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ON THE POWER MAP AND RING COMMUTATIVITY

BY
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Let R denote an associative ring with 1, let n be a positive integer, and let $k=1, 2$, or 3. The ring R will be called an (n, k)-ring if it satisfies the identities

$$(xy)^m = x^m y^m$$

for all integers m with $n \le m \le n+k-1$. It was shown years ago by Herstein (See [2], [9], and [10]) that for $n>1$, any $(n, 1)$-ring must have nil commutator ideal $C(R)$. Later Luh [12] proved that primary $(n, 3)$-rings must in fact be commutative, and Ligh and Richoux [11] recently showed that all $(n, 3)$-rings are commutative. Luh gave an example showing that $(n, 2)$-rings need not be commutative; Awtar [1] and Harmanci [5], using rather complicated combinatorial arguments, established commutativity of $(n, 1)$-rings and $(n, 2)$-rings in which the additive group R^+ is p-torsion-free for all primes $p \le n$.

The first theorem of this note improves the latter results for $(n, 2)$-rings by relaxing the torsion restrictions and, incidentally, provides a much simpler proof of Harmanci's result; and the next two theorems provide different kinds of commutativity conditions for $(n, 2)$-rings. The remainder of the paper deals with commutativity conditions for rings which are either radical over their center or satisfy the identity $x^n y - yx^n = xy^n - y^n x$ for some $n>1$.

Throughout the paper, we shall denote the commutator $xy-yx$ by $[x, y]$, the center of R by Z, and the commutator ideal by $C(R)$.

1. **Commutativity theorems for** $(n, 2)$**-rings**

Theorem 1. *Let n be any positive integer. If R is any* $(n, 2)$*-ring for which* R^+ *is n-torsion-free, then R is commutative.*

Proof. Following Ligh and Richoux, we note that $x^{n+1}y^{n+1} = (xy)^{n+1} = (xy)x^n y^n = x^n y^n xy$; hence

(1) $\qquad x[x^n, y]y^n = 0 \quad$ and $\quad x^n[x, y^n]y = 0 \quad$ for all $\quad x, y \in R.$

Replacing y by $y+1$ in the first equation of (1) and right-multiplying by y^{n-1}, we see that $x[x^n, y]y^{n-1} = 0$; a similar argument applied to the second equation of (1) gives $x^{n-1}[x, y^n]y = 0$. Repetition of this argument, together with an

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interchange of x and y in the computations involving the second equation of (1), eventually gives

(2) $$x[x^n, y]=[x^n, y]x=0 \quad \text{for all} \quad x, y \in R.$$

It follows at once that $x^n \in Z$ for all invertible elements x; and since u nilpotent implies $1+u$ is invertible, we have

(3) $$1+nu+v \in Z \quad \text{for all nilpotent elements } u,$$

where $v=\binom{n}{2}u^2+\binom{n}{3}u^3+\cdots$. Now (3) implies that for u with $u^2=0$, $nu \in Z$ and hence $u \in Z$. Proceeding inductively on the assumption that $u^j=0$ with $j<k$ implies $u \in Z$, we consider u with $u^k=0$ and note that the corresponding v satisfies $v^{k-1}=0$, so that (3) again forces $nu \in Z$ and hence $u \in Z$. Thus, all nilpotent elements are central.

Now R is an $(m, 1)$-ring for at least one $m>1$, so Herstein's result guarantees that commutators are nilpotent, hence central. It is well known, and easily provable by induction, that for rings with central commutators,

(4) $$[x^m, y]=mx^{m-1}[x, y] \quad \text{for all integers} \quad m \geq 1 \quad \text{and all} \quad x, y \in R.$$

Applying this in the case $m=n$ and recalling (2), we get

(5) $$0=x[x^n, y]=nx^n[x, y];$$

we now use the absence of n-torsion to get $x^n[x, y]=0$ for all $x, y \in R$. Finally, replacing x by $x+1$ and proceeding as at the beginning of the proof, we get $[x, y]=0$ for all $x, y \in R$.

THEOREM 2. *Let n and m be relatively prime positive integers. Then any ring R which is both an $(n, 2)$-ring and an $(m, 2)$-ring is commutative.*

Proof. The proof above needs only trivial modification. Let p and q be integers such that $1=pm+qn$. At each stage of the inductive argument involving nilpotent elements, the n and m versions of (3) show that nu and $mu \in Z$; thus $u=pmu+qnu \in Z$. Similarly, at the end of the proof we get $nx^n[x, y]=mx^m[x, y]=0$, and hence $mx^t[x, y]=nx^t[x, y]=0$, where t is the larger of m and n; thus, invoking the relative-primeness of m and n shows $x^t[x, y]=0$ and hence $[x, y]=0$ for all $x, y \in R$.

