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FLAT SUBMODULES OF FREE MODULES OVER COMMUTATIVE BEZOUT RINGS

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A ring is called Bezout if every finitely generated ideal is principal. We show that every ideal of a commutative Bezout ring R is flat if and only if every submodule of a free R-module is flat. Using this theorem we obtain Neville's theorem.

1. INTRODUCTION

Neville has proved that the topological space X is an F space if and only if every ideal of $C(X)$ is flat, or if and only if every submodule of a free $C(X)$-module is flat. This theorem is the main result in [**4**]. In this paper we define quasi-torsion-free modules, and when R is a commutative Bezout ring we show an R-module is quasi-torsion-free if and only if it is flat. We also show that every ideal of R is flat if and only if every submodule of a free R-module is flat. In Section 3, we prove Neville's theorem using these theorems.

We need to review briefly some standard terminology. In this paper R is always a commutative ring with identity and modules are unital. An R-module is flat if the tensor product is an exact functor. An ideal I of a ring R is called pure if for every $a \in I$, there exists $b \in I$, such that $a = ab$.

We denote by $\mathrm{Max}(R)$ the spectrum of maximal ideals of R. We say R is semiprimitive if $\cap\mathrm{Max}(R) = (0)$. For any ideal I of R and $a \in R$, we set

$$\mathrm{M}(a) = \{M \in \mathrm{Max}(R) : a \in M\} \text{ and } \mathrm{M}(I) = \{M \in \mathrm{Max}(R) : I \subseteq M\}.$$

Then the sets $\mathrm{M}(I) = \bigcap_{a \in I} \mathrm{M}(a)$, where I is an ideal of R, satisfy the axioms for the closed sets of a topology on $\mathrm{Max}(R)$, called the *Stone topology*, see [**3**, 7M].

Throughout, X will denote a completely regular and Hausdorff space and $C(X)$ denotes the ring of continuous real-valued functions on X. Two sets E and F are completely separated if there exists some $f \in C(X)$ such that $f = 0$ on E and $f = 1$ on F. The cozero set of a function $f \in C(X)$ is the set $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$. A space X is an F space if disjoint cozero sets are always completely separated. Several equivalent conditions for F spaces are given in [**3**, Theorem 14.25]; in particular X is an F space if and only if $C(X)$ is a Bezout ring. The reader is referred to [**3**] for undefined terms and notations.

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2. Submodules of free modules

The following lemma is proved in [**1**].

Lemma 2.1. *In a ring R, every principal ideal is flat if and only if for each $a \in R$, $\mathrm{Ann}(a)$ is a pure ideal.*

Remark. Let R be a ring. Suppose

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} A \longrightarrow 0$$

is an exact sequence of R-modules, where F is flat. If A is flat, then for every principal ideal $I = (r)$ of R we have: $K \cap FI = KI$, see [**5**, Theorem 3.55]. It is easy to see that for any $x \in F$, $xr \in K \cap FI$ implies that $xr = kr$, for some $k \in K$.

By redefining the concept of torsion, we can say something interesting about flat modules.

Definition: Consider the exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} A \longrightarrow 0 \tag{1}$$

where F is a flat submodule of a free module. A R-module A is quasi-torsion-free relative to the exact sequence (1), if the following is true: whenever $r \in R$, $x \in A$ and $rx = 0$, there are $x' \in F$ and $k \in K$ such that $\phi(x') = x$ and $rx' = rk$.

The independence of the notion of quasi-torsion-free from the exact sequence (1) follows from the following lemma.

