

Baer semigroups and lattices

D. H. Adams

Baer semigroups, semigroups with zero in which the left and right annihilators of elements are generated as left and right ideals by idempotent elements, are of increasing importance in lattice theory. They were introduced by D.J. Foulis [1] and M.F. Janowitz [3, 4], who showed that any lattice with extreme elements is isomorphic to the lattice of left annihilators of elements of a suitable Baer semigroup. Thus many problems of lattice theory can be translated into the theory of Baer semigroups. This thesis consists of results on this theme, of which the first is a Baer semigroup characterisation of orthocomplemented lattices.

We then observe that in any semigroup with zero the set of annihilators of *subsets* of the semigroup is a complete lattice. In a Baer semigroup, this observation gives us a method of completing the coordinatised lattice, and we show that, in the case of involution lattices [3], this completion is the well known completion by cuts. It has been pointed out by Professor Janowitz that the semigroup completion is always the completion by cuts. In any semigroup with no non-zero nilpotent elements, the complete lattice of annihilators of subsets is boolean.

This result applies to semilattices in particular, and we consider its relation to the boolean lattice of closed elements in a pseudo-complemented semilattice [2]; and to the generalisation of pseudo-complementedness discussed in [6]. This leads to a generalisation of Nachbin's characterisation of boolean lattices [5].

In the last chapter we return to the direct study of Baer semigroups. By regarding maps from each element to the idempotent generators of its annihilators as unary operations, Baer semigroups may be regarded as algebras of type $\langle 2, 1, 1, 0 \rangle$, called *Baer algebras*. These form an

Received 5 August 1970. Thesis submitted to Monash University, March 1970. Degree approved July 1970. Supervisor: Professor P.D. Finch.

equational class, and any equational class of lattices with extreme elements is coordinatised by an equational class of Baer algebras. We prove this by showing that Baer algebra homomorphisms, direct products and subalgebras generate similar constructions on the lattices that they coordinatise, but it is also quite straightforward to translate lattice identities into Baer algebra identities. We consider the cases of Baer semigroups with involution and Baer $*$ -semigroups, and obtain some partial converses for the latter.

References

- [1] David J. Foulis, "Baer $*$ -semigroups", *Proc. Amer. Math. Soc.* 11 (1960), 648-654.
- [2] Orrin Frink, "Pseudocomplements in semilattices", *Duke Math. J.* 29 (1962), 505-514.
- [3] M.F. Janowitz, "Baer semigroups", *Duke Math. J.* 32 (1965), 85-95.
- [4] M.F. Janowitz, "A semigroup approach to lattices", *Canad. J. Math.* 18 (1966), 1212-1223.
- [5] Leopoldo Nachbin, "Une propriété caractéristique des algèbres Booléiennes", *Portugal. Math.* 6 (1947), 115-118.
- [6] J.C. Varlet, "A generalisation of the notion of pseudo-complementedness", *Bull. Soc. Roy. Sci. Liège* 37 (1968), 149-158.