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Extreme Pick-Nevanlinna Interpolants

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Abstract. Following the investigations of B. Abrahamse [1], F. Forelli [11], M. Heins [14] and others, we continue the study of the Pick-Nevanlinna interpolation problem in multiply-connected planar domains. One major focus is on the problem of characterizing the extreme points of the convex set of interpolants of a fixed data set. Several other related problems are discussed.

Introduction

Let $z_0, \dots, z_n$ be $n+1$ distinct points in a bounded domain Ω in the complex plane. If $n+1$ complex numbers $w_0, \dots, w_n$ are given, the classic Pick-Nevanlinna interpolation problem is to determine whether there is an analytic function f on Ω that is bounded by one and that interpolates this data: that is,

$$f(z_j) = w_j, \quad j = 0, \dots, n \text{ and } \|f\|_\infty \le 1. \tag{1}$$

The linear fractional transformation $w \to \frac{1+w}{1-w}$ converts the class of analytic functions bounded by one on Ω into the class $H_+(\Omega)$ of analytic functions with positive real part on Ω. Hence, the interpolation problem (1) is equivalent to determining if the set $\mathbb{M} = \mathbb{M}(\zeta_0, \dots, \zeta_n)$ defined by

$$\mathbb{M} = \left\{ g \in H_+(\Omega) : g(z_j) = \zeta_j = \frac{1 + w_j}{1 - w_j}, j = 0, \dots, n \right\} \tag{2}$$

is non-empty. If this is the case, then $\mathbb{M}$ is both convex and compact.

When Ω is the open unit disc $\Delta = \{z : |z| < 1\}$, there is a simple necessary and sufficient condition that $\mathbb{M}$ be non-empty; moreover, it is also known when there is just one element in $\mathbb{M}$. For all of this and more, see, for instance, the book [6] by P. Duren or [8]. Under the normalizations that $z_0 = 0$ and that $g(0) = 1$, M. Heins [14] demonstrated that if $\mathbb{M}$ has more than one element, then its extreme points are precisely those members of $\mathbb{M}$ that map Δ onto the right half-plane with constant valence k, where k is any integer between $n + 1$ and $2n + 1$.

We also consider the more general case when Ω is a domain whose boundary Γ consists of $p + 1$ disjoint analytic simple closed curves, $\Gamma_0, \dots, \Gamma_p$; the case $p = 0$ corresponds, of course, to the unit disc Δ. For a multiply-connected domain Ω of this type, B. Abrahamse [1] determined the necessary and sufficient condition that $\mathbb{M}$ is non-empty and, as well, when there is just one interpolant; see also [8]. F. Forelli [11] characterized the extreme points of $\mathbb{M}$ in the case when $n = 0$ (and $\zeta_0 = 1$). His result is that the extreme

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points of $\mathbb{M}$ are also functions of constant valence $p + 1$. We investigate both of these cases further. Our analysis is most complete and our conclusions sharpest on the open unit disc and so we devote Section 1 to this. Sections 2 and 3 contain extensions and generalizations of these results to finitely-connected domains.

The Pick-Nevanlinna theorem in the unit disc has applications in circuit theory [5] and the theory of n-widths of sets of analytic functions [10], among other places. It would be interesting to see if the extensions elaborated here for multiply-connected domains have similar applications.

1 The Unit Disc

Δ denotes the open unit disc in the complex plane and $\mathbf{T}$ its boundary, the unit circle. H^∞ denotes the space of bounded analytic functions on Δ with the supremum norm. H^1 consists of those analytic functions f on Δ for which the quantity

$$\sup\left\{\int_0^{2\pi} |f(re^{i\theta})|\, d\theta : 0 < r < 1\right\}$$

is finite. Each $f \in H^1$ has boundary values on $\mathbf{T}$ and we may equivalently describe H^1 as those functions $f \in L^1(T, d\theta)$ whose negative Fourier coefficients are zero. Yet again, H^1 consists of those $L^1(T, d\theta)$ functions that have an analytic extension to the open unit disc Δ. H^1_0 consists of those elements of H^1 that vanish at the origin (or whose mean-value over $\mathbf{T}$ is zero.) The quotient space L^1/H^1_0 is the pre-dual of H^∞ when all the spaces in question are viewed on the unit circle $\mathbf{T}$. Finally, a function in H^∞ is *inner* if its boundary values have modulus one a.e.

Definition A Blaschke product of degree m, $m \geq 1$, is an analytic function B of the form

$$B(z) = \lambda \prod_{j=1}^{m} \frac{z - a_j}{1 - \overline{a}_j z}, \quad |\lambda| = 1, \quad a_j \in \Delta, \quad j = 1, \dots, m. \tag{3}$$

By definition a Blaschke product of degree 0 is a unimodular constant. We denote the set of all Blaschke products of degree n or less by $\mathfrak{B}_n$. $\mathfrak{B}_n$ is compact in the topology of uniform convergence on compact subsets of Δ.

Proposition 1

(a) *Let $R > 1$ and let $\Omega = \{z : 1/R < |z| < R\}$. Suppose that G is analytic on Ω. Then there is a unique function h that is the best approximation to G in $L^1(T, d\theta)$ from H^1_0; further, h is analytic in the disc $\{z : |z| < R\}$. Moreover, there is a Blaschke product B of some finite degree such that $B(G+h) \geq 0$ on the unit circle* T. *Any zero of $G+h$ on* T *has even order.*

(b) *Suppose $u \in L^1(T, d\theta)$ is not in H^1_0. If there is an H^1_0 function h and an inner function $I \in H^\infty$ with*

$$I(u + h) \geq 0 \quad \text{a.e. } d\theta \text{ on } \mathrm{T} \tag{4}$$

then (i) h is the best approximation in $L^1(T, d\theta)$ to u from H^1_0 and (ii) I is the unique solution to the extremal problem: $\sup\{\mathrm{Re} \int uf\, d\theta : f \in H^\infty, \|f\|_\infty \leq 1\}$.

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Proof (a) The existence and uniqueness of the best approximation in $L^1(T, d\theta)$ to G from H^1_0 is standard; see [6; Chapter 8]. If h is this best approximation, then there is an analytic function $B \in H^\infty$ with

$$B(G+h) \geq 0 \text{ and } |B| = 1, \quad \text{both a.e. } d\theta. \tag{5}$$

It now follows that both B and $G+h$ are analytic across the unit circle T (see, for instance, [22; Lemma 4.5]) and hence that B is a finite Blaschke product of some degree. Since G is analytic in the region $\{z : 1/R < |z| < 1\}$, the reflection principle establishes that $B(G+h)$ is analytic in the region $\{z : 1 < |z| < R\}$. But G is already analytic in this same region and B is rational with no zeros in this same region. Hence, $G+h$ and, therefore h, are both analytic in this region. Moreover, $B(G+h) \geq 0$ on T and hence any zeros of $B(G+h)$ on T have even order. But B has no zeros on T and so the zeros of $G+h$ (if there are any) have even order.

(b) For $f \in H^\infty$ and $\|f\| \leq 1$, we have

$$\operatorname{Re} \int uf \, d\theta = \operatorname{Re} \int (u+h) f \, d\theta \leq \int |u+h| \, d\theta = \int I(u+h) \, d\theta = \operatorname{Re} \int uI \, d\theta.$$

Moreover, for any $g \in H^1_0$, $\int |u+g| \, d\theta \geq \int I(u+g) \, d\theta = \int I(u+h) \, d\theta = \int |u+h| \, d\theta$.

The following result, which characterizes the boundary points of Λ when the domain Ω is the open unit disc Δ is well-known; see [18] and the references therein. We give a proof that highlights the role of the number of zeros of particular functions associated with the solution that will be important in Theorem 3.

Theorem 2 *Let $z_0, \ldots, z_n$ be $n+1$ distinct points in $\Delta \setminus \{0\}$ and set*

$$\Lambda = \{\big(f(z_0), \ldots, f(z_n)\big) : \|f\|_\infty \leq 1\}.$$

A point $P = \big(F(z_0), \ldots, F(z_n)\big)$ lies in the boundary of Λ if and only if F is a Blaschke product of degree n or less.

