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SLICING SURFACES AND THE FOURIER RESTRICTION CONJECTURE

FABIO NICOLA

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy (fabio.nicola@polito.it)

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Abstract We deal with the restriction phenomenon for the Fourier transform. We prove that each of the restriction conjectures for the sphere, the paraboloid and the elliptic hyperboloid in $\mathbb{R}^n$ implies that for the cone in $\mathbb{R}^{n+1}$. We also prove a new restriction estimate for any surface in $\mathbb{R}^3$ locally isometric to the plane and of finite type.

Keywords: restriction theorem; Fourier transform; conic sections

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1. Introduction and discussion of the results

Let S be a smooth hypersurface with (possibly empty) boundary in $\mathbb{R}^n$, $n \geqslant 2$, or a compact subset (with non-empty interior) of such a hypersurface and let $\mathrm{d}\sigma$ be the surface measure on S. Denote by $\hat{f}$ the Fourier transform of the function f. We deal with the so-called restriction estimate

$$\|\hat{f}|_S\|_{L^q(S,\mathrm{d}\sigma)} \leqslant C_{p,q,S}\|f\|_{L^p(\mathbb{R}^n)}$$

for all Schwartz functions f, or equivalently with the extension estimate

$$\|(u\,\mathrm{d}\sigma)^\vee\|_{L^{p'}(\mathbb{R}^n)} \leqslant C_{p,q,S}\|u\|_{L^{q'}(S,\mathrm{d}\sigma)},$$

for all smooth functions u with compact support in S. We denote these estimates by $R_S(p \to q)$ and $R_S^*(q' \to p')$, respectively. Here $(u\,\mathrm{d}\sigma)^\vee$ is the inverse Fourier transform of the measure $u\,\mathrm{d}\sigma$.

We will mostly be interested in the case in which S is the sphere, or the elliptic paraboloid, or the elliptic hyperboloid in $\mathbb{R}^n$ and also the light cone in $\mathbb{R}^{n+1}$. Precisely, we define

$$\begin{aligned} S_{\text{sphere}} &= \{\xi \in \mathbb{R}^n : |\xi| = 1\}, \\ S_{\text{parab}} &= \{(\xi', \xi_n) \in \mathbb{R}^n : \xi_n = \tfrac{1}{2}|\xi'|^2\}, \\ S_{\text{hyperb}} &= \{(\xi', \xi_n) \in \mathbb{R}^n : \xi_n = \sqrt{1 + |\xi'|^2}\}, \\ S_{\text{cone}} &= \{(\xi, \tau) \in \mathbb{R}^{n+1} : \tau = |\xi|,\ \xi \neq 0\}. \end{aligned}$$

On the sphere we take the usual surface measure, whereas the measure on the paraboloid and on the hyperboloid are defined as the pull-back under the projection $(\xi', \xi_n) \mapsto \xi'$ of the measures $\mathrm{d}\xi'$ and $\mathrm{d}\xi'/\sqrt{1+|\xi'|^2}$, respectively. Also, on the cone we consider the Lorentz invariant measure $\mathrm{d}\sigma_{\mathrm{cone}}$ defined as the pull-back under the projection $(\xi, \tau) \mapsto \xi$ of the measure $\mathrm{d}\xi/|\xi|$. Of course for the cone we mean that, in the estimates $R_S(p \to q)$ and $R^*(q' \to p')$, n must be replaced by $n+1$. The restriction estimate is conjectured to hold in the following cases (see [**13**, **15**, **23**, **28**] and especially [**21**]).

Conjecture 1.1 (restriction conjecture).

(i) *Suppose that*

$$\frac{p'}{n+1} \geqslant \frac{q}{n-1}, \quad p' > \frac{2n}{n-1}. \tag{1.1}$$

Then $R_S(p \to q)$ holds when S is any compact subset of a hypersurface in $\mathbb{R}^n$ with non-vanishing Gaussian curvature or of the cone in $\mathbb{R}^{n+1}$.

(ii) *Suppose that*

$$\frac{p'}{n+1} = \frac{q}{n-1}, \quad p' > \frac{2n}{n-1}, \tag{1.2}$$

i.e. the pair p, q is scale invariant. Then $R_S(p \to q)$ holds when S the whole paraboloid in $\mathbb{R}^n$, or the whole hyperboloid in $\mathbb{R}^n$, or the whole cone in $\mathbb{R}^{n+1}$.

The conditions (1.1) and (1.2) are known to be necessary.

This fascinating conjecture was proved for curves in the plane by Zygmund [**29**] and Fefferman [**5**], for the cone in $\mathbb{R}^3$ by Barceló [**1**] and for the cone in $\mathbb{R}^4$ by Wolff [**27**]. Partial results in higher dimensions have been obtained by many authors, culminating in the work of Wolff [**27**], who proved (i) for the cone under the additional condition $p' > 2(n+3)/(n+1)$, and Tao [**19**], who proved (i) for the sphere and the paraboloid under the additional condition $p' > 2(n+2)/n$.

