property and (5.1) (the bilinear

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development of the nucleus) follow from (3.9) but mainly because the proof given in section 3 covers also the case of transcendental Lamé functions (*cf. infra* sections 7 and 8).

The nucleus (4.4) has the remarkable property that

$$\lambda'_s = \lambda_s \; E_n^s(p) \; E_n^s(q)$$

is in this case independent of p and q [Whittaker 1915 b, equation (4)]. This is easily inferred from the way in which p, q, x, y are coupled in (3.8), but follows also from Heine's expansion [1878, p. 432 equation (73)]

$$P_n(\cos\theta') = \sum_{s=0}^{2n} b_s \; E_n^s(p) \, E_n^s(q) \, E_n^s(x) \, E_n^s(y), \tag{5.2}$$

which is completely equivalent to Whittaker's integral equation.

6. A few words may be said in connection with the " Weierstrassian " form of Lamé's equation,

$$\frac{d^2\Lambda}{du^2} + \{H - n(n+1)\wp u\}\,\Lambda = 0, \tag{6.1}$$

the doubly-periodic solutions of which will be denoted by $\Lambda_n^s(u)$. The substitution corresponding to (3.4) is

$$\cos\theta = \left\{\frac{(\wp u - e_3)(\wp v - e_3)}{(e_2 - e_3)(e_1 - e_3)}\right\}^{\frac{1}{2}},$$

$$\sin\theta\cos\phi = \left\{\frac{(\wp u - e_1)(\wp v - e_1)}{(e_1 - e_2)(e_1 - e_3)}\right\}^{\frac{1}{2}}, \; \sin\theta\sin\phi = i\left\{\frac{(\wp u - e_2)(\wp v - e_2)}{(e_2 - e_3)(e_1 - e_2)}\right\}^{\frac{1}{2}}; \tag{6.2}$$

and the integral equation

$$\Lambda_n^s(u) = \lambda_s \int_a^{a+4\omega} Y_n(\theta', \phi') \, \Lambda_n^s(v) \, dv \tag{6.3}$$

can be proved like (3.9).

Sharma's nucleus (1937)

$$\left(\wp'\frac{u}{2} \; \wp'\frac{v}{2}\right)^{-n} \left\{(e_1 - e_2)(e_3 - e_2) - \left(\wp\frac{u}{2} - e_2\right)\left(\wp\frac{v}{2} - e_2\right)\right\}^{2n} \tag{6.4}$$

may be shown to be a special nucleus of the sectorial type, *i.e.*

$$Y_n(\theta', \phi') = \{\sin\theta' \, . \, e^{i(\phi' - \alpha')}\}^n, \tag{6.5}$$

where

$$\cos\gamma = \left(\frac{e_2 - e_3}{e_1 - e_3}\right)^{\frac{1}{2}}, \; \sin\gamma = -\left(\frac{e_1 - e_2}{e_1 - e_3}\right)^{\frac{1}{2}}, \; \alpha = 0. \tag{6.6}$$

The transformation of (6.5) into (6.4) is based on the duplication formula of Weierstrass's $\wp$-function and is easily effected by means of the formula

$$\wp'\frac{u}{2}\,(\wp u - e_1)^{\frac{1}{2}} = \wp^2\frac{u}{2} - 2e_1\,\wp\,\frac{u}{2} - e_1^2 - e_2e_3$$

and the corresponding formulae for $(\wp u - e_2)^{\frac{1}{2}}$ and $(\wp u - e_3)^{\frac{1}{2}}$. The necessary calculations are, however, somewhat lengthy and are therefore omitted.

Transcendental Lamé functions

7. Besides Lamé polynomials, there are transcendental (simply-periodic) Lamé functions. This and the following section contain a brief discussion of integral equations for these functions. In the present section n will be supposed to be a non-negative integer.

Suppose that γ and α are chosen so that the closed contour described by $\cos\theta'$, when y describes the interval $-2K \leqq y \leqq 2K$ and x has any real value, does *not* enclose either of the two points $\cos\theta' = \pm 1$. Obviously such values of α and γ exist; for instance sufficiently small values of α and γ have this property.