Proof P is in the boundary of Λ if and only if there are complex scalars $c_0, \ldots, c_n$ not all of which are zero with

$$\operatorname{Re} \sum_{i=0}^{n} c_i F(z_i) \geq \operatorname{Re} \sum_{i=0}^{n} c_i f(z_i) \quad \text{for all } f \text{ in the unit ball of } H^\infty.$$

By replacing f by unimodular multiples of itself, we see that the quantity $\operatorname{Re} \sum_{i=0}^{n} c_i F(z_i)$ is positive. Set

$$G(z) = \sum_{i=0}^{n} c_i \frac{z}{z - z_i} \tag{6}$$

so that the inequality in the third line of the proof may be rewritten as

$$\int_{\mathrm{T}} F(e^{i\theta}) G(e^{i\theta}) \, d\theta \geq \operatorname{Re} \int_{\mathrm{T}} f(e^{i\theta}) G(e^{i\theta}) \, d\theta, \quad \text{for all } f \text{ in the unit ball of } H^\infty.$$

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The kernel G is analytic in an annular region $r_0 < |z| < 1/r_0$, $r_0 = \max|z_i|$. Let h be the best approximation to G in $L^1(T, d\theta)$ from H_0^1 and let B be the Blaschke product from Proposition 1(a). Proposition 1(b) implies that $F = B$. Let $\lambda_1, \ldots, \lambda_p$ be the distinct zeros of $G + h$ on T of respective orders $2m_1, \ldots, 2m_p$; let $m = \sum_{i=1}^{p} m_i$. Finally, consider

$$R(z) = B(z)\big(G(z) + h(z)\big) \prod_{i=1}^{p} \left(\frac{z}{(z - \lambda_i)(1 - \overline{\lambda_i} z)} \right)^{m_i}.$$

R is rational on a neighborhood of the closed unit disc and positive on the unit circle T. The argument principle then implies that R has equally many zeros as poles in Δ. However, R has as many poles as there are non-zero coefficients c_i and so certainly no more than $n + 1$. $G + h$ has at least one zero at the origin and s zeros on $\Delta \setminus \{0\}$. Hence, R has at least $m + 1$ zeros at the origin and s other zeros on $\Delta \setminus \{0\}$; B has d zeros on Δ. This gives $s + m + 1 + d \leq n + 1$ or

$$s + m + d \leq n. \tag{7}$$

Evidently, (7) implies that $d \leq n$. This shows that each point in the boundary of Λ arises from a Blaschke product of degree at most n.

To prove the converse, let B be a Blaschke product of degree n or less. Suppose that the point $P = \big(B(z_0), \ldots, B(z_n)\big)$ lies in the the interior of Λ. Then there is a scalar $\rho > 1$ so that $\rho P \in \partial\Lambda$. By the first part of the theorem, there is a Blaschke product C of degree n or less with $C(z_j) = \rho B(z_j)$, $j = 0, \ldots, n$. However, $|(C - \rho B) + \rho B| = 1 < \rho = |\rho B|$ on the unit circle T. By Rouché's theorem, $C - \rho B$ and ρB have equally many zeros in Δ. But $C - \rho B$ has at least $n + 1$ zeros while B has at most n. This contradiction establishes that $P \in \partial\Lambda$.

Theorem 3 *Let $\mathcal{T}_P$ be the set of supporting hyperplanes at a point $P = \big(B(z_0), \ldots, B(z_n)\big) \in \partial\Lambda$. The degree of B is n if and only if $\mathcal{T}_P$ has a single element (up to scalar multiples). If B has degree d, $d < n$, then $\mathbf{c} \in \mathcal{T}_P$ is an extreme point of $\mathcal{T}_P$ if and only if $G + h$ has no zeros on Δ and $2(n - d)$ zeros on the unit circle $\boldsymbol{T}$.*

Proof We note first that if the degree of B is precisely n, then (7) shows that $G + h$ has no zeros in $\Delta \setminus \{0\}$ and none on the unit circle T and all the coefficients c_i must be non-zero. Suppose now that B has degree exactly n and that $\mathbf{c}, \mathbf{d} \in \mathbb{C}^{n+1}$ both give supporting hyperplanes at P with $\sum_{j=0}^{n} c_j B(z_j) = \sum_{j=0}^{n} d_j B(z_j) = \mu \geq \operatorname{Re} \sum_{j=0}^{n} c_j f(z_j)$ for all f in the unit ball of H^∞. Let

$$G_1(z) = \sum_{j=0}^{n} c_j \frac{z}{z - z_j} \quad \text{and} \quad G_2(z) = \sum_{j=0}^{n} d_j \frac{z}{z - z_j}.$$

Let h_1, h_2 be the best approximation to G_1, G_2, respectively, from H_0^1. Then $B(G_i + h_i) > 0$ on the unit circle T, $i = 1, 2$, all the scalars $c_0, \ldots, c_n$ and $d_0, \ldots, d_n$ are non-zero, and both $G_1 + h_1$ and $G_2 + h_2$ are zero-free in Δ. Hence, $(G_1 + h_1)/(G_2 + h_2)$ is analytic on Δ (their poles cancel) and positive on the unit circle T. Thus, this function is identically constant.

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The constant must be one since $G_1 + h_1$ and $G_2 + h_2$ have the same L^1 norm. From this, it follows easily that $c_j = d_j$, $j = 0, \dots, n$. Suppose that $P = \big(B(z_0), \dots, B(z_n)\big)$ where B has degree d, $d < n$. Assume first that $G + h$ has $2(n - d)$ zeros on the unit circle $\mathbf{T}$ (and hence none in Δ). If $G + h = \frac{1}{2}(G_1 + h_1) + \frac{1}{2}(G_2 + h_2)$ where $G_i + h_i \in \mathcal{T}_P$, $i = 1, 2$, then $G_1 + h_1$ and $G_2 + h_2$ have the same argument on $\mathbf{T}$ and simple division shows that both $G_1 + h_1$ and $G_2 + h_2$ vanish at the zeros of $G + h$ on $\mathbf{T}$ to at least the same order as $G + h$. Hence, the zeros have exactly the same order (since $s + m + d = n$) and so $(G_1 + h_1)/(G + h)$ is analytic on Δ and postive on $\mathbf{T}$. Thus, this rational function is constant and because the L^1 norms of these functions are the same, the functions coincide. Therefore, $G = G_1 = G_2$ and $G + h$ is an extreme point of $\mathcal{T}_P$.

Suppose next that B has degree $d < n$. Let $\xi_0, \dots, \xi_d$ be any $d + 1$ points from among $z_0, \dots, z_n$. By Theorem 2, there are scalars $c_0, \dots, c_d$ (none of which are zero) such that

$$\sum_{j=0}^{d} c_j B(\xi_j) \geq \operatorname{Re} \sum_{j=0}^{d} c_j f(\xi_j)$$

for all f in the unit ball of H^∞. Since the points $\xi_0, \dots, \xi_d$ may be chosen in $\binom{n+1}{d+1}$ ways, there are many different supporting hyperplanes at $\big(B(z_0), \dots, B(z_n)\big)$.