For the paraboloid, any scale-invariant result for compact subsets extends automatically to the whole paraboloid, whereas, for the cone, this is not so immediate. The above-mentioned results by Wolff should, however, extend (in the scale-invariant case) to the whole cone (possibly after conceding an epsilon in the exponents) by using the techniques in [**16**, **17**] (T. Tao, personal communication). Moreover, it is well known (see, for example, [**17**] and [**21**, Problem 1.1]) that the restriction conjecture for the sphere (or any other hypersurface with n positive principal curvatures) implies the restriction conjecture for the paraboloid. (See [**21**] for a detailed survey.)

As one sees, the numerology for the sphere, the paraboloid and the hyperboloid in $\mathbb{R}^n$ agrees with that of the cone in $\mathbb{R}^{n+1}$. Heuristically, this is explained by observing that the cone has one extra dimension, which, nevertheless, being flat, is not expected to produce any contribution in the restriction estimate. A deeper investigation of this link, in the case of the paraboloid and the cone, has been also exploited in [**18**, Proposition 17.5] (see also the discussion in [**21**, pp. 7–8]). However, to our knowledge, proofs of formal implications have not appeared in the literature and do not seem to be known (see [**22**, p. 12]).

The first result of this paper shows that each of the restriction conjectures for the sphere, the paraboloid and the hyperboloid in $\mathbb{R}^n$ implies the restriction conjecture for the cone in $\mathbb{R}^{n+1}$. More precisely, we have the following result.

Theorem 1.2.

(a) *Assume (for some p, q satisfying (1.1)) one of the following hypotheses:*

(i) $R_S(p \to q)$ *holds for the sphere S in $\mathbb{R}^n$;*

(ii) $R_S(p \to q)$ *holds for every compact subset S of the paraboloid in $\mathbb{R}^n$;*

(iii) $R_S(p \to q)$ *holds for every compact subset S of the hyperboloid in $\mathbb{R}^n$.*

Then $R_S(p \to q)$ holds for every compact subset S of the cone in $\mathbb{R}^{n+1}$.

(b) *Assume one of the following hypotheses:*

(i′) $R_S(p \to q)$ *holds, for some p, q satisfying (1.2), for the sphere S in $\mathbb{R}^n$;*

(ii′) $R_S(p \to q)$ *holds, for some p, q satisfying (1.2), for the whole paraboloid S in $\mathbb{R}^n$;*

(iii′) $R_S(p \to q)$*holds, for some p, q satisfying (1.2), for the whole hyperboloid S in $\mathbb{R}^n$.*

Then $R_S(p \to q)$ holds for the whole cone S in $\mathbb{R}^{n+1}$.

Notice that, even in the case of the whole cone, there is no loss in the exponents. Also, by combining Theorem 1.2 with the sharp restriction theorem for the circle in the plane, we obtain another proof of the sharp restriction theorem for the cone in $\mathbb{R}^3$.

As might be expected, the proof exploits the fact that the sphere, the paraboloid and the hyperboloid are sections of the light cone. We observe that a similar trick goes back to [**6**], and has been used, in different contexts, by Vega [**26**], Carbery [**3**], Mockenhaupt *et al*. [**8**], Tataru [**7**, Appendix B], Tao [**20**, Proposition 4.1] and Burq *et al*. [**2**, Theorem 2]. Also, this trick works for more general surfaces of revolution than the cone, e.g. the paraboloid, but one no longer obtains sharp estimates. The second result of this paper consists in a restriction theorem for surfaces in $\mathbb{R}^3$ with Gaussian curvature vanishing everywhere. Here we say that a point P_0 of a hypersurface S is of type k if k is the order of contact of S_0 with its tangent plane at P_0.*

Theorem 1.3. *Let S be a surface in $\mathbb{R}^3$ and let $P_0 \in S$ be a point of type k. Suppose that S has Gaussian curvature vanishing identically near P_0. Then there exists a compact neighbourhood $S_0 \subset S$ of P_0 and a constant C such that*

$$\|(u\,\mathrm{d}\sigma)^{\vee}\|_{L^{p'}(\mathbb{R}^3)} \leqslant C\|u\|_{L^{q'}(S,\mathrm{d}\sigma)} \tag{1.3}$$

for every $p' > 4$, $p' \geqslant k+2$, $p' \geqslant (k+1)q$ and every smooth u supported in S_0.

* In general, if S is a smooth m-dimensional submanifold of $\mathbb{R}^n$, $1 \leqslant m \leqslant n-1$, and $\phi : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is a local parametrization of S, with $\phi(x_0) = P_0$, then P_0 is called of type k if k is the smallest integer such that, for each unit vector η, there exists an α with $|\alpha| \leqslant k$ for which $\partial^\alpha[\phi(x) \cdot \eta]|_{x=x_0} \neq 0$ (see [**13**, p. 350] for more details).

Notice that the numerology agrees with the sharp restriction theorem by Sogge [**11**] for curves of finite type in the plane, according to the fact that S has one principal curvature identically zero near P_0. We also recall that the hypothesis of the vanishing of the Gaussian curvature near P_0 is equivalent to saying that a convenient neighbourhood of P_0 is isometric to the plane. Classical examples of surfaces with such a property are given by the developable surfaces (hence cones, cylinders, tangent developables); see Spivak [**12**] for details. We point out that, when instead the Gaussian curvature vanishes on a one-dimensional submanifold and there are no umbilic points, decay estimates for $(\mathrm{d}\sigma)^\vee$ have recently been obtained by Erdős and Salmhofer [**4**].

