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AN IMPROVED SUBGROUP THEOREM FOR HNN GROUPS WITH SOME APPLICATIONS

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1. Introduction. In [**4**], a subgroup theorem for HNN groups was established. The theorem was proved by embedding the given HNN group in a free product with amalgamated subgroup and then applying the subgroup theorem of [**3**]. In this paper we obtain a sharper form of the subgroup theorem of [**4**] by applying the Reidemeister-Schreier method directly, using an appropriate Schreier system of coset representatives. Specifically, we prove (in Theorem 1) that if *H is a subgroup of the* HNN *group*

(1) $$G = \langle t, K; tLt^{-1} = M\rangle,$$

then H is an HNN *group whose base is a tree product of vertices* $cKc^{-1} \cap H$ *where c ranges over a double coset representative system for G* mod (H, K); *the amalgamated and associated subgroups are contained in vertices of this base and are of the form* $dMd^{-1} \cap H$ *where d ranges over a double coset representative system for G* mod (H, M).

This improved subgroup theorem for HNN groups was obtained independently by D. E. Cohen [**1**] using Serre's theory of groups acting on trees.

Using the present version of the subgroup theorem, several proofs in [**4**] can be simplified and results strengthened (see, e.g., [**1**]). Here we give two new applications of the improved subgroup theorem.

Our first application deals with subgroups with non-trivial center of one-relator groups.

Definition. A *treed* HNN *group* is an HNN group whose base is a tree product and whose associated subgroups are contained in vertices of the tree product base.

Let H be a f.g. (finitely generated) subgroup with center Z $(\neq 1)$ *of a torsion-free one-relator group G. Then H is a free abelian group of rank two, or H is a treed* HNN *group with infinite cyclic vertices and with center contained in the center of the base* (see Theorem 2).

Two corollaries are the following:

If H is a subgroup with center Z $(\neq 1)$ *of a torsion-free one-relator group, then Z is infinite cyclic unless H is free abelian of rank two or H is locally infinite cyclic.*

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If H is a f.g. subgroup of the centralizer C of an element x in a torsion-free one-relator group and $H \cap gp(x) = 1$, then H is a free group.

The first corollary was obtained independently by Mahimovski [**8**]. Theorem 2 generalizes Pietrowski's [**12**] characterization of one-relator groups having non-trivial centers. The centralizer of an element in a one-relator group with torsion is always cyclic (see Newman [**11**] or [**4**, p. 956]).

Our second application connects the structure of a subgroup of finite index of a certain type of treed HNN group to its index. Classical examples of such a connection are given by the Schreier rank formula for free groups, the Euler characteristic for fundamental groups of orientable compact surfaces as compared with that of a j-sheeted covering space, and the Riemann-Hurwitz formula for Fuchsian groups. Each of these cases may be viewed as associating a number $\chi(G)$ to each group G in the class so that if $G:H = j$, then $\chi(H) = j \cdot \chi(G)$; indeed, we take this property as the defining property of a characteristic defined on a class of groups closed under taking subgroups of f.i. (finite index). Specifically, for the free group G take $\chi(G) = 1$-rank G, for the fundamental group $G = \langle a_1, b_1, \ldots, a_g, b_g, \prod[a_i, b_i]\rangle$ let $\chi(G) = 2 - 2g$, and for the Fuchsian group

$$G = \langle c_1, \ldots, c_t, a_1, b_1, \ldots, a_g, b_g; c_1^{\gamma_1}, \ldots, c_t^{\gamma_t}, c_1^{-1} \ldots c_t^{-1}[a_1, b_1] \ldots [a_g, b_g]\rangle$$

let $\chi(G) = 2g - 2 + \sum(1 - \gamma_i^{-1})$. In all three cases if $\chi(G) \neq 0$, then isomorphic subgroups of f.i. must have the same index; indeed, in the first two cases $\chi(H)$ determines H (up to isomorphism). In any case, knowing the index of the subgroup H determines $\chi(H)$, and therefore limits the structure of H.