To see the assertion about the extreme points, let B have degree d, $d < n$, and suppose that $\mathbf{c}$ gives an extreme point of $\mathcal{T}_P$. The function $R = B(G+h)$ is rational and non-negative on the unit circle. Hence, it has the form

$$R(z) = Az \frac{\prod_{j=1}^{d}(z - \zeta_j)(1 - z\overline{\zeta}_j) \prod_{k=1}^{n-d}(z - w_k)(1 - z\overline{w}_k)}{\prod_{i=0}^{n}(z - z_i)(1 - z\overline{z}_i)} \tag{8}$$

where $\zeta_1, \dots, \zeta_d$ are the zeros of B, $w_1, \dots, w_{n-d}$ are some points in the closed unit disc, and $A > 0$. For a polynomial $P = Az^m \prod_{j=1}^{d}(z - z_j)$, $z_j \neq 0$, we introduce the notation

$$P^*(z) = z^{d+m}\overline{P(1/\overline{z})} = \overline{A} z^m \prod_{j=1}^{d} (1 - z\overline{z_j}). \tag{9}$$

Let $Q(z) = \prod_{k=1}^{n-d}(z - w_k)$; we write $Q = Q_1 Q_2$ where all the zeros of Q_1 lie on $\mathbf{T}$ and all the zeros of Q_2 lie in Δ. Suppose the degree of Q_2 is $r > 0$; consequently, there is a positive number δ so that $|Q_2|^2 - \delta \geq 0$ on $\mathbf{T}$. Let $\tau(e^{it}) = a_0 + a_r \cos rt$ where a_0, a_r are real, not both zero, and chosen so that

$$|Q_2|^2 \pm \tau \geq 0 \quad \text{on } \mathbf{T} \text{ and } \int_{\mathbf{T}} \tau |Q_1|^2 \, |D| = 0 \tag{10}$$

where

$$D(z) = Az \frac{\prod_{j=1}^{d}(z - \zeta_j)(1 - z\overline{\zeta}_j)}{\prod_{i=0}^{n}(z - z_i)(1 - z\overline{z}_i)}.$$

By the Riesz-Fejer theorem, there are polynomials (in z) S_1 and S_2 of degree r with all their zeros in Δ such that $|Q_2|^2 + \tau = |S_1|^2$ and $|Q_2|^2 - \tau = |S_2|^2$ on $\mathbf{T}$. Hence,

$$S_1(e^{it})S_1^*(e^{it}) = e^{irt}|S_1(e^{it})|^2 = Q_2(e^{it})Q_2^*(e^{it}) + e^{irt}\tau(e^{it}) \tag{11}$$

and

$$S_2(e^{it})S_2^*(e^{it}) = e^{irt}|S_2(e^{it})|^2 = Q_2(e^{it})Q_2^*(e^{it}) - e^{irt}\tau(e^{it}). \tag{12}$$

Define $R_1 = DQ_1Q_1^*S_1S_1^*$ and $R_2 = DQ_1Q_1^*S_2S_2^*$. Then R_1 and R_2 both give supporting hyperplanes at P, $\frac{1}{2}(R_1+R_2) = R$, and $R_j = B(G_j+h_j)$, $j = 1, 2$ for appropriate coefficients and functions $h_1, h_2 \in H_0^1$. This contradiction shows that Q_2 must be constant; that is, all the zeros of $G+h$ lie on the unit circle if $\mathbf{c}$ is an extreme point of $\mathcal{T}_P$. To complete the proof we note that if Q_1 has $r < n-d$ zeros on $\mathbf{T}$, then there is a non-constant polynomial S of degree $n-d-r$ with $1 \pm |S|^2 \geq 0$ and so (just as above) there are polynomials S_1, S_2 of degree $n-d-r$ with all their zeros in Δ so that $|S_1|^2 = (1+|S|^2)$ and $|S_2|^2 = (1-|S|^2)$; set $R_1 = DQ_1Q_1^*S_1S_1^*$ and $R_2 = DQ_1Q_1^*S_2S_2^*$; then $\frac{1}{2}(R_1+R_2) = R$. This contradicts the fact that R is an extreme point.

Conversely, suppose that $G+h$ has $n-d$ zeros on $\mathbf{T}$ (and hence no zeros in $\Delta \setminus 0$). If $G+h = \frac{1}{2}(G_1+h_1) + \frac{1}{2}(G_2+h_2)$ where both G_1+h_1 and G_2+h_2 produce supporting hyperplanes at P, then $B(G_1+h_1) \geq 0$ and $B(G_2+h_2) \geq 0$ on $\mathbf{T}$ and thus the zeros of both G_1+h_1 and G_2+h_2 lie at the zeros of $G+h$. This also implies that G_1+h_1 and G_2+h_2 have no zeros in Δ. The quotient $(G_1+h_1)/(G_2+h_2)$ is therefore analytic in a neighborhood of the closed disc and real (in fact, positive) on the unit circle. Thus, it is constant and so G_1+h_1 and G_2+h_2 are both multiples of $G+h$.

2 Finitely-Connected Domains

Let Ω be a bounded domain whose boundary Γ consists of $p+1$ disjoint analytic simple closed curves. We fix a point $t_0 \in \Omega$ and let ω denote harmonic measure on Γ for t_0. On Γ we have

$$d\omega = \frac{i}{2\pi}Q'(z)\,dz \tag{13}$$

where $Q = G+iH$, G is the Green's function for Ω with pole at t_0, and H is the harmonic conjugate of G; see [8, p. 89]. Q' has precisely p zeros in Ω at, say, $\{\zeta_1, \ldots, \zeta_p\}$; these are called the *critical points of G*. Q' has a single pole of order one at t_0. H^∞ denotes the space of bounded analytic functions on Ω with the supremum norm. Each function in H^∞ has boundary values a.e. ω on Γ. H^1 consists of those analytic functions f on Ω whose modulus has a harmonic majorant on Ω. The norm of f is the value of its (unique) least harmonic majorant at the point t_0. Each $f \in H^1$ has boundary values a.e. ω on Γ and the mapping from f to its boundary values is an isometry of H^1 onto a closed subspace of $L^1(\Gamma,\omega)$. Hence, we may equivalently describe H^1 as those functions $f \in L^1(\Gamma,\omega)$ that have an analytic extension to Ω. H_0^1 consists of those functions $f \in H^1$ that vanish at t_0; equivalently, the mean-value of f over Γ with respect to ω is zero. Here a significant

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difference between Δ and Ω appears: there is a linear space N of dimension p that consists of all bounded measurable functions u on Γ that satisfy

$$\int_\Gamma u \operatorname{Re}(h)\, d\omega = 0 \tag{14}$$

for all $h \in H^1$. N is the *Schottky space* of Ω and is spanned by p functions $Q_1, \ldots, Q_p$ called the *Schottky functions*. Each Q_j is real on Γ and has a meromorphic extension to a neighborhood of the closure of Ω. Indeed, we can be more specific. Set $P_0(z) = \prod_{j=1}^p (z - \zeta_j)$ where $\{\zeta_1, \ldots, \zeta_p\}$ are the critical points of the Green's function with pole at t_0. Then

$$Q_k = H_k / P_0, \quad k = 1, \ldots, p \tag{15}$$

where H_k is analytic on the closure of Ω and vanishes at t_0. The predual of H^∞ is $L^1(\Gamma, \omega)/(N + H_0^1)$ when all these spaces are considered on Γ.

Definition A Blaschke product of degree r on Ω is a bounded analytic function B whose modulus satisfies

$$-\log|B(z)| = \sum_{k=1}^r g(z; w_k) \tag{16}$$

where $g(z; w)$ is the Green's function for Ω with pole at $w \in \Omega$. We let $\mathfrak{B}_r$ denote the set of Blaschke products of degree r or less.

In contrast to the open unit disc Δ, the location and number of the zeros of a finite Blaschke product on Ω are not arbitrary. The radial Cauchy-Riemann equations imply that the argument of finite Blaschke product is an increasing function on each component of the boundary of Ω. Hence, the argument must increase by an integer multiple of 2π. Thus, it is necessary that $r \geq p + 1$. Next, the increase in the argument of B along a component Γ_j of Γ is $2\pi \sum_{k=1}^r \omega_j(w_k)$ where $\omega_j(w)$ is the harmonic measure for Γ_j relative to $w \in \Omega$; that is, the value at w of the harmonic function whose boundary values are 1 on Γ_j and zero on the other components of Γ. Hence, in order to be single-valued it is necessary (and evidently sufficient) that

$$\sum_{k=1}^r \omega_j(w_k) \text{ is a positive integer}, \quad j = 0, \ldots, p. \tag{17}$$

Quite clearly, (17) can not hold for all selections of points $w_1, \ldots, w_r$ in Ω even when $r \geq p + 1$.