The proof of Theorem 1.3 uses the simple idea, as above, of transferring restriction estimates from slices of a surface (given here by Sogge's result mentioned above) to the surface itself. However, to this end we need to prove a new normal form for S near P_0 (see Proposition 3.2), defined in terms of an *orthogonal* transformation in $\mathbb{R}^3$ and hence particularly convenient for the restriction problem. We refer the reader to Remark 3.3 for a comparison with the normal forms in [**10**].

The next two sections are devoted to the proof of Theorems 1.2 and 1.3, respectively.

2. Proof of Theorem 1.2

We first fix the notation and recall some preliminary results which are needed in the proof of Theorem 1.2. Given a measure space $X = (X, \mathcal{B}_X, \mu_X)$, we denote by $L^{\alpha,\beta} = L^{\alpha,\beta}(X)$, $0 < \alpha < \infty$, $0 < q \leqslant \infty$, the Lorentz spaces on X. Hence,

$$\|f\|_{L^{\alpha,\beta}} = \|\lambda\mu(\{|f| \geqslant \lambda\})^{1/\alpha}\|_{L^\beta(\mathbb{R}^+,\mathrm{d}\lambda/\lambda)}.$$

We recall (see, for example, [**14**]) that $L^{\alpha,\alpha} = L^\alpha$, and $L^{\alpha,\beta_1} \hookrightarrow L^{\alpha,\beta_2}$ if $\beta_1 \leqslant \beta_2$. Moreover, Hölder's inequality for Lorentz spaces reads as follows: if $0 < \alpha_1, \alpha_2, \alpha < \infty$ and $0 < \beta_1, \beta_2, \beta \leqslant \infty$ obey

$$\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \quad \text{and} \quad \frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2},$$

then

$$\|fg\|_{L^{\alpha,\beta}} \leqslant \|f\|_{L^{\alpha_1,\beta_1}} \|f\|_{L^{\alpha_2,\beta_2}}. \tag{2.1}$$

We also recall that there is a sharp version of the Hausdorff–Young inequality in terms of Lorentz spaces in $\mathbb{R}^n$, with the Lebesgue measure [**14**, Corollary 3.16, p. 200]. Namely, if $1 < p \leqslant 2$, we have*

$$\|\hat{u}\|_{L^{p'}} \lesssim \|u\|_{L^{p,p'}}. \tag{2.2}$$

We will need the following lemma on the interchange of norms. Consider two measure spaces $X = (X, \mathcal{B}_X, \mu_X)$ and $Y = (Y, \mathcal{B}_Y, \mu_Y)$ and a function $f(x, y)$ on the product space $(X \times Y) = (X \times Y, \mathcal{B}_X \times \mathcal{B}_Y, \mu_X \times \mu_Y)$. We define the mixed norms $L_x^{\alpha,\beta} L_y^{\gamma,\delta}(X \times Y)$ of f as

$$\|f\|_{L_x^{\alpha,\beta} L_y^{\gamma,\delta}(X\times Y)} = \|\|f(x,\cdot)\|_{L^{\gamma,\delta}(Y)}\|_{L^{\alpha,\beta}(X)}.$$

* We write $A \lesssim B$ if $A \leqslant CB$ for some constant $C > 0$ which may depend on parameters like Lebesgue exponents or the dimension n.

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Lemma 2.1. *If* $1 < p \leqslant 2$, *we have*

$$\|u\|_{L_x^{p'} L_y^{p,p'}} \lesssim \|u\|_{L_y^{p,p'} L_x^{p'}}. \tag{2.3}$$

Proof. By Minkowski's inequality we have

$$\|u\|_{L_x^{p'} L_y^1} \leqslant \|u\|_{L_y^1 L_x^{p'}}. \tag{2.4}$$

Therefore, the desired estimate follows by real interpolation from (2.4) and the trivial estimate $\|u\|_{L_x^{p'} L_y^{p'}} = \|u\|_{L_y^{p'} L_x^{p'}}$. Indeed, if

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{p'},$$

we have

$$[L_x^{p'} L_y^1, L_x^{p'} L_y^{p'}]_{\theta,p'} = L_x^{p'} L_y^{p,p'}$$

by [**25**, (3), p. 128] and [**25**, (16), p. 134], whereas

$$[L_y^1 L_x^{p'}, L_y^{p'} L_x^{p'}]_{\theta,p'} = L_y^{p,p'} L_x^{p'},$$

again by [**25**, (16), p. 134].