Lemma 2.2. *Let R be a ring. If an R-module is quasi-torsion-free relative to one exact sequence, it is quasi-torsion-free relative to every exact sequence.*

Proof: Suppose that A is quasi-torsion-free relative to the exact sequence

$$0 \longrightarrow K_2 \longrightarrow F_2 \xrightarrow{\phi_2} A \longrightarrow 0$$

where F_2 is a flat submodule of a free module. Use the fact that every module is a quotient of a free module to find a free module F_1 and an onto map $\psi_1 : F_1 \longrightarrow F_2$. Define the following diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \psi_1 & & \| & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & F_2 & \xrightarrow{\phi_2} & A & \longrightarrow & 0 \end{array}$$

by letting $\phi_1 = \phi_2 \circ \psi_1$ and $K_1 = \mathrm{Ker}(\phi_1)$. Consider the exact sequence

$$0 \longrightarrow K_3 \longrightarrow F_1 \xrightarrow{\psi_1} F_2 \longrightarrow 0$$

Suppose $rx = 0$, where $r \in R$ and $x \in A$. Then there exist $x_2 \in F_2$ and $k_2 \in K_2$ such that $\phi_2(x_2) = x$ and $rx_2 = rk_2$. Let $x_1 \in F_1$ and $k_1 \in K_1$ be such that $\psi_1(x_1) = x_2$ and

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$\psi_1(k_1) = k_2$. Therefore $r(x_1 - k_1) \in K_3$. Therefore by the above remark, $r(x_1 - k_1) = rk_3$, some $k_3 \in K_3$. Now $K_3 \subseteq K_1$, so $k_1 + k_3 \in K_1$. Thus we have $\phi_1(x_1) = \phi_2 \circ \psi_1(x_1) = x$ and $rx_1 = r(k_1 + k_3)$. Hence A is quasi-torsion-free relative to the top exact sequence. We note that the middle term of the top exact sequence is free. Now suppose that

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} A \longrightarrow 0 \tag{2}$$

is an arbitrary exact sequence of R-modules, where F is a flat submodule of a free module. By the projectivity of free modules, there exists the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \psi & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{\phi} & A & \longrightarrow & 0 \end{array}$$

As in the above proof, it follows that A is quasi-torsion-free relative to (2). This means that A is quasi-torsion-free relative to every exact sequence. ∎

THEOREM 2.3. *Let R be a ring. Then every flat R-module is quasi-torsion-free. If R is Bezout, every quasi-torsion-free R-module is flat.*

PROOF: Suppose that A is a R-module and consider the exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} A \longrightarrow 0$$

where F is a submodule of a free module. First we claim that A is quasi-torsion-free if and only if $K \cap FJ = KJ$, for all principal ideals J. Let $K \cap FJ = KJ$, for all principal ideals J and let $r \in R$, $x \in A$ and $rx = 0$, then there exists $x' \in F$ such that $\phi(x') = x$. So $rx' \in K \cap FJ$, where $J = (r)$. Hence by the remark, there exists $k \in K$ such that $rx' = rk$ and this implies that A is quasi-torsion-free. Conversely, let A be quasi-torsion-free and $rx \in K \cap FJ$, where $r \in R$ and $x \in F$. Then $r\phi(x) = 0$, so there are $x' \in F$ and $k \in K$ such that $\phi(x') = \phi(x)$ and $rx' = rk$. Since $x = x' + k'$, for some $k' \in K$, then $rx = r(k + k')$. Therefore $rx \in KJ$ and this proves the claim. Now the proof follows from [5, Theorem 3.55]. ∎

We come now to the main result of this section. But we first need the following lemma.

LEMMA 2.4. *Let R be a ring and let A be a submodule of $\coprod_{j \in T} R$. Let π_j be the canonical projection map onto the jth coordinate, and assume the ideal $\pi_n(A) = I_n$ is principal with generator a_n. Then the exact sequence*

$$0 \longrightarrow K_n \xrightarrow{i} A \xrightarrow{\pi_n} I_n \longrightarrow 0$$

splits if and only if there exists $x_n \in A$ with $\pi_n(x_n) = a_n$ and $\mathrm{Ann}(a_n) \subseteq \mathrm{Ann}(x_n)$.

PROOF: Assume that the above exact sequence splits. Let $\psi_n : I_n \longrightarrow A$ be the splitting homomorphism and let $x_n = \psi_n(a_n)$. If $a \in I_n$, then $\psi_n(a) = \psi_n(ba_n)$

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$= b\psi_n(a_n) = bx_n$, for any b such that $ba_n = a$. In other words, $ba_n = 0$ implies that $bx_n = 0$, for any $b \in R$. Consequently, $\operatorname{Ann}(a_n) \subseteq \operatorname{Ann}(x_n)$.