The *double* of Ω, denoted by $\widehat{\Omega}$, is formed by gluing a second copy Ω^* of Ω to Ω along their common edges. $\widehat{\Omega}$ is a compact Riemann surface of genus p. A meromorphic function h on Ω that is real-valued on Γ extends to be meromorphic on $\widehat{\Omega}$ by $f(z^*) = \overline{f(z)}$. Likewise, if g is meromorphic (or analytic) on a neighborhood of the closure of Ω and unimodular on Γ, then g has an extension to $\widehat{\Omega}$ given by the rule $g(z^*) = 1/\overline{g(z)}$.

The following theorem is a partial analogue of Theorem 3. Its main result is well-known; see [13], [18], and [8, Theorem 5.4.1, p. 130].

Theorem 4 *Let $z_0, \dots, z_n$, $n \geq 1$ be distinct points in $\Omega \setminus \{\zeta_1, \dots, \zeta_p\}$ and set*

$$\Lambda = \{ \big(f(z_0), \dots, f(z_n) \big) : \|f\|_\infty \leq 1 \}. \tag{18}$$

A point $P = (w_0, \dots, w_n)$ lies in the boundary of Λ if and only if there is exactly one function in the unit ball of H^∞ that interpolates the data $w_0, \dots, w_n$. If this is the case, then $P = \big(F(z_0), \dots, F(z_n)\big)$ where either F is a unimodular constant or F is a Blaschke product of degree at most $n + p$. If the degree of F is $p + n$, then there is a unique tangent functional to the boundary of Λ at $P = \big(F(z_0), \dots, F(z_n)\big)$.

Proof The first equivalence is Theorem 5.4.1, p. 130 of [8]. Let us assume that P is not the same unimodular constant repeated $n + 1$ times. If $P = \big(F(z_0), \dots, F(z_n)\big)$ lies in the boundary of Λ, then there are scalars $c_0, \dots, c_n$, not all of which are zero, with

$$\operatorname{Re} \sum_{j=0}^{n} c_j F(z_j) \geq \operatorname{Re} \sum_{j=0}^{n} c_j f(z_j), \quad \|f\|_\infty \leq 1. \tag{19}$$

Let

$$G(z) = \sum_{j=0}^{n} c_j \frac{1}{z - z_j}.$$

Use the Cauchy integral formula and the relationship $Q' dz = -2\pi i \, d\omega$, to rewrite this as

$$\int_\Gamma (GF/Q') \, d\omega \geq \left| \int_\Gamma (Gf/Q') \, d\omega \right|, \quad \|f\|_\infty \leq 1. \tag{20}$$

Let u be the best approximation to G/Q' in $L^1(\Gamma, \omega)$ from $H^1_0 + N$. Then

$$F(G/Q' + u) \geq 0 \text{ a.e. } d\omega \text{ on } \Gamma. \tag{21}$$

However, we know that $u = h/Q'$ where $h \in H^1_0$. Thus, we learn that

$$F(G + h)/Q' \geq 0 \text{ a.e. } d\omega \text{ on } \Gamma. \tag{22}$$

It is standard that (22) then implies that h and F are analytic in a neighborhood of the closure of Ω and that F is unimodular on Γ; that is, F is a finite Blaschke product of some degree d. Let $G + h$ have $2m$ zeros on Γ and s' zeros on Ω. Let n' be the number of non-zero coefficients c_i, so that $n' \leq n + 1$ and n' is the degree of the rational function G. Then $R = (G + h)/Q'$ has $n' + p$ poles, s' zeros on Ω, and $2m$ zeros on Γ. Because FR is meromorphic on the double $\widehat{\Omega}$, a compact Riemann surface, we find that

$$2m + 2(s' + d) = 2(n' + p). \tag{23}$$

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Hence, $s' + d + m \leq p + n + 1$. We let s be the number of zeros of R on $\Omega \setminus t_0$ so that $s' \geq s + 1$ and we obtain

$$s + d + m \leq p + n. \tag{24}$$

Evidently, this implies that $d \leq p + n$. Since a Blaschke product on Ω is single-valued only if its degree is $p + 1$ or more, we see that each boundary point of Λ arises from a Blaschke product of degree d, $p + 1 \leq d \leq p + n$.

Suppose now that the degree of F is exactly $p + n$. Then (23) implies that $m = 0$, $s' = 1$ and $n' = n + 1$. If there is another supporting hyperplane at P, then the corresponding rational function R_1 has no zeros in $\Omega \setminus t_0$ and, as well, none on Γ. The ratio R/R_1 is then analytic on Ω and positive on Γ and therefore constant.

Example 1 When $p \geq 1$, the converse implication of Theorem 4 may fail; that is, it is possible to find a Blaschke product B of degree $p + n$ so that the point $P = \big(B(z_0), \ldots, B(z_n)\big)$ lies in the interior of Λ rather than on the boundary. For instance, on Ω there is a Blaschke product B of degree $p + 1$; one such Blaschke product is the Ahlfors function; see the end of Section 3 or [8, Section 5.1]. Take $n = p$ and let the points $z_0, \ldots, z_p$ be the zeros of B. Then the degree of B is $p + 1 \leq 2p = n + p$ while $P = \big(B(z_0), \ldots, B(z_n)\big) = (0, \ldots, 0)$ lies in the interior of Λ. (Recall that the origin is always interior to Λ since, for instance, it has many different interpolants from the unit ball of H^∞; *cf.* Theorem 4.)

Remarks 1. It would be very interesting to give an intrinsic characterization of those Blaschke products B of degree $n + p$ or less for which the point $P = \big(B(z_0), \ldots, B(z_n)\big)$ lies in the boundary of Λ.

2. Suppose that $n \geq p + 1$. A simple application of Rouché's theorem shows that if B has degree n or less, then $P = \big(B(z_0), \ldots, B(z_n)\big)$ lies in the boundary of Λ.

3. A different way of formulating the Pick-Nevanlinna interpolation problem on multiply-connected domains is explored in [9].

The convex compact subset Λ of $\mathbb{C}^{n+1}$ defined in (18) is carried homeomorphically onto a closed (unbounded) convex subset Λ' in $\mathbb{C}^{n+1}$ by the map

$$\Phi(w_0, \ldots, w_n) = \left(\frac{1 + w_0}{1 - w_0}, \ldots, \frac{1 + w_n}{1 - w_n}\right).$$

Moreover, each point of Λ' arises from an analytic function g whose real part is positive on Ω and, conversely, each such function gives a point of Λ'. The homeomorphism carries the boundary of Λ onto the boundary of Λ'. Thus, investigating those Blaschke products that give rise to boundary points of Λ is equivalent to investigating those analytic functions with positive real part that give rise to boundary points of Λ'. This is what we now set out to do.

We begin with a discussion of the Poisson kernel for a point $z \in \Omega$. Let $d\omega$ be the harmonic measure on Γ for the point t_0; then the harmonic measure $d\omega_z$ for a point $z \in \Omega$ has the form

$$d\omega_z(\xi) = P(\xi, z)\, d\omega(\xi) = \left(\frac{1}{\xi - z} + F_z(\xi)\right) \frac{d\xi}{2\pi i}$$

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where F_z is analytic on a neighborhood of the closure of Ω; for this, see [8]. We now apply (13) to obtain

$$P(\xi, z) = \frac{1}{Q'(\xi)}\left(\frac{1}{\xi - z} + F_z(\xi)\right).$$

This implies that $P(\xi, z)$ has a meromorphic extension to $\widehat{\Omega}$ with a zero at t_0 and poles at z and the critical points of the Green's function $\{\zeta_1, \ldots, \zeta_p\}$ and corresponding zeros and poles at the reflections of these points.