This concludes the proof. □

We now prove Theorem 1.2 separately in the three cases, namely for the sphere, the paraboloid and the hyperboloid. Although the three proofs follow a similar pattern, and, in fact, the part for the sphere and the hyperboloid (part (b)) follows from that for the paraboloid combined with results in [**17**], for the convenience of the reader we present each proof in a self-contained form. With abuse of notation we always identify functions on the cone with functions in $\mathbb{R}^n_{\xi'}$. Moreover, we will make use, both in the hypotheses and in the conclusion, of the formulation $R_S^*(q' \to p')$, which is easily seen to be equivalent to $R_S(p \to q)$. We also suppose that $p > 1$, since the case when $p = 1$ is trivial.

Proof that spherical restriction ⇒ conical restriction. Here we prove the conclusions of Theorem 1.2, under (i) or (i′). We use polar coordinates (r, ω) in $\mathbb{R}^n_\xi$, and we denote by $\mathrm{d}\omega$ the measure on the sphere and by $L^{\alpha,\beta}$ the Lorentz spaces on $\mathbb{R}^+$ with respect to Lebesgue measure.

Assume (i′). We have

$$\begin{aligned}(u\,\mathrm{d}\sigma_{\mathrm{cone}})^\vee(x,t) &= \int \mathrm{e}^{2\pi\mathrm{i}(x\xi + t|\xi|)} u(\xi) \frac{\mathrm{d}\xi}{|\xi|} \\ &= \int_0^{+\infty} \mathrm{e}^{2\pi\mathrm{i}tr} r^{n-2} \int_{\mathbb{S}^{n-1}} \mathrm{e}^{2\pi\mathrm{i}rx\omega} u(r\omega)\,\mathrm{d}\omega\,\mathrm{d}r. \end{aligned} \tag{2.5}$$

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Hence, by the Hausdorff–Young inequality (2.2) and (2.3) we have

$$\begin{aligned}\|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}} &= \|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}_x L^{p'}_t}\\ &\lesssim \bigg\| r^{n-2}\int_{\mathbb{S}^{n-1}} \mathrm{e}^{2\pi \mathrm{i} r x\omega} u(r\omega)\,\mathrm{d}\omega\bigg\|_{L^{p'}_x L^{p,p'}_r}\\ &\lesssim \bigg\| r^{n-2}\int_{\mathbb{S}^{n-1}} \mathrm{e}^{2\pi \mathrm{i} r x\omega} u(r\omega)\,\mathrm{d}\omega\bigg\|_{L^{p,p'}_r L^{p'}_x}.\end{aligned}$$

Now, a change of variables and the hypothesis give

$$\begin{aligned}\bigg\|\int_{\mathbb{S}^{n-1}} \mathrm{e}^{2\pi \mathrm{i} r x\omega} u(r\omega)\,\mathrm{d}\omega\bigg\|_{L^{p'}_x} &= r^{-n/p'}\bigg\|\int_{\mathbb{S}^{n-1}} \mathrm{e}^{2\pi \mathrm{i} x\omega} u(r\omega)\,\mathrm{d}\omega\bigg\|_{L^{p'}_x}\\ &\lesssim r^{-n/p'}\|u(r\cdot)\|_{L^{q'}(\mathbb{S}^{n-1})}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}} &\lesssim \|r^{n-2-(n/p')}\|u(r\cdot)\|_{L^{q'}(\mathbb{S}^{n-1})}\|_{L^{p,p'}}\\ &= \|r^{(n-2)/q-(n/p')}\cdot r^{(n-2)/q'}\|u(r\cdot)\|_{L^{q'}(\mathbb{S}^{n-1})}\|_{L^{p,p'}}\\ &\lesssim \bigg\|\underbrace{r^{(n-2)/q-(n/p')}}_{F(r)}\cdot\underbrace{r^{(n-2)/q'}\|u(r\cdot)\|_{L^{q'}(\mathbb{S}^{n-1})}}_{G(r)}\bigg\|_{L^{p,q'}},\end{aligned}\tag{2.6}$$

where the last inequality follows because $p' > q'$. Now, let α be defined by $1/\alpha + 1/q' = 1/p$. A direct computation shows that $F \in L^{\alpha,\infty}$ (since the pair p, q is scale invariant). Hence, by Hölder's inequality for Lorentz spaces (2.1), we see that the last expression is not greater than

$$\|F\|_{L^{\alpha,\infty}}\|G\|_{L^{q',q'}} = C\|G\|_{L^{q'}} = C\|u\|_{L^{q'}(\mathbb{R}^n,\mathrm{d}\xi/|\xi|)}.$$

This concludes the proof of the restriction estimates for the whole cone.

The proof of the restriction estimate for compact subsets of the cone, under assumption (i), is even easier, since this amounts to proving the extension estimate for u supported where $r \approx 1$, so that one concludes using (2.6), the embedding $L^p \hookrightarrow L^{p,p'}$ and Hölder's inequality for L^p spaces (since $p < q'$). □

Proof that parabolic restriction ⇒ conical restriction. Here we prove the conclusions in Theorem 1.2, assuming (ii) or (ii′).