Conversely, suppose that there exists $x_n \in A$ such that $\pi_n(x_n) = a_n$ and $\operatorname{Ann}(a_n) \subseteq \operatorname{Ann}(x_n)$. Define the splitting homomorphism $\psi_n : I_n \longrightarrow A$ by $\psi_n(ba_n) = bx_n$. Now if $ba_n = ca_n$, then $b - c \in \operatorname{Ann}(a_n) \subseteq \operatorname{Ann}(x_n)$, so ψ_n is well defined. Clearly, ψ_n is a module homomorphism. Finally, $\pi_n \circ \psi_n(ba_n) = b\pi_n \circ \psi_n(a_n) = b\pi_n(x_n) = ba_n$, so ψ_n is indeed a splitting homomorphism. $\square$

THEOREM 2.5. *Let R be a Bezout ring. Then every principal ideal of R is flat if and only if every finitely generated submodule of a free R-module is flat.*

PROOF: Suppose that every principal ideal of R is flat. Let A be a finitely generated submodule of a free module. Then A can be embedded in a finitely generated free module. So without loss of generality $A \subseteq \coprod_1^n R$. The proof is by induction on n. If $n = 1$ then A is principal ideal, and so is flat by the hypothesis. Suppose $n > 1$ and the theorem has been proof for all finitely generated modules contained in $\coprod_1^{n-1} R$. Let F_n be the free module $\coprod_1^n R$. Let π_j and I_n be as in the lemma. Since R is Bezout and I_n is finitely generated, $I_n = (a_n)$, for some $a_n \in R$. Consider the homomorphism $\phi : F_n \to F_n$ defined by $\phi(b_1, \dots, b_{n-1}, b_n) = (b_1, \dots, b_{n-1}, b_n a_n)$. We want to consider a suitable submodule B of F_n (or rather $\phi^{-1}(A)$), so that we can apply Lemma 2.4 with $\pi_n(x_n) = 1$. Since A is finitely generated, there exists a finitely generated submodule B of F_n such that $\phi(B) = A$. Since $a_n \in \pi_n(A)$, we may assume without loss of generality that there exists $x'_n \in B$ such that $\pi_n(x'_n) = 1$ (If no such x'_n exists, consider any $x' = (b_1, \dots, b_{n-1}, b_n) \in B$ such that $\pi_n \circ \phi(x') = a_n$, that is, $a_n b_n = a_n$. Let $x'_n = (b_1, \dots, b_{n-1}, 1)$ and enlarge B to include x'_n). Clearly, $\pi_n(B) = R$. Consider the exact sequence

$$0 \longrightarrow L_n \longrightarrow B \xrightarrow{\pi_n} R \longrightarrow 0.$$

Trivially, the hypothesis of Lemma 2.4 is satisfied, and so $B = L_n \oplus R$. Clearly $L_n = \{(b_1, \dots, b_{n-1}, 0) \in B\}$ is embedable in $\coprod_1^{n-1} R$, so that L_n is flat by the inductive hypothesis. Thus B is flat. Now consider the exact sequence

$$0 \longrightarrow K \longrightarrow B \xrightarrow{\phi} A \longrightarrow 0 \tag{3}$$

We shall prove that A is flat by proving that A is quasi-torsion-free relative to the exact sequence (3). First note that the middle term B is a flat submodule of the free module F_n. Now assume that $x \in A$, $r \in R$ and $rx = 0$. We must find $x' \in B$ and $k \in K$ such that $\phi(x') = x$ and $rx' = rk$. Let $x' = (b_1, \dots, b_{n-1}, b_n) \in B$ be such that $\phi(x') = (b_1, \dots, b_{n-1}, b_n a_n) = x$. Therefore $(rb_1, \dots, rb_{n-1}, rb_n a_n) = 0$, that is,

$$r \in \operatorname{Ann}(b_1) \cap \dots \cap \operatorname{Ann}(b_{n-1}) \cap \operatorname{Ann}(b_n a_n).$$