Let $u(z)$ be a positive harmonic function on Ω. Then there is a unique positive measure μ on the boundary Γ of Ω such that

$$u(z) = \int_\Gamma P(\xi, z)\, d\mu(\xi). \tag{25}$$

If $u = \operatorname{Re} g$ where g is analytic on Ω, then there is another restriction on μ. Because g is analytic on Ω, u has a single-valued harmonic conjugate on all of Ω and so we must have

$$\int_\Gamma Q_k(\xi)\, d\mu(\xi) = 0, k = 1, \ldots, p \tag{26}$$

where $Q_1, \ldots, Q_p$ are the Schottky functions described earlier. The function g therefore has the representation

$$g(z) = \int_\Gamma \mathcal{P}(\xi, z)\, d\mu(\xi) \tag{27}$$

where

$$\mathcal{P}(\xi, z) = P(\xi, z) + i\widetilde{P}(\xi, z) \tag{28}$$

and $\widetilde{P}(\xi, z)$ is the real-valued function on Γ satisfying

$$\int_\Gamma \big(\operatorname{Re} f(\xi)\big)\widetilde{P}(\xi, z)\, d\omega_0(\xi) = \operatorname{Im} f(z), \quad f \in H^2(\Omega) \text{ and } \operatorname{Im} f(t_0) = 0. \tag{29}$$

The function $\widetilde{P}(\xi, z)$ has the form:

$$\widetilde{P}(\xi, z) = \frac{1}{Q'(\xi)}\left(\frac{i}{\xi - z} + H(\xi)\right) \tag{30}$$

where H is analytic on the closure of Ω. To see this, note that (29) gives

$$\int_\Gamma h(\xi)\widetilde{P}(\xi, z)\, d\omega(\xi) = -ih(z), \quad h \in H_0^2(\Omega)$$

and so $\widetilde{P} - iP$ is orthogonal to H_0^2 and therefore has the form

$$\widetilde{P} - iP = g + \sum_{j=1}^{p} c_j Q_j$$

where g lies in H^2. When we use (15) and the known form of P, we obtain (30). In particular, we learn that $\widetilde{P}(\xi, z)$ has a meromorphic extension to $\widehat{\Omega}$ with poles at $\{z, \zeta_1, \dots, \zeta_p\}$, a zero at t_0, and corresponding poles and zero at the reflections of these points, since $\widetilde{P}(\xi, z)$ is real on Γ.

Notations and Remarks (1) $\mathcal{M}_0^+$ denotes the convex cone of those non-negative measures on Γ that satisfy the p homogeneous conditions (26). We shall henceforth assume that all the analytic functions g with non-negative real part on Ω are normalized by the condition $\operatorname{Im} g(t_0) = 0$. With this assumption, every analytic function on Ω whose real part is positive is obtained from an element of $\mathcal{M}_0^+$ by convolution with the family of Poisson kernels and visa versa.

(2) We shall assume that the first interpolation point z_0 coincides with the base point t_0; this does not reduce the generality of our results but does simplify some of the notation since the Poisson kernel for t_0 is identically 1. This assumption and (1) imply that the first entry in the $(n+1)$-tuple used in determining Λ' is a positive real number.

(3) Recall again that the points $z_0, \dots, z_n$ lie in $\Omega \setminus \{\zeta_1, \dots, \zeta_p\}$. We define $\mathcal{S}$ to be the real span of the $2n + p + 1$ linearly independent functions

$$1, P(\xi, z_1), \dots, P(\xi, z_n), \widetilde{P}(\xi, z_1), \dots, \widetilde{P}(\xi, z_n), Q_1(\xi), \dots, Q_p(\xi).$$

Every function in $\mathcal{S}$ is real on Γ and has a meromorphic extension to $\widehat{\Omega}$ with at most simple poles among the points $z_1, \dots, z_n, \zeta_1, \dots, \zeta_p$ and their reflections. Conversely, if h is meromorphic on $\widehat{\Omega}$ with simple poles among $z_1, \dots, z_n, \zeta_1, \dots, \zeta_p$ and their reflections and h is real-valued on Γ, then $h \in \mathcal{S}$.

The following result is a special case of Lemma 2 of [18].

Proposition 5 *There are $p + 1$ points $x_0, \dots, x_p$ in Γ with this property: each p-tuple of real numbers $(r_1, \dots, r_p)$ has the form*

$$r_j = \sum_{k=0}^{p} c_k Q_j(x_k), \quad j = 1, \dots, p$$

for some choice of non-negative scalars $c_0, \dots, c_p$. There is a constant M with the property that $\sum c_j^2 \le M \sum r_j^2$.

Theorem 6 *A point $P = (\xi_0, \xi_1, \dots, \xi_n)$, $\xi_0 > 0$, lies in the boundary of Λ' if and only if there is a unique g with positive real part on Ω with $g(z_j) = \xi_j$, $j = 0, \dots, n$. If this is the case, then the measure $\lambda \in \mathcal{M}_0^+$ corresponding to g is supported in the set of zeros in Γ of a function $h \in \mathcal{S}$ that is non-negative on Γ. Conversely, suppose that the measure $\lambda \in \mathcal{M}_0^+$ is supported in the zero set of a function $h \in \mathcal{S}$ that is non-negative on Γ; let $g(z) = \int_\Gamma \mathcal{P}(\xi, z)\, d\lambda(\xi)$ be the analytic extension of λ to Ω. Then $P = \big(g(z_0), \dots, g(z_n)\big)$ lies in the boundary of Λ'.*

Proof Recall that the function $\mathcal{P}(\xi; z)$ is the complex Poisson kernel, defined in (28). Suppose that the measure $\lambda \in \mathcal{M}_0^+$ produces a boundary point of Λ'. Then there is a non-zero vector $(c_0, \dots, c_n) \in \mathbb{C}^{n+1}$ such that

$$\text{(31)} \qquad \operatorname{Re} \sum_{k=0}^{n} c_k \int_\Gamma \mathcal{P}_k \, d\lambda \le \operatorname{Re} \sum_{k=1}^{n} c_k \int_\Gamma \mathcal{P}_k \, d\rho$$

for all measures $\rho \in \mathcal{M}_0^+$. Since $\mathcal{M}_0^+$ is a cone, we evidently learn that the lefthand side of (31) is zero. Let $G = \operatorname{Re} \sum_{k=0}^{n} c_k \mathcal{P}_k$ so that $\int_\Gamma G \, d\rho \ge 0$ for all $\rho \in \mathcal{M}_0^+$. We now show that the hypotheses of Theorem 2.6.2 of [7] are valid. Let E be the space of real measures on Γ in the weak-star topology; E is an ordered vector space using the cone of non-negative measures to determine the partial order. Let M be the subspace of those measures that are orthogonal to the Schottky functions $Q_1, \dots, Q_p$. The linear functional $\ell(\mu) = \int G \, d\mu$ is non-negative on M by (31). Let $\rho \in E$; by Proposition 5 there is a non-negative measure ν such that $\rho + \nu \in M$. Hence, by Theorem 2.6.2 of [7], ℓ may be extended to a non-negative linear functional on all of E; that is, there is a non-negative continuous function h on Γ so that

$$\int G \, d\mu = \int h \, d\mu, \quad \text{for all } \mu \in M.$$

Thus, $G - h$ is a real linear combination of the Schottky functions $Q_1, \dots, Q_p$; equivalently, $h = G + H$ where H is a linear combination of $Q_1, \dots, Q_p$ and

$$\int_\Gamma (G + H) \, d\nu \ge 0, \quad \text{for all } \nu \in \mathcal{M}^+$$

with equality when $\nu = \lambda$. Moreover, $h(\xi) = \sum_{j=0}^{n} \big(a_j P(\xi, z_j) + \widetilde{a}_j \widetilde{P}(\xi, z_j) \big) + \sum_{k=1}^{p} b_k Q_k(\xi)$ for some real scalars $a_0, \dots, a_n$, $\widetilde{a}_0, \dots, \widetilde{a}_n$, $b_1, \dots, b_p$ so $h \in \mathcal{S}$ and $\operatorname{supp}(\lambda)$is a subset of the zero set of h on Γ.

