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THE GROUP OF UNITS OF THE INTEGRAL GROUP RING ZS_3

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We denote by ZG the integral group ring of the finite group G. We call $\pm g$, for g in G, a *trivial unit* of ZG. For G abelian, Higman [4] (see also [3, p. 262 ff]) showed that every unit of finite order in ZG is trivial. For arbitrary finite G (indeed, for a torsion group G, not necessarily finite), Higman [4] showed that every unit in ZG is trivial if and only if G is

(i) abelian and the order of each element divides 4, or

(ii) abelian and the order of each element divides 6, or

(iii) the direct product of the quaternion group of order 8 and an abelian group of exponent 2.

Subsequently Berman [1] showed that, for a finite group G, every unit of finite order in ZG is trivial if and only if either G is abelian or G is the direct product of the quaternion group of order 8 and an elementary abelian 2-group.

In this note we investigate the group of units $U(ZS_3)$ of ZS_3, where S_3 is the symmetric group on three symbols. It is a consequence of Berman's result [1] that ZS_3 contains nontrivial units of finite order. Taussky [9, p. 341 ff] has listed some nontrivial units of order 2 and given some information about $U(ZS_3)$. Our study is guided by the following three interrelated question which we formulate for an arbitrary finite group G.

(a) Is every unit of finite order in ZG conjugate to a trivial unit? (This question was suggested to us by Professor H. Zassenhaus.)

(b) What are the finite subgroups of $U(ZG)$?

(c) Is every normalized automorphism of ZG the product of an inner automorphism and an automorphism of G (see Sehgal [6])?

(An automorphism τ of ZG is said to be *normalized* (see [6]) if $g\tau\sigma=1$ for all g in G, where $\sigma:ZG\to Z$ is the homomorphism such that $g\sigma=1$ for all g in G. There is little loss of generality in considering only normalized automorphisms, since if τ is an automorphism of ZG then τ' given by $g\tau'=(g\tau\sigma)g\tau$ is a normalized automorphism).

We answer these three questions for ZS_3 and also describe the structure of $U(ZS_3)$.

For arbitrary G, we denote $\{\pm 1\}$ in ZG by $C(ZG)$ and by $V(ZG)$ the subgroup of units u of ZG with $u\sigma=1$; clearly $U(ZG)=V(ZG)\times C(ZG)$. Note that each

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conjugacy class C in $V(ZG)$ gives rise to exactly two conjugacy classes C and $-C$ in $U(ZG)$. We summarize our results in the following.

THEOREM. (1) $U(ZS_3) \simeq \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, Z) \mid a+c \equiv b+d \pmod 3.\right\}$

(2) *A nontrivial element of finite order in* $V(ZS_3)$ *has order* 2 *or* 3. *In* $V(ZS_3)$ *all elements of order* 3 *are conjugate, while there are* 2 *conjugacy classes of elements of order* 2, *with generic elements* (12) *and* $t=(12)+3(13)-3(23)-3(123)+3(132)$ *respectively. However all elements of order* 2 *in* $V(ZS_3)$ *are conjugate in* QS_3.

(3) *Every maximal finite subgroup of* $V(ZS_3)$ *is either conjugate to* S_3 *or to* $\{1, t\}$.

(4) *Every normalized automorphism of* ZS_3 *is inner.*

On page 341 of [9], Taussky has given two nontrivial units of order 2 in $V(ZS_3)$. If in these one takes a to be (123) and b to be (12) they are $-(12)+(13)+(23)\pm[(123)-(132)]$ and each is conjugate in $V(ZS_3)$ to t of the theorem.

1. **The group of units.** For a ring R we denote by R_2 the total matrix ring of degree 2 over R. The map θ given below gives an isomorphism from QS_3 into $S=Q\oplus Q\oplus Q_2$ where Q is the field of rational numbers:

$$(12)\quad \theta = \left(1, -1, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\right),$$

$$(123)\quad \theta = \left(1, 1, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\right).$$