In this case $Q_n^m(\cos\theta')\,e^{\pm im(\phi'-\alpha')}$, which is a solution of (3.5), is a periodic function of x and y and hence the integral equation (3.9) still holds if Y_n contains associated Legendre functions of the second kind. Now, $Q_n^m(\cos\theta')\,e^{\pm im(\phi'-\alpha')}$ is *not* a polynomial in $\operatorname{sn} x \operatorname{sn} y$, $\operatorname{cn} x \operatorname{cn} y$ and $\operatorname{dn} x \operatorname{dn} y$, and hence it is clear that this nucleus must have other eigen-functions beside Lamé polynomials. *These other eigen-functions are transcendental Lamé functions, belonging to integral values of n*, of which there is an infinity. Hence the integral equation

$$E_n^s(x) = \lambda_s \int_{-2K}^{2K} Q_n^m(\cos\theta')\,e^{\pm i(\phi'-\alpha')m}\,E_n^s(y)\,dy, \tag{7.1}$$

in which α and γ are supposed to have such values that both $\cos\theta' = 1$ and $\cos\theta' = -1$ are outside the simply connected domain of the $\cos\theta'$-plane containing all points corresponding to *real* values of x and y, and n and m are any non-negative integers, is valid for any non-negative integer s.

At first it would seem that this conclusion is not entirely justified and that the fact (Ince 1940 *b* p. 88) that E_n^{2s-1} and E_n^{2s} belong to the same value of h if $s = n+1,\ n+2,\ \ldots.$ would cause additional difficulties (Ince 1921-22 p. 46). This is, however, not so in the present case. The nucleus having the property that it does *not* change when *both* x and y change their signs, it is clear that multiplied by an even function, $E_n^{2s}(y)$, and integrated with respect to y, it will yield an *even* function, proportional to $E_n^{2s}(x)$, and not a linear combination of $E_n^{2s}(x)$ and $E_n^{2s-1}(x)$, which could not be an even function of x, unless the term with $E_n^{2s-1}(x)$ vanishes identically.

Plainly nuclei may be constructed the eigen-functions of which consist *only* of *transcendental* Lamé functions (for integral n). I do not propose, however, to go into further details concerning these nuclei. I believe that (7.1) is the first integral equation to be published which is satisfied by transcendental Lamé functions of integral degree n.

8. In this section n may be any real or complex number. Then there are still Lamé functions $E_n^s(x)$ of period $4K$, all of which are transcendental unless n is an integer. A brief indication of the results concerning these functions will be sufficient.

Equation (3.9) will still hold (for $s = 0, 1, 2, \ldots$) with a kernel

$$Y_n = P_n^m(\cos\theta')\, e^{\pm im\phi'} \tag{8.1}$$

provided that a and γ are suitably chosen, *i.e.* so that $\cos\theta'$ moves inside a simply-connected domain of the complex $\cos\theta'$-plane not enclosing $\cos\theta' = -1$. Instead of this we shall say shortly "provided that $\cos\theta'$ does not reach -1." Also the nucleus

$$Y_n = P_n^m(-\cos\theta')\, e^{\pm im\phi'} \tag{8.2}$$

will be permissible if $\cos\theta'$ does not reach 1. If a and γ are chosen so that $\cos\theta'$ reaches neither $+1$ nor -1, then any linear combination of (8.1) and (8.2) can be used in connection with (3.9). In the latter case the nucleus will be expressible alternatively as a linear combination of

$$Q_n^m(\cos\theta')\, e^{\pm im\phi'} \text{ and } Q_{-n-1}^m(\cos\theta')\, e^{\pm im\phi'}. \tag{8.3}$$

Clearly sums of such nuclei are again permissible nuclei. Non-integral values of m, real or complex, may be used provided that, in addition to the conditions stated above, $e^{\pm i\phi'}$ does not reach zero.

Simple nuclei of the type (8.2), namely $P_n(k \operatorname{sn} x \operatorname{sn} y)$ and the nuclei (4.6) for non-integral values of n have been discussed by Ince (1940 *a*, section 10). Obviously the nucleus (4.5) will still be valid for general values of n, and the same remark applies to (4.2), both with suitable restrictions imposed upon the parameters involved.