Conversely, if a non-negative function h lies in $\mathcal{S}$ and the measure $\lambda \in \mathcal{M}_0^+$ is supported within the zero set of h on Γ, then

$$\text{(32)} \qquad \int_\Gamma h \, d\rho \ge 0, \quad \text{for all } \rho \in \mathcal{M}_0^+$$

and equality holds for $\lambda = \rho$. Since $h \in \mathcal{S}$ there are real numbers $a_0, \dots, a_n$, $\widetilde{a}_0, \dots, \widetilde{a}_n$ and $b_1, \dots, b_p$ so that $h = \sum_{k=0}^{n} (a_k P_k + \widetilde{a}_k \widetilde{P}_k) + \sum_{j=1}^{p} b_j Q_j$. Let $c_k = a_k - i \widetilde{a}_k$. Thus, (32) implies that

$$\text{(33)} \qquad \operatorname{Re} \sum_{k=0}^{n} c_k g(z_k) \ge 0$$

for all analytic functions g whose real part is positive on Ω. Moreover, equality holds for the function g_0 determined by the measure λ. Evidently, (33) implies that $P = \big(g_0(z_0), \dots, g_0(z_n) \big)$ lies in the boundary of Λ'.

Pick Bodies and Interpolation Let $z_0, \dots, z_n$ be distinct points in the open unit disc Δ and let

$$\Lambda = \{ \big(f(z_0), \dots, f(z_n) \big) : f \in H^\infty(\Delta), \|f\| \le 1 \}.$$

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Λ has been termed a *Pick body* by B. Cole, J. Lewis, and J. Wermer. In a series of papers [2], [3], and [4], these authors studied Pick bodies from the perspective of Banach algebras and operator theory. They note that a Pick body $\mathcal{K}$ is *hyperconvex*; that is, for every positive integer m and every polynomial P of m complex variables that is bounded by one in the unit polydisc in $\mathbb{C}^m$ and every set of m points $\mathbf{z}_1, \dots, \mathbf{z}_m$ in $\mathcal{K}$, the point

$$\big(P(z_{11}, \dots, z_{m1}), P(z_{21}, \dots, z_{m2}), \dots, P(z_{m1}, \dots, z_{mm})\big)$$

lies in $\mathcal{K}$. They characterized Pick bodies as compact, hyperconvex subsets of $\mathbb{C}^{n+1}$ with the property that $\partial\mathcal{K}$ contains some point $\mathbf{w} = (w_0, \dots, w_n)$ with

$$(34) \qquad (a)\ |w_i| < 1 \text{ for all } i; \quad (b)\ w_i \neq w_j, \text{ if } i \neq j; \quad (c)\ \mathbf{w}^2, \dots, \mathbf{w}^n \in \partial\mathcal{K}.$$

Specifically, what we mean by this is that if $\mathcal{K}$ is a convex compact subset of $\mathbb{C}^{n+1}$ that satisfies the conditions listed in (34), then there are points $z_0, \dots, z_n$ in Δ so that

$$\mathcal{K} = \big\{\big(f(z_0), \dots, f(z_n)\big) : f \in H^\infty(\Delta), \|f\| \leq 1\big\}.$$

J. Wermer asked if a similar sort of characterization holds when the unit disc Δ is replaced by a domain Ω of the type we have considered here. The answer to this is no. Indeed, consider the case $p = 1, n = 2$, that is, 3 point interpolation on an annulus. Theorem 4 shows that any non-constant boundary point of Λ arises from a Blaschke product of degree at least $p+1 = 2$ and at most $n+p = 3$. The Cole-Lewis-Wermer condition would say that there is some (non-constant) point $\mathbf{w} = (w_0, w_1, w_2)$ in $\partial\Lambda$ such that $\mathbf{w}^2 = (w_0^2, w_1^2, w_2^2)$ lies in $\partial\Lambda$, too. But boundary points of Λ are characterized by having unique interpolants. Hence, if ϕ is the unique interpolant from the unit ball of H^∞ of the data (w_0, w_1, w_2), then ϕ^2 must be the unique interpolant of the data (w_0^2, w_1^2, w_2^2). However, ϕ^2 is a Blaschke product of degree at least 4, contradicting the fact that boundary points of Λ come from Blaschke products of degree at most 3.

It would be most interesting to characterize Pick bodies in the multiply-connected case; in particular, is the boundary a subset of the zero set of a real analytic function?

3 Interpolation of Fixed Data

Because of our assumptions that the first interpolation point is t_0 and that all analytic functions with non-negative real part are strictly real at t_0, the first interpolating condition $g(t_0) = \zeta_0$ involves only a real datum ζ_0. With this in mind, let $\{\zeta_0, \dots, \zeta_n\}$ be given data and suppose that there is at least one analytic function g on Ω with positive real part satisfying $g(z_j) = \zeta_j$, $j = 0, \dots, n$. We denote the set of all such interpolating functions by $\mathbb{M} = \mathbb{M}(\zeta_0, \dots, \zeta_n)$. Evidently, $\mathbb{M}$ is a convex compact set. Each function in $\mathbb{M}$ arises from a unique positive measure μ on Γ via the representation (27) and we use the same letter $\mathbb{M}$ to denote the corresponding set of positive measures on Γ. We wish to determine the extreme points of $\mathbb{M}$. This was done by Heins [14] when Ω is the open unit disc Δ. He demonstrated that if $\mathbb{M}$ has more than one element, then its extreme points are precisely those functions that map Δ onto the right half-plane with constant valence k, where k is any integer between $n+1$ and $2n+1$. Of course, if $\mathbb{M}$ has just one element, then it also maps

Δ onto the right-half plane with constant valence k, $0 < k \leq n$. We shall obtain analogs of these results in some cases and point out significant differences in other cases.

Remark The similar problem of determining the extreme points of the convex compact set

$$\mathbb{B} = \{f \in H^\infty(\Omega) : \|f\|_\infty \leq 1 \text{ and } f(z_j) = w_j,\ j = 0, \dots, n\}$$

is actually far less interesting than the one we consider. A moment's thought shows that a function $f \in \mathbb{B}$ is an extreme point of $\mathbb{B}$ if and only if it is an extreme point of the unit ball of H^∞ (and lies in $\mathbb{B}$, of course). The sets $\mathbb{B}$ and $\mathbb{M}$ are homeomorphic under the correspondence $f \to \frac{1+f}{1-f}$ and so their boundaries are sent one to the other under this mapping, but the extreme points of the sets $\mathbb{B}$, $\mathbb{M}$ are not preserved by this (non-linear) correspondence.

Theorem 7

(a) *Each extreme point of $\mathbb{M}$ arises from a discrete measure with at most $2n + p + 1$ points of support. A discrete measure in $\mathbb{M}$ that is not an extreme point of $\mathbb{M}$ has at least $2n + 2$ points of support.*
(b) *A discrete measure μ with $2n + p + 1$ or fewer points of support gives rise to an extreme point of $\mathbb{M}$ if and only if the restriction of $\mathcal{S}$ to the support of μ is linearly independent.*
(c) *A discrete measure $\mu \in \mathbb{M}$ is an extreme point of $\mathbb{M}$ if and only if it has minimal support; that is, if $\beta \in \mathbb{M}$ and $\operatorname{supp}(\beta) \subset \operatorname{supp}(\mu)$, then $\beta = \mu$.*
(d) *If $\mathbb{M}$ has just one element, then the number of points in the support of the corresponding measure is at most $n + p$.*

Proof (a) Suppose that μ is an extreme point of $\mathbb{M}$ and that there are $2n + 2 + p$ disjoint sets in Γ of positive μ measure. A simple linear algebra argument then shows that there is a real-valued piecewise-constant function v that is not identically zero supported on the union of these sets that satisfies the $2n + p + 1$ real conditions:

$$\begin{cases} \int_\Gamma v(\xi) Q_k(\xi)\, d\mu(\xi) = 0, & k = 1, \dots, p, \\ \int_\Gamma v(\xi) \mathcal{P}(\xi, z_j)\, d\mu(\xi) = 0, & j = 1, \dots, n, \\ \int_\Gamma d\mu(\xi) = 0. \end{cases}$$

Thus, the measure $(1 + \epsilon v)\mu$ lies in $\mathbb{M}$ for all small ϵ, ϵ positive or negative. This clearly contradicts the extremality of μ. Consequently, $2n + p + 2$ such sets do not exist and so μ must be the sum of at most $2n + p + 1$ point masses.