We are using here the so-called "natural" irreducible representation of S_3 (see Boerner [2, p. 119]). (In S_3 we multiply thus: (12)(123)=(13).) In fact if $\alpha=(\alpha_1 \cdots \alpha_6)$ denotes the element

$$\alpha_1 . 1+\alpha_2(12)+\alpha_3(13)+\alpha_4(23)+\alpha_5(123)+\alpha_6(132)$$

of QS_3 and if $x=(x_1 \cdots x_6)$ denotes the element

$$\left(x_1, x_2, \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix}\right)$$

of $Q\oplus Q\oplus Q_2$ and we think of α and x as row vectors then $x=\alpha\theta=\alpha A$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

and

$$A^{-1} = \tfrac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 2 & 2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & 2 & -2 & 2 \\ 0 & -2 & 0 & 2 & 2 & -2 \\ 2 & -2 & 2 & 0 & 0 & -2 \end{pmatrix}.$$

In particular, note that

$$t\theta = \left(1, -1, \begin{pmatrix} -5 & 2 \\ -12 & 5 \end{pmatrix}\right).$$

It is clear that $ZS_3\theta \subset Z\oplus Z\oplus Z_2$. It follows from $x\theta^{-1}=xA^{-1}$ that $x\theta^{-1}\in ZS_3$ if and only if

$$x_1+x_2+2x_3+2x_6 \equiv 0 \pmod 6,$$
$$x_1-x_2+2x_3-2x_5-2x_6 \equiv 0 \pmod 6,$$

and four other congruences modulo 6 (obtained from the columns of A^{-1}) are satisfied by $x_1, \ldots, x_6$. It is not difficult to show (e.g., by reducing to a kind of echelon form, being careful to divide equations only by numbers relatively prime to 6) that these 6 congruences are satisfied if and only if each x_i is in Z and

(1) $$x_1+x_2 \equiv 0 \pmod 2, \qquad x_2 \equiv x_6-x_5 \pmod 3,$$

(2) $$x_1 \equiv x_3+x_5 \equiv x_4+x_6 \pmod 3.$$

If we denote the projection of S into Q_2 by ϕ, we see, using the congruences above, that

$$(ZS_3)\theta\phi = \left\{\begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \,\middle|\, x_3+x_5 \equiv x_4+x_6 \pmod 3\right\}$$
$$= Y \quad \text{(say)}.$$

Suppose $x=(x_1 \cdots x_6)\in ZS_3\theta$. Then since $x_6\equiv x_3+x_5-x_4 \pmod 3$ it follows that $x_3x_6-x_4x_5=\delta$ implies that $(x_3-x_4)(x_3+x_5)\equiv\delta \pmod 3$. It then follows from the congruences above that x^{-1} exists and is in $ZS_3\theta$ if and only if

(3) $$x_3x_6-x_4x_5 = \delta = \pm 1, \qquad x_1 = \pm 1 \quad \text{and} \quad x_2 = \delta x_1.$$

The mapping $\theta\phi$ is a ring homomorphism from ZS_3 into Y and so induces a homomorphism of $U(ZS_3)$ into the group $U(Y)$ of units of Y. In fact this induced mapping is an isomorphism onto. For let

$$z = \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \in U(Y).$$

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Then $\delta=x_3x_6-x_4x_5=\pm 1$ and, if x_1, x_2 lying in $\{-1, 0, 1\}$ are defined by (1) and (2) respectively, it follows that neither x_1 nor x_2 is 0 and, in fact, that (3) is satisfied. Thus $\alpha=x\theta^{-1}$ is a unit in ZS_3 with $\alpha\theta\phi=z$. Further it is a consequence of (1), (2) and (3) that α is the unique unit of ZS_3 with $\alpha\theta\phi=z$. We have now proved (1) of the theorem.

A simple calculation shows that

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \tag{4}$$

is a complete set of left coset representatives of $U(Y)$ in $Gl(2, Z)$.

2. **The conjugacy problem.** We denote $(V(ZS_3))\theta\phi$ by $V(Y)$. Then $U(Y)=V(Y)\times C$, where $C=\{\pm I\}$, and of course $V(Y)$ is isomorphic to $V(ZS_3)$. Since $V(Y)$ is isomorphic to a subgroup of $Gl(2, Z)/C$, its nontrivial elements of finite order can have orders 2 and 3 only. Let v in $V(Y)$ have order 3. In $Gl(2, Z)$ any two elements of order 3 are conjugate [3]. Thus v is conjugate in $Gl(2, Z)$ to $(123)\theta\phi=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}=u$. Now if $w\neq I$ is any of the left coset representatives of of $U(Y)$ in $Gl(2, Z)$, given by (4), then by calculation $w^{-1}uw\notin U(Y)$. This means that if $x\notin U(Y)$ then $x^{-1}ux\notin U(Y)$. Thus u and v are conjugate in $U(Y)$ and so also in $V(Y)$. Thus there is only one conjugacy class of elements of order 3 in $V(Y)$ as stated in (2) of the theorem.