Only Lamé functions of period $4K$ (or $2K$) have been considered. The same method applies obviously to Lamé functions (transcendental or algebraical) whose fundamental period is any multiple of $4K$. The difficulty caused by the coexistence of two periodic solutions in this case (Ince 1940 *b*, section 7) can be overcome in precisely the same manner as in section 7.

Products of Lamé polynomials

9. Returning now to Lamé polynomials, *i.e.* to integral values $n \geqq 0$ and $0 \leqq s \leqq 2n$, let us consider an ellipsoidal surface harmonic $E_n^s(\beta)\, E_n^s(\gamma)$. In order to cover the whole surface of the ellipsoid, γ has to take all real values between $-2K$ and $2K$, and β all values between K and $K + 2iK'$ (Whittaker and Watson 1927 § 23.5). Introducing polar coordinates

$$(9.1)\quad \cos\theta = k \operatorname{sn}\beta \operatorname{sn}\gamma,\ \sin\theta\cos\phi = \frac{1}{k'} \operatorname{dn}\beta \operatorname{dn}\gamma,\ \sin\theta\sin\phi = i\frac{k}{k'}\operatorname{cn}\beta\operatorname{cn}\gamma,$$

it is easily proved that

$$(9.2)\qquad E_n^s(\beta)\, E_n^s(\gamma) = Y_n(\theta, \phi).$$

For Y_n contains $2n+1$ arbitrary constants and (there being $2n+1$ linearly independent spherical harmonics of degree n) it is always possible to determine their $2n$ ratios so that all but one of the g_s in (5.1) vanish. (9.2) was given by Heine (1878 p. 376 Schema *B*).

Now, according to Funk (1916; *cf.* also Hecke 1918 and Erdélyi 1938) the spherical harmonics are eigen-functions of any integral equation

$$(9.3)\qquad Y_n(\theta, \phi) = \lambda_n' \int f(\mu)\, Y_n(\theta', \phi')\, d\omega',$$

the nucleus of which depends only on the spherical distance

$$(9.4)\qquad \mu = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'),$$

$f(\mu)$ is an arbitrary function of μ the modulus of which is Lebesgue-integrable in $(-1, 1)$, and $d\omega'$ is the element of the surface of the unit sphere over which the integration has to be extended.

Introducing

$$\cos\theta' = k \operatorname{sn}\beta' \operatorname{sn}\gamma',\ \sin\theta'\cos\phi' = \frac{1}{k'}\operatorname{dn}\beta'\operatorname{dn}\gamma',\ \sin\theta'\sin\phi' = i\frac{k}{k'}\operatorname{cn}\beta'\operatorname{cn}\gamma',$$

we have

$$d\omega' = ik^2(\operatorname{sn}^2\beta' - \operatorname{sn}^2\gamma')\, d\beta'\, d\gamma'$$

and

$$(9.5)\quad \mu = k^2 \operatorname{sn}\beta\operatorname{sn}\beta'\operatorname{sn}\gamma\operatorname{sn}\gamma' + \frac{1}{k'^2}\operatorname{dn}\beta\operatorname{dn}\beta'\operatorname{dn}\gamma\operatorname{dn}\gamma' - \frac{k^2}{k'^2}\operatorname{cn}\beta\operatorname{cn}\beta'\operatorname{cn}\gamma\operatorname{cn}\gamma'.$$

Thus the integral equation becomes

$$(9.6)\qquad E_n^s(\beta)\,E_n^s(\gamma) = \lambda_n \int_K^{K+2iK'}\int_{-2K}^{2K} f(\mu)\,E_n^s(\beta')\,E_n^s(\gamma')\,(\mathrm{sn}^2\beta' - \mathrm{sn}^2\gamma')\,d\beta'\,d\gamma'.$$

Since λ_n depends on n only (and not on s) this integral equation is equally satisfied by any (composite) ellipsoidal surface harmonic

$$E_n(\beta, \gamma) = \sum_{s=0}^{2n} g_s\,E_n^s(\beta)\,E_n^s(\gamma).$$

Again the above integral equation is implicitly contained in Heine's *Handbuch* comprising so many unnoticed details which had to be re-discovered by subsequent workers. Assuming an expansion of $f(\mu)$ in a series of Legendre polynomials and using Heine's development (5.2), we have

$$(9.7)\qquad f(\mu) = \sum_{n=0}^{\infty}\sum_{s=0}^{2n} c_{ns}\,E_n^s(\beta)\,E_n^s(\beta')\,E_n^s(\gamma)\,E_n^s(\gamma'),$$

from which (9.6) follows by the well-known orthogonal property of ellipsoidal surface harmonics. Conversely, from (9.6) the orthogonal property of ellipsoidal surface harmonics and (9.7), the bilinear development of the nucleus, immediately follow.

