A QUESTION OF C. R. HOBBY ON REGULAR p-GROUPS

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In (**2**) a finite p-group G is said to be nearly regular if it has the following two properties:

(i) There exists a central subgroup Z of order p and G/Z is regular.

(ii) If $x \in G$ and $y \in \gamma_2(G)$, then $gp\{x, y\}$ is regular.

(For unfamiliar notation we refer to (**1**).) C. R. Hobby proved in (**2**) that (i) implies (ii) when $p = 2$ or 3, and an open question is whether (i) implies (ii) for $p \geqq 5$. It has been suggested in (**3**) and (**4**) that this is a deep problem, comparable to the Hughes problem perhaps, though the solution is in fact quite simple; it seems worth while to set the record straight with the present note. We shall exhibit, for any $p \geqq 5$, a finite *metabelian* p-group with (i) but not (ii).

In considering the structure of a metabelian p-group G with property (i) but not (ii) we shall be guided by the necessary and sufficient conditions given in (**1**) for a nilpotent metabelian p-group to be regular, namely that $\gamma_p(H) \leqq \gamma_2(H)^p$ for every two-generator subgroup H. Such a group has of course $p \geqq 5$ and G/Z will not have exponent p.

Let us suppose that G is generated by $\{a, b\}$ and that G has class $p+1$, more precisely that $Z = \gamma_{p+1}(G)$. It follows according to (i) that

$$\gamma_p(G) \leqq \gamma_2(G)^p Z,$$

which implies that $\gamma_{p+1}(G) \leqq \gamma_3(G)^p$, and so $\gamma_p(G) \leqq \gamma_2(G)^p$. Let us define the subgroup K as $gp\{x, y\}$, where $x = a$ and $y = [a, b]$; K is to be the non-regular subgroup appearing in (ii). Let us make K non-regular by arranging that $\gamma_p(K) \leqq \gamma_2(K)^p$ is false. We have $\gamma_p(K) \leqq \gamma_{p+1}(G) \leqq \gamma_3(G)^p$ and so we must not allow $\gamma_3(G)^p \leqq \gamma_2(K)^p$; in particular $\gamma_3(G)^p \neq 1$. Since G/Z is regular we have $\gamma_p(K) \leqq \gamma_2(K)^p Z$, however, and the fact that Z has order p now implies that $Z \leqq K$. Note that $\gamma_p(K)$ cannot be 1 as K is non-regular, so it seems reasonable to put $Z = gp\{z\}$, where $z = [a, b, (p-1)a]$.

We return to $\gamma_3(G)^p \nleqq \gamma_2(K)^p$. Since

$$\gamma_2(K)^p = gp\{[a, b, ia]^p : 1 \leqq i < p\}$$

we shall aim to have $[a, b, b]^p \notin \gamma_2(K)^p$. Earlier remarks indicate that we must avoid $z \in \gamma_2(K)^p$. Since $\gamma_p(G) \leqq \gamma_2(G)^p$ we face the problem of specifying $[a, b, ia, (p-2-i)b]$ as an element of $\gamma_2(G)^p$, for $0 \leqq i \leqq p-2$.

We put

$$[a, b, (p-2)a] = [a, b, b]^p z,$$

a relation which implies $z = [a, b, a, b]^p$ and therefore $[a, b, (p-2)a] \in \gamma_2(G)^p$, without obviously entailing $z \in \gamma_2(K)^p$. We note the further consequence

$$[a, b, a, b]^{p^2} = 1.$$

Next we put

$$[a, b, b, b] = [a, b, a, a, b] = 1$$

and this trivially yields $[a, b, ia, (p-2-i)b] = 1$ for $0 \leqq i < p-2$. Such relations as we have mentioned do not imply that G is a p-group, and so we put $a^{p^2} = b^{p^2} = 1$.

If H is a proper subgroup of G and if H has 2 generators, then the relations give the fact that, modulo Z, H has class $p-1$; so G/Z is regular by the criterion of (**1**), and we have (i), if Z has order p. It therefore remains to establish that $z \neq 1$ and that $[a, b, b]^p \notin \gamma_2(K)^p$, in order to prove (i) and disprove (ii).

At this point a construction, which we shall merely outline, is called for. We start with symbols $c_{00}, c_{10}, \ldots, c_{p-1,0}, c_{01}, c_{11}$ which we suppose generate an abelian group of exponent p^2, and we impose the further relations

$$c_{20}^p = c_{30}^p = \ldots = c_{p-1,0}^p = 1,$$

$$c_{p-2,0} = c_{01}^p c_{11}^p, \; c_{p-1,0} = c_{11}^p.$$

There results a group of order p^{p+4}. From this we may obtain the required example G by adjoining elements a and b, using extension theory, so that

$$a^{p^2} = b^{p^2} = 1, \quad [a, b] = c_{00},$$

$$[c_{ij}, a] = c_{i+1,j}, \quad [c_{ij}, b] = c_{i,j+1},$$

where $c_{i+1,j}$ and $c_{i,j+1}$ are 1 if not in the initial set of symbols. Then G will have order p^{p+8}. Once this is established it is clear that $z = c_{p-1,0}$ has order p and that $[a, b, b]^p = c_{01}^p \notin \gamma_2(K)^p$ where $K = gp\{a, c_{00}\}$ and $\gamma_2(K)^p = gp\{c_{10}^p\}$. Hence:

Theorem. *There is a metabelian p-group, for each $p \geqq 5$, that satisfies* (*i*) *and does not satisfy* (*ii*).

This group does not satisfy the conclusion of Hobby's theorem in (**2**) about nearly regular p-groups either; a fact which may be verified directly by means of Corollary 2.3 of (**1**) for instance.

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