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Since R is Bezout, then

$$\operatorname{Ann}(b_1) \cap \cdots \cap \operatorname{Ann}(b_{n-1}) \cap \operatorname{Ann}(b_n a_n) = \operatorname{Ann}(b),$$

for some $b \in R$. According to Lemma 2.1, $\operatorname{Ann}(b)$ is pure, hence there exists $c \in \operatorname{Ann}(b)$ such that $r = rc$. Set $k = cx'$. Clearly $k \in B$, $rx' = rk$ and $\phi(k) = \phi(cb_1, \ldots, cb_{n-1}, cb_n a_n) = 0$, that is, $k \in K$. Thus A is quasi-torsion-free relative to the exact sequence (3). Hence A is flat, by Theorem 2.3. ∎

It is well-known that a R-module A is flat if and only if every finitely generated submodule of A is flat. Thus we have:

Corollary 2.6. *Let R be a Bezout ring. Then every ideal of R is flat if and only if every submodule of a free R-module is flat.*

3. Gelfand rings

The purpose of this section is to prove Neville's theorem, by the theorems of the previous section. We first give some results about semiprimitive Gelfand rings.

A ring R is called Gelfand (*pm*-ring) if every prime ideal of R is contained in a unique maximal ideal. When the Jacobson radical and the nilradical of ring R coincide, DeMarco and Orsatti [**2**] show that R is Gelfand if and only if $\operatorname{Max}(R)$ is Hausdorff; and if and only if $Spec(R)$ is normal (in general, not Hausdorff). This class of rings contains the classes of regular ring, local rings, zero-dimension rings and $C(X)$.

Definition. Two subsets E and F of $\operatorname{Max}(R)$ are said to be almost separated in $\operatorname{Max}(R)$ if there exists $a \in R$ such that $E \subsetneq \mathrm{M}(a)$ and $F \subsetneq \mathrm{M}(a-1)$.

Proposition 3.1. *Let R be a semiprimitive ring, then every principal ideal in R is flat if and only if for any non-zero $a, b \in R$, $\operatorname{Max}(R) - \mathrm{M}(a)$ and $\operatorname{Max}(R) - \mathrm{M}(b)$ are almost separated whenever $ab = 0$.*

Proof. Suppose every principal ideal is flat and $a, b \in R$ such that $ab = 0$. By Lemma 2.1, $\operatorname{Ann}(a)$ is pure, so there exists $c \in \operatorname{Ann}(a)$ such that $bc = b$. Hence $ac = 0$, $b(c-1) = 0$. Thus

$$\operatorname{Max}(R) - \mathrm{M}(a) \subsetneq \mathrm{M}(c), \ \operatorname{Max}(R) - \mathrm{M}(b) \subsetneq \mathrm{M}(c-1).$$

Conversely, let $a \in R$. We want to show that $\operatorname{Ann}(a)$ is pure. Let $b \in \operatorname{Ann}(a)$. If $b = 0$, then there exists $b = 0 \in \operatorname{Ann}(a)$ such that $b^2 = b = 0$. So we can assume $b \neq 0$. Because $ab = 0$, $\operatorname{Max}(R) - \mathrm{M}(a)$ and $\operatorname{Max}(R) - \mathrm{M}(b)$ are almost separated. Hence there exists $c \in R$ such that

$$\operatorname{Max}(R) - \mathrm{M}(a) \subsetneq \mathrm{M}(c) \ \text{ and } \ \operatorname{Max}(R) - \mathrm{M}(b) \subsetneq \mathrm{M}(c-1)$$

Thus $ac = 0$ and $bc = b$. Hence $\operatorname{Ann}(a)$ is pure. ∎

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Lemma 3.2. *Let R be a Gelfand ring, then the subsets E and F of* $\operatorname{Max}(R)$ *are completely separated if and only if they are almost separated in* $\operatorname{Max}(R)$.