Mathieu functions

10. When k approaches zero and at the same time n tends to infinity so that $\lim n^2 k^2 = -4\theta$, Lamé polynomials, $E_n^s(x)$, degenerate into the functions of the elliptic cylinder which will be denoted here shortly by $\mathrm{e}_s(x)$, being, in Ince's notation,

$$\mathrm{e}_{2m}(x) = \mathrm{ce}_m(x, \theta) \text{ and } \mathrm{e}_{2m-1}(x) = \mathrm{se}_m(x, \theta).$$

For small values of k we have

$$\mathrm{sn}\,x \sim \sin x,\ \mathrm{cn}\,x \sim \cos x,\ \mathrm{dn}\,x \sim 1 - \tfrac{1}{2}k^2\sin^2 x,\ \frac{1}{k'^2} \sim 1 + k^2,$$

and hence from (3.8),

$$(10.1)\qquad \cos\theta' \sim 1 - \frac{z^2}{2n^2},\qquad e^{-i(\phi'-\alpha')} \sim w,$$

where

$$(10.2)\qquad \begin{aligned} z^2 &= 4\theta\,\{\cos(x+y) - \cos(p+q)\}\{\cos(x-y) - \cos(p-q)\},\\ w^2 &= \{\cos(x+y) - \cos(p+q)\}/\{\cos(x-y) - \cos(p-q)\}.\end{aligned}$$

Now (see *e.g.* Whittaker and Watson 1927 § 17.4) for large values of n

$$P_n^m\left(1 - \frac{z^2}{2n^2}\right) \sim n^m\,J_m(z),$$

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where J_m denotes the Bessel function of the first kind, and hence

$$P_n^m (\cos \theta') \, e^{\pm im(\phi' - a')} \sim n^m J_m (z) \, w^{\pm m}.$$

Thus (3.9) yields, by the above-mentioned limiting process, the integral equation for the functions of the elliptic cylinder,

$$(10.3) \qquad \mathrm{e}_s (x) = \lambda_s \int_{-\pi}^{+\pi} J_m (z) \, w^{\pm m} \, \mathrm{e}_s (y) \, dy.$$

Here m is quite arbitrary. If m is an integer, (10.3) is valid without any restrictions on p and q. If m is not an integer, then (in order that neither z nor $w^{\pm 1}$ shall reach zero) $p + q$ and $p - q$ must be supposed not real.

Either by performing the same limiting process with Legendre functions of the second kind, or else by combining (10.3) with the corresponding equation with J_{-m}, the more general integral equation

$$(10.4) \qquad \mathrm{e}_s (x) = \lambda_s \int_{-\pi}^{+\pi} \mathscr{C}_m (z) \, w^{\pm m} \, \mathrm{e}_s (y) \, dy$$

is obtained in which $\mathscr{C}_m$ denotes any solution of Bessel's differential equation. It is supposed that $p + q$ and $p - q$ are not real.

The general form of the integral equation with exponential nucleus of the elliptic cylinder functions [*cf. e.g.* Whittaker 1915 *a*, equation (10)] is

$$(10.5) \qquad \mathrm{e}_s (x) = \lambda_s \int_{-\pi}^{+\pi} e^{i\sqrt{(8\theta)}(\cos x \cos y \cos \beta - i \sin x \sin y \sin \beta)} \, \mathrm{e}_s (y) \, dy;$$

and this can be deduced from (10.4) either by making the imaginary part of p (or q) tend to infinity and using the asymptotic formulae of Bessel functions, or by the superposition of an infinity of kernels of the type (10.4).