We introduce orthogonal coordinates

$$(a_1,\dots,a_{n-1},a_n,b) = \left(\xi_1,\dots,\xi_{n-1},\frac{\tau+\xi_n}{\sqrt{2}},\frac{\tau-\xi_n}{\sqrt{2}}\right),$$

and we use the notation $a = (a', a_n)$, $a' = (a_1,\dots,a_{n-1})$. With such coordinates the cone $\tau = |\xi|$ has the equation

$$b = \frac{|a'|^2}{2a_n},\quad a_n > 0,$$

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and the Lorentz invariant measure becomes the pull-back under the projection $(a,b) \mapsto a$ of the measure $\mathrm{d}a/\sqrt{2}a_n$. Again we denote by $L^{\alpha,\beta}$ the Lorentz spaces on the real semi-axis $\mathbb{R}^+$ with respect to the Lebesgue measure. Moreover, set $x = (x', x_n)$.

Suppose (ii$'$). We have

$$\begin{aligned}(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}(x,t) &= \frac{1}{\sqrt{2}}\int \exp\biggl\{2\pi\mathrm{i}\biggl(x\cdot a + \frac{|a'|^2}{2a_n}t\biggr)\biggr\}u(a)\frac{\mathrm{d}a}{a_n}\\ &= \frac{1}{\sqrt{2}}\int \mathrm{e}^{2\pi\mathrm{i}x_n a_n}\int \exp\biggl\{2\pi\mathrm{i}\biggl(x'\cdot a' + \frac{|a'|^2}{2a_n}t\biggr)\biggr\}u(a)\frac{\mathrm{d}a'}{a_n}\,\mathrm{d}a_n.\end{aligned}$$

Hence, by (2.2) and (2.3),

$$\begin{aligned}\|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}} &= \|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}_{x',t}L^{p'}_{x_n}}\\ &\lesssim \biggl\|\int \exp\biggl\{2\pi\mathrm{i}\biggl(x'\cdot a' + \frac{|a'|^2}{2a_n}t\biggr)\biggr\}u(a)\frac{\mathrm{d}a'}{a_n}\biggr\|_{L^{p'}_{x',t}L^{p,p'}_{a_n}}\\ &\lesssim \biggl\|\int \exp\biggl\{2\pi\mathrm{i}\biggl(x'\cdot a' + \frac{|a'|^2}{2a_n}t\biggr)\biggr\}u(a)\frac{\mathrm{d}a'}{a_n}\biggr\|_{L^{p,p'}_{a_n}L^{p'}_{x',t}}.\end{aligned}$$

Now, changing variables and the hypothesis gives

$$\begin{aligned}&\biggl\|\int \exp\biggl\{2\pi\mathrm{i}\biggl(x'\cdot a' + \frac{|a'|^2}{2a_n}t\biggr)\biggr\}u(a)\frac{\mathrm{d}a'}{a_n}\biggr\|_{L^{p'}_{x',t}}\\ &\qquad= a_n^{1/p'}\biggl\|\int \exp\biggl\{2\pi\mathrm{i}\biggl(x'\cdot a' + \frac{|a'|^2}{2}t\biggr)\biggr\}u(a)\frac{\mathrm{d}a'}{a_n}\biggr\|_{L^{p'}_{x',t}}\\ &\qquad\lesssim a_n^{(1/p')-1}\|u(\cdot,a_n)\|_{L^{q'}}.\end{aligned}$$

We deduce that

$$\begin{aligned}\|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}} &\lesssim \|a_n^{(1/p')-1}\|u(\cdot,a_n)\|_{L^{q'}}\|_{L^{p,p'}}\\ &= \|a_n^{1/q'-1/p}a_n^{1/q'}\|u(\cdot,a_n)\|_{L^{q'}}\|_{L^{p,p'}}\\ &\lesssim \|a_n^{1/q'-1/p}\cdot a_n^{1/q'}\|u(\cdot,a_n)\|_{L^{q'}}\|_{L^{p,q'}}\end{aligned} \tag{2.7}$$

for $p' > q'$. Then one concludes by applying Hölder's inequality (2.1), since $L^{qp'/(p'-q),\infty}\cdot L^{q',q'} \hookrightarrow L^{p,q'}$.

Let us now assume (ii). By symmetry and the triangle inequality we can take u supported in the sector $1 \lesssim |a'| \leqslant a_n \lesssim 1$. Then one can obtain the desired estimate by using (2.7), the embedding $L^p \hookrightarrow L^{p,p'}$ and Hölder's inequality for L^p spaces (since $p < q'$). □

Proof that hyperbolic restriction ⇒ conical restriction. Now we prove the conclusions of Theorem 1.2 when (iii) or (iii$'$) are satisfied. Denote by $L^{\alpha,\beta}$ the Lorentz spaces on $\mathbb{R}$ with the Lebesgue measure, and set $x = (x', x_n)$.