Proof: Assume E and F are completely separated in $\operatorname{Max}(R)$. So $\operatorname{cl} E \cap \operatorname{cl} F = \emptyset$. Hence there exists the ideals I and J such that $\operatorname{cl} E = \mathrm{M}(I)$ and $\operatorname{cl} F = \mathrm{M}(J)$. We claim that $I + J = R$. Otherwise there exists $M \in \operatorname{Max}(R)$ such that $I + J \subseteq M$. So $M \in \mathrm{M}(I) \cap \mathrm{M}(J)$, and this is a contradiction. Therefore $a + b = 1$, for some $a \in I$ and $b \in J$. Thus

$$\mathrm{M}(I) \subseteq \mathrm{M}(a) \ \text{ and } \ \mathrm{M}(J) \subseteq \mathrm{M}(a-1).$$

Conversely, Assume E and F are almost separated in $\operatorname{Max}(R)$. Then there exists $a \in R$ such that

$$E \subseteq \mathrm{M}(a) \ \text{ and } \ F \subseteq \mathrm{M}(a-1).$$

Thus by Urysohn's Lemma there exists the function $f : \operatorname{Max}(R) \longrightarrow R$ such that

$$f\big(\mathrm{M}(a)\big) = 0 \ \text{ and } \ f\big(\mathrm{M}(a-1)\big) = 1.$$

This shows that E and F are completely separated. ∎

The following result is a generalisation of [**1**, Theorem 4].

Theorem 3.3. *Let R be a semiprimitive Gelfand ring. Then every principal ideal in R is flat if and only if for any non-zero $a, b \in R$,* $\operatorname{Max}(R) - M(a)$ *and* $\operatorname{Max}(R) - M(b)$ *are completely separated whenever $ab = 0$.*

Proof: It is obvious from Proposition 3.1 and Lemma 3.2. ∎

Lemma 3.4. *X is an F space if and only if for any non-zero $f, g \in C(X)$,* $\operatorname{Max}\big(C(X)\big) - \mathrm{M}(f)$ *and* $\operatorname{Max}\big(C(X)\big) - \mathrm{M}(g)$ *are completely separated whenever $fg = 0$.*

Proof: We consider the map $\psi : \beta X \to \operatorname{Max}\big(C(X)\big)$ such that $\forall x \in \beta X$, $\psi(x) = M^x$, where βX is the stone-Čech compactification of X. It is well-known that ψ is a homeomorphism, and hence $\operatorname{Max}\big(C(X)\big) \cong \beta X$, see [**3**, Section 6]. Therefore by [**3**, Theorem 7.3], for any $f \in C(X)$, we have:

$$\psi\big(\operatorname*{cl}_{\beta X} \mathrm{Z}(f)\big) = \Big\{M^x \in \operatorname{Max}\big(C(X)\big) : \ f \in M^x\Big\} = \mathrm{M}(f).$$

Consequently, X is an F space if and only if for any non-zero $f, g \in C(X)$, $X - \mathrm{Z}(f)$ and $X - \mathrm{Z}(g)$ are completely separated whenever $fg = 0$ (see [**3**, 14N.4]); if and only if for any non-zero

$$f, g \in C(X), \ \operatorname*{cl}_{\beta X}\big(X - \mathrm{Z}(f)\big) = \beta X - \operatorname*{cl}_{\beta X} \mathrm{Z}(f)$$

and

$$\operatorname*{cl}_{\beta X}\big(X - \mathrm{Z}(g)\big) = \beta X - \operatorname*{cl}_{\beta X} \mathrm{Z}(g)$$

are completely separated in βX, whenever $fg = 0$; and if and only if for any non-zero

$$f, g \in C(X), \ \operatorname{Max}\big(C(X)\big) - \mathrm{M}(f)$$

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and

$$\text{Max}(C(X)) - \text{M}(g)$$

are completely separated whenever $fg = 0$. ∎

Theorem 3.5. *The following are equivalent:*

(1) *X is an F space.*

(2) *every submodule of a free $C(X)$-module is flat.*

(3) *every ideal of $C(X)$ is flat.*

Proof: To prove (1) $\Longrightarrow$ (2) suppose X is an F space, then $C(X)$ is a Bezout ring, by [**3**, Theorem 14.25]. Thus (2) follows from Corollary 2.6, Theorem 3.3 and Lemma 3.4. It is trivial to show (1) $\Longrightarrow$ (2). Finally (3) $\Longrightarrow$ (1) follows from Theorem 3.3 and Lemma 3.4. ∎

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