Suppose that $\mu \in \mathbb{M}$ is a discrete measure with support at the points $x_j \in \Gamma$, $j = 1, \dots, m$. If μ is not an extreme point of $\mathbb{M}$, then there are measures $\nu_1, \nu_2 \in \mathbb{M}$, $\nu_1 \neq \nu_2$ with $\mu = \frac{1}{2}(\nu_1 + \nu_2)$. The support of both ν_1 and ν_2 lies in that of μ. Let g_1, g_2 be the analytic functions on Ω determined according to (25) by ν_1, ν_2, respectively. The function $g = g_1 - g_2$ is not identically zero and is meromorphic on $\widehat{\Omega}$ with at most m poles and at least $2n + 2$ zeros: at the points $z_0, \dots, z_n$ and their reflections. Hence, $m \geq 2n + 2$.

(b) Let $x_1, \dots, x_m \in \Gamma$ be the support of $\mu \in \mathbb{M}$. μ is an extreme point of $\mathbb{M}$ if and only if there is not a (non-zero) measure ν supported on $x_1, \dots, x_m$ with $\mu \pm \nu \in \mathbb{M}$.

This is equivalent to saying that there is not a measure ν supported on $x_1, \ldots, x_m$ that is orthogonal to $\mathcal{S}$.

(c) Suppose that μ is an extreme point of $\mathbb{M}$, $\beta \in \mathbb{M}$, and $\operatorname{supp}(\beta) \subset \operatorname{supp}(\mu)$. Then $\nu = \mu - \beta$ is orthogonal to $\mathcal{S}$ and the support of ν is a subset of that of μ. By (b), $\nu = 0$. Hence, μ has minimal support. Conversely, suppose that $\mu \in \mathbb{M}$ has minimal support. If $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ where $\mu_1, \mu_2 \in \mathbb{M}$, then evidently the support of both μ_1 and μ_2 is a subset of that of μ. By the minimality assumption, $\mu_1 = \mu_2 = \mu$.

(d) If $\mathbb{M}$ has just one element, then this function must be the unique interpolant described in Theorem 4 and so the measure μ has no more than $n + p$ points of support.

Example 2 There are non-extreme points in $\mathbb{M}$ with as few as $2n+2$ points of support. To see this, let ϕ be an analytic function on Ω that is a $p + 1$-fold covering of Δ; the Ahlfors function (see [6, Section 5.1], for instance) is one such function and others may be obtained from Proposition 8 below. We may assume with no loss of generality that ϕ has $p+1$ distinct zeros in Ω, say at $z_0, \ldots, z_p$ and we take $n = p$. Let λ_1, λ_2 be distinct points on the unit circle and set $g_j(z) = \frac{\lambda_j - \phi(z)}{\lambda_j + \phi(z)}$, $j = 1, 2$. Then g_j has positive real part on Ω; in fact, $\operatorname{Re} g_j$ is the Poisson extension of a positive measure μ_j on Γ with exactly $p + 1$ points of support. Moreover, $g_1(z_k) = g_2(z_k) = 1$, $k = 0, \ldots, p$. The function $g = \frac{1}{2}(g_1 + g_2)$ is then not an extreme point of the set $\mathbb{M}$ of functions with positive real part that interpolate the data $1, \ldots, 1$ at the points $z_0, \ldots, z_p$. Moreover, g is the Poisson extension of discrete measure on Γ with at most $2p + 2$ points of support. Since $2p + 2 \le 2n + p + 1 = 3p + 1$ as soon as $p \ge 1$, we see that there are measures with as few as $2n + 2$ points of support that are not extreme points of $\mathbb{M}$.

The following is another example of the phenomena displayed in Example 2.

Example 3 Let Ω be the annulus $\{z : R < |z| < 1\}$ so that $p = 1$; we shall take $n = 1$ and consequently $2n + p + 1 = 4$. We shall construct a measure μ supported on four points in Γ that is not an extreme point of $\mathbb{M}$. We take the four points on the boundary to be $x_1 = i$, $x_2 = iR$, $x_3 = -iR$, $x_4 = -i$; we let μ_1 be the measure determined by placing masses at the points x_1, x_3 with weights $w_1 = 1$, $w_3 = R$, respectively, and let μ_2 be the measure determined by placing masses at the points x_2, x_4 with weights $w_2 = R$, $w_4 = 1$. The (single) Schottky function Q for Ω is

$$Q(x) = \begin{cases} \frac{1}{R \log R} & \text{if } |x| = R \\ \frac{-1}{\log R} & \text{if } |x| = 1. \end{cases}$$

Thus, both μ_1 and μ_2 are orthogonal to Q. Let $\nu = \mu_1 - \mu_2$; clearly, ν is orthogonal to the function that is identically 1. Let u, u_1, u_2 denote the Poisson extensions to Ω of ν, μ_1, μ_2, respectively, so that $u = u_1 - u_2$. Symmetry considerations show that $u(t) = 0$, $R < |t| < 1$. Let v be a harmonic conjugate of u in Ω and set $g = u + iv$. Then g is purely imaginary on the real axis and so by Schwarz reflection satisfies $g(z) = -\overline{g(\overline{z})}$, $z \in \Omega$. In particular, $g(z) = 0$ if and only if $g(\overline{z}) = 0$. Next,

$$u(iy) \to \infty \text{ as } y \uparrow 1 \quad \text{and} \quad u(iy) \to -\infty \text{ as } y \downarrow R.$$

Therefore, there is a y_0, $R < y_0 < 1$ at which $u(iy_0) = 0$. We now specify that the harmonic conjugate v of u be chosen to be zero at iy_0. Hence, $g(iy_0) = 0$ and $g(-iy_0) = -\big(\overline{g(iy_0)}\big) = 0$, as well. We denote by g_1, g_2 the analytic functions on Ω whose real parts are u_1, u_2, respectively, and whose imaginary parts vanish at iy_0. Evidently, $g = g_1 - g_2$. Set $z_1 = iy_0$ and $z_2 = -iy_0$; we then have

$$g_1(z_1) = g_2(z_1) = \zeta_1 \quad \text{and} \quad g_1(z_2) = g_2(z_2) = \zeta_2.$$

Therefore, the function $f = \frac{1}{2}(g_1 + g_2)$ is not an extreme point of $\mathbb{M}(\zeta_1, \zeta_2)$ but yet is the Poisson integral of a measure $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ on Γ with just $4 = 2n + p + 1$ points of support.

The case when $n = 0$ can be worked out fully. We shall need the following simple result.

Lemma 8 *If μ is any non-negative discrete measure on Γ that is orthogonal to $Q_1, \ldots, Q_p$, then μ has support on each component $\Gamma_0, \ldots, \Gamma_p$ of Γ. In particular, if μ is discrete, then it has at least $p + 1$ points in its support.*

Proof Let g be the analytic function on Ω obtained by extending the measure μ according to (25); g has non-negative real part on Ω. Moreover, g extends analytically across any component Γ_k of Γ on which μ has no support and $\operatorname{Re} g$ vanishes identically there. The function $f = \frac{g-1}{g+1}$ is analytic on Ω and is bounded by one. Moreover, f extends continuously to Γ_k and has unit modulus there. The Cauchy-Riemann equations then imply that the argument of f is (strictly) increasing on Γ_k. Since f is single-valued, this means that the argument of f must increase by an integer multiple of 2π on Γ_k and so f must take on the value 1 on Γ_k. However, $f(x) = 1$ at some point $x \in \Gamma_k$ only if the function g has a discontinuity at x. That is, μ has a point of support at x.