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First assume (iii′). We write the Fourier extension operator as

$$(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}(x,t) = \int \mathrm{e}^{2\pi\mathrm{i}(x\cdot\xi+t|\xi|)}u(\xi)\frac{\mathrm{d}\xi}{|\xi|} = \int \mathrm{e}^{2\pi\mathrm{i}x_n\xi_n}\int \mathrm{e}^{2\pi\mathrm{i}(x'\cdot\xi'+t|\xi|)}u(\xi)\frac{\mathrm{d}\xi'}{|\xi|}\,\mathrm{d}\xi_n.$$

By applying (2.2) and (2.3),

$$\|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}} = \|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}_{x',t}L^{p'}_{x_n}} \lesssim \biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x'\cdot\xi'+t|\xi|)}u(\xi)\frac{\mathrm{d}\xi'}{|\xi|}\biggr\|_{L^{p'}_{x',t}L^{p,p'}_{\xi_n}} \lesssim \biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x'\cdot\xi'+t|\xi|)}u(\xi)\frac{\mathrm{d}\xi'}{|\xi|}\biggr\|_{L^{p,p'}_{\xi_n}L^{p'}_{x',t}}.$$

Now, a change of variables and the hypothesis gives

$$\biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x'\cdot\xi'+t|\xi|)}u(\xi)\frac{\mathrm{d}\xi'}{|\xi|}\|_{L^{p'}_{x',t}} = |\xi_n|^{n-2-(n/p')}\biggl\|\int \exp\{2\pi\mathrm{i}(x'\cdot\xi'+t\sqrt{1+|\xi'|^2})\}u(\xi_n\xi',\xi_n)\frac{\mathrm{d}\xi'}{\sqrt{1+|\xi'|^2}}\biggr\|_{L^{p'}_{x',t}} \lesssim |\xi_n|^{n-2-(n/p')}\|u(\xi_n\xi',\xi_n)(1+|\xi'|^2)^{-1/2q'}\|_{L^{q'}_{\xi'}}. \tag{2.8}$$

It follows that

$$\|(u\,\mathrm{d}\sigma_{\mathrm{cone}})^{\vee}\|_{L^{p'}} \lesssim \|\,|\xi_n|^{n-2-(n/p')}\|u(\xi_n\xi',\xi_n)(1+|\xi'|^2)^{-1/2q'}\|_{L^{q'}_{\xi'}}\|_{L^{p,p'}_{\xi_n}} = \|\,|\xi_n|^{(n-2)/q-(n/p')}\|u(\xi)|\xi|^{1/q'}\|_{L^{q'}_{\xi'}}\|_{L^{p,p'}_{\xi_n}}. \tag{2.9}$$

Again one concludes by using the embedding $L^{p,q'}_{\xi_n} \hookrightarrow L^{p,p'}_{\xi_n}$ and Hölder's inequality (2.1) (for the pair p,q is scale invariant).

Assume now (iii). By symmetry and the triangle inequality we can take u supported where $|\xi'| \lesssim 1$ and $\xi_n \approx 1$. Hence, the desired conclusion follows from (2.9), the embedding $L^p \hookrightarrow L^{p,p'}$ and Hölder's inequality for L^p spaces (since $p < q'$). □

3. Proof of Theorem 1.3

We need the following result on the normal form of a hypersurface S in $\mathbb{R}^n$, which is proved in [**9**, Proposition 2.2].

For $P \in S$, denote by $\nu(P)$ the number of principal curvatures which vanish at P (i.e. the dimension of the kernel of the second fundamental form at P).

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Proposition 3.1. *Let S be a hypersurface in $\mathbb{R}^n$, let $P_0 \in S$ and define*

$$\underline{\nu} := \liminf_{P \to P_0} \nu(P) \neq 0, n-1.$$

There is an orthogonal system of coordinates (ξ', ξ'', ξ_n), $\xi' = (\xi_1, \dots, \xi_{n-1-\underline{\nu}})$, $\xi'' = (\xi_{n-\underline{\nu}}, \dots, \xi_{n-1})$ with the origin at P_0 such that, in a neighbourhood of P_0, S is the graph of a function $\xi_n = \phi(\xi', \xi'')$ of the type

$$\phi(\xi', \xi'') = \langle M(\xi', \xi'')\xi', \xi' \rangle, \tag{3.1}$$

where M is a square matrix of size $n - 1 - \underline{\nu}$ with smooth entries.

We now prove a finer result for a surface in $\mathbb{R}^3$ with Gaussian curvature identically zero near a point of type greater than or equal to k.

Proposition 3.2. *Let S be a surface in $\mathbb{R}^3$ and let $P_0 \in S$ be a point of type greater than or equal to k. Suppose that the Gaussian curvature of S vanishes identically near P_0. Then there is an orthogonal system of coordinates (ξ_1, ξ_2, ξ_3) with the origin at P_0 such that, in a neighbourhood of P_0, S is the graph of a function $\xi_3 = f(\xi_1, \xi_2)$ of the type*

$$f(\xi_1, \xi_2) = a(\xi_1, \xi_2)\xi_1^k \tag{3.2}$$

for some smooth function a defined in an open neighbourhood of 0.

We emphasize that the transformation which brings S into the desired form is an orthogonal one, and not merely smooth. This will be essential for applications to the restriction problem. Incidentally, we also see that the notion of 'point of type k' propagates along a segment containing P_0 as interior point (in particular, the set of points of type k does not have isolated points).