It was shown in [4] that for $n>1$, if R is a ring generated by its nth powers and if the map $x \to x^n$ is an *additive* endomorphism, then R is commutative. It is natural to inquire whether a similar result holds if the nth-power map is an endomorphism of the multiplicative semigroup—i.e. if R is an $(n, 1)$-ring. Luh's example of a non-commutative $(3, 2)$-ring shows that this is not the case, for it is a $(4, 1)$-ring generated by its fourth powers; however, for $(n, 2)$-rings, a result of this kind does hold.

THEOREM 3. *Let n be any positive integer. Then any $(n, 2)$-ring which is generated as a ring by either its n^2-powers or its $n(n+1)$-powers is commutative.*

Proof. Consider first the case of R generated by its n^2-powers. Replacing y by y^n in (2), we get $x^n(x^ny^n - y^nx^n) = (x^ny^n - y^nx^n)x^n = 0$; thus, for arbitrary nth-powers a and b we have $a^2b = aba = ba^2$. It follows at once that $a^nb = ba^n$, so that n^2-powers commute and R is commutative.

Now suppose R is generated by its $n(n+1)$-powers. By applying (2), we get

$$[x^{n+1}, y] = x[x^n, y] + [x, y]x^n = [x, y]x^n;$$

replacing x by x^n and again using (2) gives $[x^{n(n+1)}, y] = 0$ so that R is commutative.

2. **Further commutativity theorems.** The use of equation (4) in the proofs of Theorems 1 and 2 depends on the fact that $(n, 2)$-rings have nil commutator ideal. Among other classes of rings in which $C(R)$ is known to be nil are (i) rings radical over their center—i.e. rings in which some power of each element is central [7]; (ii) rings satisfying the identity $[x^n, y] = [x, y^n]$ for some $n > 1$ [4]. (The latter class includes the rings for which the nth-power map is an additive endomorphism.) The remaining theorems state sufficient conditions for full commutativity of certain of these rings. The proof of Theorem 4 is omitted, since it is very similar to those of Theorems 1 and 2.

THEOREM 4. *Let R be a ring with 1 which satisfies one of the following conditions:*

(A) *R is radical over its center and R^+ is torsion-free;*

(B) *For a fixed integer $n > 1$, R^+ is n-torsion-free; and for each $x \in R$, there exists an integer $k = k(x)$ such that $x^{n^k} \in Z$;*

(C) *For each $x \in R$, there exists a pair p, q of relatively prime positive integers for which $x^p \in Z$ and $x^q \in Z$.*

Then R is commutative.

THEOREM 5. *Let R be a ring with 1 and $n > 1$ a fixed positive integer. If R^+ is n-torsion-free and R satisfies the identity*

$$(\dagger) \qquad x^ny - yx^n = xy^n - y^nx,$$

then R is commutative.

Proof. As in our previous proofs, we show by induction that nilpotent elements are central. Note first that if u is nilpotent and y is arbitrary,

$$(6) \qquad [u, y^n] = [u^n, y]$$

and

$$(7) \qquad [1+u, y^n] = [1 + nu + \tbinom{n}{2}u^2 + \cdots u^n, y].$$

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Thus, if $u^2=0$, u commutes with all nth-powers by (6); and (7) then shows that $[nu, y]=0$ and hence $[u, y]=0$. Now suppose that if $u^j=0$ with $j<k$, then u is central; consider u with $u^k=0$. Then $u^2, u^3, \ldots, u^n$ are all central, so (6) shows u commutes with nth-powers, and (7) then yields the result that $nu\in Z$, hence $u\in Z$.

Since we now know that $C(R)\subseteq Z$, we shall routinely use equation (4) without explicit mention. In particular, the following properties of R, and hence of any homomorphic image of R, follow as in [4]:

$$n[x^n, y](x^{n(n-1)}-x^{n-1})=0 \quad \text{for all} \quad x, y\in R; \tag{8}$$

$$x^q\in Z \quad \text{for all} \quad x\in R, \quad \text{where} \quad q=n(2^n-2). \tag{9}$$

Represent R as a subdirect sum of a family $\{R_\alpha\}$ of subdirectly irreducible rings which are homomorphic images of R. Clearly, each R_α has 1, satisfies (†), has central commutator ideal, and satisfies (8) and (9); however, we cannot assume that R_α^+ is n-torsion-free. It is our immediate aim to show that each R_α satisfies the identity $[x^m-x, y^{n^2}]=0$, where $m=q(n-1)+1$, q being as in (9) above.