The first method is quite obvious and may be left to the reader. In the second case use is made of the fact that since $J_m (z) \, w^{\pm m}$ is a suitable nucleus for any m, so is any linear combination of such nuclei. Hence (β being any constant) we have the nucleus

$$\sum_{-\infty}^{+\infty} J_m (z) \, (iwe^{i\beta})^m = \exp \{\tfrac{1}{2} z \, (iwe^{i\beta} + iw^{-1} \, e^{-i\beta})\},$$

using the generating function of Bessel coefficients (Whittaker and Watson 1927 § 17.1); and this is

$$\exp \{(-8\theta)^{\frac{1}{2}} (\cos x \cos y \cos \beta - i \sin x \sin y \sin \beta - \cos p \cos q \cos \beta + i \sin p \sin q \sin \beta)\},$$

that is, apart from a constant factor the nucleus of (10.5).

11. There is a close connection between the integral equations of the preceding section and wave problems. This connection becomes clearer if we replace y by iy and take x and (the new) y to be elliptic coordinates in the plane, suitably altering the conditions imposed upon p and q. (10.5) is then a relation equivalent to the representation of plane waves in elliptic coordinates or, in other words, equivalent to the expansion of a plane wave in an infinite series of elliptic waves. Similarly (10.4) is equivalent to analysing a cylindrical wave into a series of elliptic-cylindrical waves. The twofold transition from (10.4) to (10.5) in the preceding section corresponds to the two possibilities of transition from (ordinary) cylindrical waves to plane waves, either by taking the source of the cylindrical waves to be very far away so that the wave-front becomes practically plane, or else by building up a plane wave, according to the expansion of Jacobi, by the superposition of cylindrical waves.

It would be interesting and perhaps not unprofitable to follow up these ideas, to develop the nuclei of our integral equations *qua* functions of x, y, p, q and to interpret the results in terms of wave functions. The developments obtained thus could be utilised for the investigation of the diffraction of a cylindrical wave on an elliptic cylinder, or on a slit (the limiting case of a hyperbolic cylinder).

REFERENCES TO LITERATURE.

A. Erdélyi, 1938, Die Funksche Integralgleichung der Kugelflächenfunktionen und ihre Uebertragung auf die Ueberkugel. *Math. Annalen*, **115**, 356-365.

P. Funk, 1916, Beiträge zur Theorie der Kugelfunktionen. *Math. Annalen*, **77**, 136-152.

E. Hecke, 1918, Ueber orthogonalinvariante Integralgleichungen. *Math. Annalen*, **78**, 398-404.

E. Heine, 1878, *Handbuch der Kugelfunktionen.* 2nd edition, vol. i. (Berlin.)

E. W. Hobson, 1931, *Spherical and ellipsoidal harmonics.* (Cambridge.)

E. L. Ince, 1921-22, On the connection between linear differential systems and integral equations. *Proc. Royal Soc. Edinburgh*, **42**, 43-53.

E. L. Ince, 1940 *a*, The periodic Lamé functions. *Proc. Royal Soc. Edinburgh*, **60**, 47-63.

E. L. Ince, 1940 *b*, Further investigations into the periodic Lamé functions. *Proc. Royal Soc. Edinburgh*, **60**, 83-99.

C. G. Lambe and D. R. Ward, 1934, Some differential equations and associated integral equations. *Quart. J. of Math.* (Oxford), **5**, 81-97.

J. L. Sharma, 1937, An integral equation satisfied by the Lamé functions. *Journal de Math.*, **16**, 199-203.

E. T. Whittaker, 1912, On the functions associated with the elliptic cylinder in harmonic analysis. *Intern. Congress of Math. Cambridge.* Vol. 1, 366-371.

E. T. Whittaker, 1915 *a*, On an integral equation whose solutions are the functions of Lamé. *Proc. Royal Soc. Edinburgh*, **35**, 70-77.

E. T. Whittaker, 1915 *b*, On Lamé's differential equation and ellipsoidal harmonics. *Proc. London Math. Soc.* (2), **14**, 260-268.

E. T. Whittaker, 1915 *c*, On a class of differential equations whose solutions satisfy integral equations. *Proc. Edinburgh Math. Soc.*, **33**, 14-23.

E. T. Whittaker and G. N. Watson, 1927, *Modern Analysis.* (Cambridge.)

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