Proof of Proposition 3.2. The proof uses induction on k. The statement is true for $k = 2$. This follows from Proposition 3.1 with $n = 3$, if $\underline{\nu} = 1$, whereas if $\underline{\nu} = 2$, then a neighbourhood of P_0 lies on a plane, and the result is trivial. One could also obtain the result for $k = 2$ as a consequence of [**12**, Corollary 6, p. 359] if P_0 is of type 2 and [**12**, Corollary 7, p. 361] if P_0 is of type greater than 2. Suppose then that the statement is true with $k - 1$ in place of k and let P_0 be a point of type greater than or equal to $k \geqslant 3$. By the inductive hypothesis there are orthogonal coordinates (ξ_1, ξ_2, ξ_3) for which S coincides, near the origin, with the graph of a function

$$f(\xi_1, \xi_2) = a(\xi_1, \xi_2)\xi_1^{k-1}. \tag{3.3}$$

Observe that the hypothesis on the Gaussian curvature can be expressed by the equation

$$f_{\xi_1\xi_1} f_{\xi_2\xi_2} = f^2_{\xi_1\xi_2}. \tag{3.4}$$

After substituting in (3.4) the expression for f given in (3.3), we divide by ξ_1^{2k-4} and let $\xi_1 \to 0$. Upon setting $\phi(t) = a(0, t)$ we find the following (singular) Cauchy problem

$$\phi\phi'' = \frac{k-1}{k-2}(\phi')^2, \quad \phi(0) = 0. \tag{3.5}$$

The initial condition in (3.5) comes from the fact that P_0 is of type greater than or equal to k and $\nabla f(0) = 0$, so that $\partial^\alpha f(0) = 0$ for every $\alpha \in \mathbb{Z}^2_+$, $|\alpha| \leqslant k-1$. To finish the proof it suffices to verify that the only solution to (3.5) is the trivial one: $\phi(t) = 0$ for every t. In fact, a Taylor expansion of $a(\xi_1, \xi_2)$ at $\xi_1 = 0$ then gives $a(\xi_1, \xi_2) = b(\xi_1, \xi_2)\xi_1$ for some smooth b, and therefore $f(\xi_1, \xi_2) = b(\xi_1, \xi_2)\xi_1^k$.

To this end we observe that the maximal non-constant solutions to the equation in (3.5) (in the region where the Cauchy well-posedness theorem applies) are of the form

$$\phi(t) = \pm(At + B)^{1/(1-\alpha)}, \quad \alpha = \frac{k-1}{k-2}, \quad A \neq 0,$$

defined on $(-B/A, +\infty)$ if $A > 0$, or $(-\infty, -B/A)$ if $A < 0$. At any rate, since $\alpha > 1$, they blow up at $t = -B/A$ and an elementary continuity argument then shows that there is no solution $\phi \not\equiv 0$ with $\phi(0) = 0$. □

Remark 3.3. We point out that useful normal forms were obtained by Schulz [**10**] for convex hypersurfaces S of finite type, in the sense (different from that in the present paper) that S has no tangents of infinite order. In particular, we see that this condition is never satisfied here, because in (3.2) we have $f(0, \xi_2) \equiv 0$. Moreover, it is worth noting that the normal forms in [**10**] are expressed in terms of a Taylor expansion at a given point P_0, whereas here we deal with the geometry of S in a whole neighbourhood of P_0.

We also recall the following result by Sogge [**11**] (see also [**13**, p. 418]).

Theorem 3.4. *Let ψ be a smooth function on an interval $[-a, a]$, with $\psi^{(j)}(0) = 0$ for $1 \leqslant j \leqslant k-1$ and $\psi^{(k)}(0) \neq 0$. Then there exist constants $0 < \delta < a$ and $C > 0$ such that*

$$\left\| \int_{-\delta}^{\delta} \mathrm{e}^{2\pi \mathrm{i}(tx_1 + \psi(t)x_2)} g(t)\, \mathrm{d}t \right\|_{L^{p'}(\mathbb{R}^2)} \leqslant C \|g\|_{L^{q'}(-\delta,\delta)}$$

for every $p' > 4$, $p' \geqslant k+2$, $p' \geqslant (k+1)q$ and $g \in L^{q'}(-\delta, \delta)$. Moreover, the constants δ and C depend only on a, p, q, k, on upper bounds for finitely many derivatives of ψ on $[-a, a]$ and a lower bound for $\psi^{(k)}(0)$.

The remark on the uniformity of the constants δ and C did not appear in the statement of [**11**], but it easily follows from the proof of that result. Also, the case $k = 2$ was not explicitly considered there (because it had already been treated in [**5**, **29**]), but in any case it also follows from the same proof, under the additional (and necessary) condition $p' > 4$.