Let S be the heart of R_α—that is, the intersection of all non-zero ideals; and note that if d is a central zero divisor, then $Sd=0$, since the annihilator of d is a two-sided ideal and must therefore contain S. Now let a be an arbitrary zero divisor in R_α. (There is no distinction between left and right zero divisors since commutators are central.) For arbitrary $y\in R$, we get from (8) the result that $n[a^n, y](a^{n(n-1)}-a^{n-1})=0$. Multiplying this by appropriate powers of $a^{(n-1)^2}$ and subtracting, we see that

$$n[a^n, y]a^{n-1}f=0, \tag{10}$$

where $f=1-a^{(n-1)^2q}$. Let $T=\{x\in R_\alpha \mid xyf=0 \text{ for all } y\in R_\alpha\}$; note that T is a two-sided ideal and that, in view of (10) and the centrality of $C(R)$, $n[a^n, y]a^{n-1}\in T$ for all $y\in R_\alpha$. If T is non-trivial, then $S\subseteq T$; and since S annihilates central zero divisors, for each non-zero $s\in S$ we get $0=sf=s-s(a^q)^{(n-1)^2}=s$—a contradiction. Thus, $T=\{0\}$ and $n[a^n, y]a^{n-1}=0$ for all $y\in R_\alpha$. It follows that

$$[a, y^{n^2}]=[a^{n^2}, y]=[(a^n)^n, y]=n[a^n, y]a^{n(n-1)}=0 \tag{11}$$

for all $y\in R_\alpha$ and all zero divisors $a\in R_\alpha$.

Suppose now that there exists some $b\in R_\alpha$ which does not commute with n^2-powers. Then b is not a zero divisor, and there exists $r\in R_\alpha$ for which $[b, r^n]\neq 0$. For arbitrary $z\in Z$, replacing x by zx in (†) yields $(z^n-z)[x, y^n]=0$ for all $x, y\in R$; in particular, $(b^{nq}-b^q)[b, r^n]=0$, so that $b^{nq}-b^q$, and hence also $b^{q(n-1)+1}-b$ is a zero divisor. Thus, if $m=q(n-1)+1$, it follows from (11) that $[x^m-x, y^{n^2}]=0$ for all $x, y\in R_\alpha$.

It is now clear that our original ring R also satisfies the identity

$$[x^m - x, y^{n^2}] = 0. \tag{12}$$

Moreover, since R^+ is n-torsion-free, $[w, y^{n^2}] = 0 = n^2 y^{n^2-1}[w, y]$ for all $y \in R$ implies $y^{n^2-1}[w, y] = 0$ for all $y \in R$; employing the device of replacing y by $y+1$ as in our earlier proofs, we get the result that $w \in Z$. From (12) it follows that $x^m - x \in Z$ for all $x \in R$; by a well-known theorem of Herstein (See [3] or [6]), this forces R to be commutative.

Harmanci showed in [5] that if $n > 1$ and R is a ring with 1 which satisfies the identities $[x^n, y] = [x, y^n]$ and $[x^{n+1}, y] = [x, y^{n+1}]$, then R must be commutative. The methods of our last proof yield the following generalization of that result.

THEOREM 6. *Let m and n be relatively prime integers greater than* 1. *If R is any ring with* 1 *satisfying both the identities $[x^m, y] = [x, y^m]$ and $[x^n, y] = [x, y^n]$, then R is commutative.*

Proof. The beginning of the proof of Theorem 5 can easily be modified to show that nilpotent elements are central under the present hypotheses. The argument for subdirectly irreducible rings can then be carried out for both m and n, yielding integers $j, k > 1$ such that R satisfies the identities $[x^j - x, y^{m^2}] = 0$ and $[x^k - x, y^{n^2}] = 0$. Letting $p(x) = (x^j - x)^k - (x^j - x)$, we see that $0 = [p(x), y^{m^2}] = m^2 y^{m^2-1}[p(x), y]$ and $0 = [p(x), y^{n^2}] = n^2 y^{n^2-1}[p(x), y]$ for all $x, y \in R$. The relative primeness of m and n yields $y^t[p(x), y] = 0$ for all $x, y \in R$, where $t = \max\{m^2 - 1, n^2 - 1\}$; and it follows as usual that $p(x)$ is central. Since $p(x)$ has form $x - x^2 q(x)$ with q having integer coefficients, the theorem of [8] shows that R is commutative.

REMARK. In Theorem 5, the restriction on n-torsion is essential. To see this, begin with Harmanci's Example 1 [5, p. 29] and use the Dorroh construction (with the ring of integers mod. 2) to obtain a ring R with 1. This ring R is non-commutative and satisfies the identity $[x^2, y] = [x, y^2]$.

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