Apart from $\{-1\}$ there are two conjugacy classes of elements of order 2 in $Gl(2, Z)$ [3, §74.3]. In fact it is not difficult to see that one of them is

$$C_1=\left\{X=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\in Z_2 \mid a^2+bc=1 \text{ with } a \text{ odd and } b,\, c \text{ even}\right\}.$$

Indeed a simple calculation shows that

$$\{X\in Z_2 \mid X^2=I\}=\left\{X=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\in Z_2 \mid a^2+bc=1\right\}\cup\{\pm I\}.$$

If $B=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, a calculation shows that the conjugacy class of B is contained in C_1. Conversely let $X=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\in C_1$ and let $a=2x+1$, $b=2y$, $c=2z$ where $x, y, z\in Z$. If $y=z=0$, X is obviously conjugate to B. If $y\neq 0$ and if d is the greatest common divisor of x and y then $a^2+bc=1$ means $x(x+1)+yz=0$ and it follows that $[(x+1)d]/y\in Z$, also if

$$Y=\begin{pmatrix}-y/d & d\\ x/d & -[(x+1)d]/y\end{pmatrix}$$

then $Y\in Sl(2, Z)$ and $YBY^{-1}=X$. A similar result holds if $z\neq 0$.

Now $(12)\theta\phi=\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}$ is not in C_1 while $t\theta\phi=\begin{pmatrix}-5 & 2\\ -12 & 5\end{pmatrix}$ is and so these are not even conjugate in $Gl(2, Z)$, but both are in $V(Y)$. By conjugating $(12)\theta\phi$ and $t\theta\phi$ by each matrix in (4) we see that any unit in $V(Y)$ which is conjugate in $Gl(2, Z)$ to $(12)\theta\phi$ or $t\theta\phi$ is in fact conjugate in $V(Y)$ to $(12)\theta\phi$ or $t\theta\phi$ respectively. Thus there are precisely two conjugacy classes of elements of order 2 in $V(Y)$. Because $(12)\theta\phi$ and $t\theta\phi$ are however conjugate in $Gl(2, Q)$ (as $\lambda+1$ and $\lambda-1$ are the elementary divisors of each of them) it is clear that (12) and t are conjugate in QS_3. This completes the proof of (2) of the theorem.

The nontrivial units of order 2 given by Taussky in [9] and referred to above correspond to $\begin{pmatrix}-3 & 4\\ -2 & 3\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}$ in $V(Y)$ and so each is conjugate in $V(ZS_3)$ to t since both of these matrices are in C_1.

Berman [1, Lemma 2] and Takahashi [8] have shown that if G is any finite group and if v is a unit of finite order in ZG then either v is ± 1 or else the coefficient of 1 in v is 0. In the special case we have here this can easily be verified directly from the information given above.

3. **The finite subgroups.** Let $W=Sl(2, Z)\cap U(Y)$ and $T=W\cap V(Y)$; then $W=T\times C$ and $[V(Y):T]=2$. Now $PSl(2, Z)$ is the free product of a group of order 2 and one of order 3 [5, Appendix B]. Because $V(Y)$ has only one conjugacy class of elements of order 3 it follows that T has at most two such classes. Also T contains no elements of order 2 since $-I$ is the only element of order 2 in $Sl(2, Z)$. Since $W/C\cong T$ it then follows from the subgroup theorem [5, §34] that W/C is the free product of a group of order 3 and a free group F. (In fact we can see by using Takahashi's form of the subgroup theorem [7] that F is infinite cyclic, although we don't need to use this extra information here.) In particular the only finite subgroups of T are of order 1 or 3. Now if H is a finite subgroup of $V(Y)$ then, since $H/H\cap T\cong HT/T$, we see that $[H:H\cap T]$ is 1 or 2 and so H has order 1, 2, 3 or 6. (3) of the theorem now follows from (2) of the theorem.

4. **The automorphisms.** Let τ be a normalized automorphism of ZS_3. Then τ induces an automorphism of $V(ZS_3)$. By (3) of the theorem, $S_3\tau=w^{-1}S_3w$ for some w in $V(ZS_3)$. This implies that τ is an inner automorphism and (4) of the theorem is proved.

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