Proof of Theorem 1.3. We consider the orthogonal system of coordinates given in Proposition 3.2. Hence, near the point P_0 (which now coincides with the origin), S is the graph of a function $\xi_3 = f(\xi_1, \xi_2)$ of the form (3.2). Let $S_0 = \{(\xi_1, \xi_2, f(\xi_1, \xi_2)) \in \mathbb{R}^3 : |\xi_1| \leqslant \delta,\ |\xi_2| \leqslant \delta\}$, where $\delta > 0$ is a small constant that will be chosen later on. Let u be any smooth function on S supported in S_0. As usual we will think of u as a function of the variables ξ_1, ξ_2. Then we have

$$(u\,\mathrm{d}\sigma)^{\vee}(x_1,x_2,x_3) = \int \mathrm{e}^{2\pi \mathrm{i} x_2\xi_2}\biggl(\int \mathrm{e}^{2\pi\mathrm{i}(x_1\xi_1+x_3 f(\xi_1,\xi_2))}u(\xi_1,\xi_2)\phi(\xi_1,\xi_2)\,\mathrm{d}\xi_1\biggr)\,\mathrm{d}\xi_2, \quad (3.6)$$

where $\phi(\xi_1,\xi_2) = (1+|\nabla f(\xi_1,\xi_2)|^2)^{1/2}$.

Since $p' > 2$, by the Hausdorff–Young inequality we have

$$\begin{aligned}\|(u\,\mathrm{d}\sigma)^{\vee}\|_{L^{p'}(\mathbb{R}^3)} &\lesssim \biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x_1\xi_1+x_3 f(\xi_1,\xi_2))}u(\xi_1,\xi_2)\phi(\xi_1,\xi_2)\,\mathrm{d}\xi_1\biggr\|_{L^{p'}_{x_1,x_3}L^{p}_{\xi_2}} \\ &\leqslant \biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x_1\xi_1+x_3 f(\xi_1,\xi_2))}u(\xi_1,\xi_2)\phi(\xi_1,\xi_2)\,\mathrm{d}\xi_1\biggr\|_{L^{p}_{\xi_2}L^{p'}_{x_1,x_3}}. \qquad (3.7)\end{aligned}$$

Now we apply Theorem 3.4 with $\psi(t) = f(t,\xi_2) = a(t,\xi_2)t^k$. We see that, if δ is small enough, the hypothesis in Theorem 3.4 is satisfied uniformly with respect to the parameter ξ_2. Hence, we deduce that, if δ is small enough, the norm in $L^{p'}$ with respect to x_1, x_3 of the integral on the right-hand side of (3.7) is not greater than

$$C\|u(\cdot,\xi_2)\phi(\cdot,\xi_2)\|_{L^{q'}},$$

uniformly with respect to ξ_2. It follows that

$$\|(u\,\mathrm{d}\sigma)^{\vee}\|_{L^{p'}(\mathbb{R}^3)} \lesssim \|u\|_{L^{p}_{\xi_2}L^{q'}_{\xi_1}},$$

which gives the desired estimate, since $L^{q'}(-\delta,\delta) \hookrightarrow L^{p}(-\delta,\delta)$ because $q' > p$. □

Remark 3.5. By combining the normal form (3.2) for the function f which defines S near P_0 with Knapp-type scaling arguments (see, for example, [**24**,**28**]), one can see that the condition $p' \geqslant (k+1)q$ is always necessary for (1.3) to hold under the hypotheses of Theorem 1.3.

Indeed, (1.3) implies that

$$\biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x_1\xi_1+x_2\xi_2+x_3 f(\xi_1,\xi_2))}u(\lambda\xi_1,\lambda^{\varepsilon}\xi_2)\phi(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\biggr\|_{L^{p'}(\mathbb{R}^3)} \lesssim \|u(\lambda\cdot,\lambda^{\varepsilon}\cdot)\|_{L^{q'}(\mathbb{R}^2)},$$

where $u \not\equiv 0$ is a fixed test function, λ is a large parameter and $\varepsilon > 0$. As above, $\phi(\xi_1,\xi_2) = (1+|\nabla f(\xi_1,\xi_2)|^2)^{1/2}$.

Changing variables gives

$$\begin{aligned}\biggl\|\int \mathrm{e}^{2\pi\mathrm{i}(x_1\xi_1+x_2\xi_2+\lambda^k x_3 f(\xi_1/\lambda,\xi_2/\lambda^{\varepsilon}))}u(\xi_1,\xi_2)\phi(\xi_1/\lambda,\xi_2/\lambda^{\varepsilon})\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\biggr\|_{L^{p'}(\mathbb{R}^3)} \\ \lesssim \lambda^{(1+\varepsilon)/q-(1+k+\varepsilon)/p'}\|u\|_{L^{q'}(\mathbb{R}^2)}. \qquad (3.8)\end{aligned}$$

If one assumes, by contradiction, that $p' < (k+1)q$, then there is $\varepsilon > 0$ such that the exponent of λ on the right-hand side of (3.8) is negative. Hence, since $\lambda^k f(\xi_1/\lambda,\xi_2/\lambda^{\varepsilon}) \to$

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$a(0,0)\xi_1^k$ as $\lambda \to \infty$, by applying dominated convergence to the integral in ξ and the Fatou lemma to the integral in x, we obtain

$$\left\| \int \mathrm{e}^{2\pi \mathrm{i}(x_1\xi_1 + x_2\xi_2 + x_3 a(0,0)\xi_1^k)} u(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \right\|_{L^{p'}(\mathbb{R}^3)} = 0,$$

which is a contradiction because of the uniqueness of the Fourier transform.

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