Proposition 9 *A measure $\mu \in \mathcal{M}_0^+$ lies in an extremal ray of $\mathcal{M}_0^+$ if and only if μ has $p + 1$ points of support.*

Proof Suppose first that $\mu \in \mathcal{M}_0^+$ lies in an extremal ray of $\mathcal{M}_0^+$. If there are $p + 2$ disjoint sets of positive μ-measure, we may construct a bounded piecewise constant function v that is not identically 1 such that $v\,d\mu$ is orthogonal to $Q_1, \ldots, Q_p$. Thus, for a sufficiently small ϵ, we have $1 = \frac{1}{2}[(1 + \epsilon v) + (1 - \epsilon v)]$ and so μ fails to be extremal, a contradiction. Hence, the support of μ has at most $p + 1$ points. Since the support has at least $p + 1$ points, it must have exactly $p + 1$ points. Conversely, suppose $\mu \in \mathcal{M}_0^+$ has $p + 1$ points of support. If $\mu = \frac{1}{2}(\nu_1 + \nu_2)$ where $\nu_1, \nu_2 \in \mathcal{M}_0^+$, then the support of ν_1, ν_2 is a subset of that of μ and so is the exact same set of $p + 1$ points. Suppose that $\nu_1 \neq \mu$. Then there is a constant A with $\mu - A\nu_1 \geq 0$ and $\mu - A\nu_1$ has p or fewer points of support. But this contradicts Lemma 7. Hence, μ is extremal.

Ahlfors' Functions Let Ω be a domain in the complex plane that supports non-constant bounded analytic functions. Fix some point $z_0 \in \Omega$ and consider the extremal problem

$$\gamma = \sup_{f \in H^\infty} \operatorname{Re} f'(z_0). \tag{35}$$

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It is known (*cf.* [8, Theorem 5.1.1]) that there is a unique solution F to this problem, called the *Ahlfors function* for Ω and z_0 and $F(z_0) = 0$. In the case when Ω is bounded by $p + 1$ disjoint smooth simple closed curves, the Ahlfors' function may be extended analytically across Γ and maps each component Γ_j of Γ one-to-one onto the unit circle. As a consequence, it is a $p+1$-fold cover of the unit disc Δ and the associated function G with positive real part $G = (1+F)/(1-F)$ is the (complex) Poisson integral of a positive measure μ_G on Γ with precisely $p + 1$ points of support, one in each Γ_j.

We now demonstrate that the converse of this statement does not hold. That is, there is an analytic function on Ω with positive real part determined by a positive measure on Γ with exactly one point of support in each component Γ_j that is not of the form $(1 + F)/(1 - F)$ where F is the Ahlfors function for some point in Ω. Equivalently, not every Blaschke product on Ω of degree $p + 1$ is an Ahlfors function. To see this, we suppose the contrary. Let z_0, z_1 be distinct points of Ω. Theorem 4 tells us that each point in the boundary of

$$\Lambda = \{\big(f(z_0), f(z_1)\big) : f \in H^\infty, \|f\| \le 1\}$$

arises from a Blaschke product of degree $p + 1$, unless $f(z_0) = f(z_1) \in \mathbf{T}$, the unit circle. In particular, if we (forever) fix two non-zero complex numbers c_0, c_1 with different arguments, then the solution to the extremal problem

$$\sup \operatorname{Re}\{c_0 f(z_0) + c_1 f(z_1) : f \in H^\infty, \|f\| \le 1\} \tag{36}$$

is a Blaschke product of degree exactly $p + 1$. Suppose that for each choice of $z_0, z_1 \in \Omega$, there is some point $z_2 \in \Omega$ so that the solution of the extremal problem (36) is the Ahlfors function F for z_2. Let R_0 be the kernel for the extremal problem (36) and let R_1 be the kernel for the extremal problem (35); that is, for the Ahlfors function. We know that R_0 has poles of order 1 at z_0, z_1 and at the critical points of the Green's function for t_0; further, from (23), R_0 has no zeros on $\Omega \cup \Gamma$ except at t_0. Likewise, R_1 has a pole of order 2 at z_2, poles of order one at the critical points of the Green's function for t_0, and no zeros on $\Omega \cup \Gamma$ except at t_0. Finally, we also know that

$$R_0 F > 0 \quad \text{and} \quad R_1 F > 0 \text{ on } \Gamma.$$

Hence,

$$R = R_0/R_1 = FR_0/FR_1 > 0 \text{ on } \Gamma. \tag{37}$$

R has poles of order one at z_0, z_1 and a double zero at z_2. R extends to be meromorphic on the double $\widehat{\Omega}$ since it is real on Γ; thus, it is a 4-fold cover of the Riemann sphere with poles at z_0, z_1 and their reflections z_0^*, z_1^* across Γ. We now show this can not be the case for arbitrary z_0, z_1.

$\widehat{\Omega}$ is a compact Riemann surface of genus p. Let ω_j, $j = 1, \ldots, p$, be the harmonic function on Ω whose boundary values are identically one on Γ_j and identically zero on $\Gamma \setminus \Gamma_j$; see the material preceding (17). Let $\tilde{\omega}_j$ be the (multiple-valued) harmonic conjugate of ω_j, $s_j = \omega_j + i\tilde{\omega}_j$ and $b_j = s_j'\,dz$. Then $b_1, \ldots, b_p$ are a basis of the holomorphic

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differentials on the double $\widehat{\Omega}$ that are real on Γ. According to Theorem 18.2 of [12], we must have

$$\sum_z \operatorname{Res}[R(z)b_j(z)] = 0, \quad j = 1, \dots, p \tag{38}$$

where the sum is taken over all points in $\widehat{\Omega}$. Near the point z_k, the 1-form b_j has the expansion

$$b_j(z) = a_{kj}\, dz + O(z - z_k)\, dz, \quad k = 0, 1; \quad j = 1, \dots, p.$$

By symmetry at the points z_0^*, z_1^*, we have

$$b_j(z) = \overline{a_{kj}}\, dz + O(z - z_k^*)\, dz, \quad k = 0, 1; \quad j = 1, \dots, p.$$

Let w_0, w_1 be the residues of R at z_0, z_1, respectively. Thus,

$$\operatorname{Res}[Rb_j; z_k] = w_k a_{kj} \text{ and } \operatorname{Res}[Rb_j; z_k^*] = w_k \overline{a_{kj}}, \quad k = 0, 1; \quad j = 1, \dots, p. \tag{39}$$

Using (39) in (38) we find that

$$\operatorname{Re}(w_0 a_{0j} + w_1 a_{1j}) = 0, \quad j = 1, \dots, p. \tag{40}$$

This implies that

$$\operatorname{Re} s_j'(z_1) = \operatorname{Re} c s_j'(z_0), \quad j = 1, \dots, p \tag{41}$$

where c is a complex number depending on z_0, z_1. Fix z_1 and set $z = z_0$. Define $\mathbf{V}_1 = \big(\operatorname{Re} s_1'(z_1), \dots, \operatorname{Re} s_p'(z_1)\big)$ and $\mathbf{V}_2 = \big(\operatorname{Im} s_1'(z_1), \dots, \operatorname{Im} s_p'(z_1)\big)$. We then note that equation (41) implies that the vector $\big(\operatorname{Re} s_1'(z), \dots, \operatorname{Re} s_p'(z)\big) \in \mathbb{R}^p$ lies in the two dimensional plane spanned by $\mathbf{V}_1$, $\mathbf{V}_2$ for every $z \in \Omega$. Therefore, this continues to be true when $z \to \xi \in \Gamma$. If we let ξ be in turn a point $\xi_k \in \Gamma_k$, $k = 1, \dots, p$, we obtain p vectors $\mathbf{W}_k = \big(s_1'(\xi_k), \dots, s_p'(\xi_k)\big)$, $k = 1, \dots, p$ that lie in the span of $\mathbf{V}_1$, $\mathbf{V}_2$. However, on Γ we have $s_j' = \partial\omega_j/\partial n$, $j = 1, \dots, p$, which is purely real. Moreover, these p vectors are linearly independent; see [18, Lemma 1]. This is surely a contradiction if $p \geq 3$.

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