Wall [**15**] introduced a "rational Euler characteristic" for finite extensions of discrete groups which admit a finite complex as classifying space. For these groups, not only does $\chi(H) = j \cdot \chi(G)$ when $G:H = j$, but also the formula $\chi(A * B) = \chi(A) + \chi(B) - 1$ holds. The class of groups considered by Wall includes finite extensions of f.g. free groups, and for these groups Stallings [**14**] generalized Wall's formula to $\chi(A * B;\, U) = \chi(A) + \chi(B) - |U|^{-1}$, where U is a finite group (of order $|U|$), and A, B are finite extensions of f.g. free groups. We generalize this further to show that *if G is a treed* HNN *group with finitely many vertices $A_1, \ldots, A_r$ each of which is a finite extension of a free group, and there are finite amalgamated subgroups $U_1, \ldots, U_{r-1}$ and finitely many pairs of finite associated subgroups $L_1, M_1, \ldots, L_n, M_n$, then Wall's characteristic $\chi(G)$ is given by*

$$(2) \qquad \chi(G) = \chi(A_1) + \ldots + \chi(A_r) - |U_1|^{-1} - \ldots - |U_{r-1}|^{-1} - |M_1|^{-1} - \ldots - |M_n|^{-1}$$

(see Theorem 3).

We then extend the formula (2) using the more general notion of characteristic (indicated above) to other classes of treed HNN groups (see Theorem 4).

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The generalized formula applies to (Kleinian) function groups (certain discontinuous subgroups of LF (2, C)).

2. The subgroup theorem for HNN groups. Let G be as in (1). We may suppose that a set of generating symbols is chosen for K which includes a subset $\{m_i\}$ which generates M and a corresponding subset $\{l_i\}$ where $l_i = t^{-1}m_i t$, which generates L. A K-symbol is one of the chosen K-generators or its inverse; an M-symbol is one of the m_i or its inverse.

Let H be a subgroup of G. In the proof of Theorem 1 below we shall show that there exists a Schreier coset representative system for G mod H of the form $\{D_k \cdot E_m \cdot Q(m_i)\}$ where $Q(m_i)$ is a word in M-symbols, $E_m \cdot Q(m_i)$ is a word in K-symbols, D_k does not end in a K-symbol, $D_k \cdot E_m$ does not end in an M-symbol, and in no representative does t follow a non-empty M-symbol. Moreover, $\{D_k\}$ is a representative system for G mod (H, K), and $\{D_k \cdot E_m\}$ is a representative system for G mod (H, M).

THEOREM 1. *Let G be as in* (1), *let H be a subgroup of G, and let a Schreier representative system for G* mod *H be chosen as described above. Then H is a treed* HNN *group whose vertices are of the form $D_kKD_k^{-1} \cap H$ (where D_k ranges over the full double coset representative system for G* mod *(H, K)) and whose amalgamated and associated subgroups are of the form $D_kE_mKE_m^{-1}D_k^{-1} \cap H$ (where D_kE_m ranges over the full double coset representative system for G* mod *(H, M)).*

Proof. The proof of the theorem is analogous to that of the proof of the subgroup theorem (Theorem 5) of [**3**], and so we merely sketch the argument.

First we construct a Schreier representative system for G mod H of the type described. For this purpose define the length of an (H, K) double coset as the shortest length of any word in it. For the (H, K) coset of length 0, we choose the empty word 1 as its K-double coset representative. To obtain the Schreier representatives for the H-cosets of H in HK, we supplement the double coset representative 1 with a special Schreier system (defined after Lemma 5, page 240 of [**3**]) for K mod $K \cap H$ with respect to M.

Assume we have defined Schreier representatives (in this manner) for all cosets of H contained in a double coset of (H, K) of length less than r. Let HWK and W have length $r > 0$. Now W ends in a t-symbol; hence $W = Vt^\epsilon$, $\epsilon = \pm 1$. Moreover, the Schreier representative V^* of V has already been defined and has the form $V^* = D_k \cdot E_m \cdot Q(m_i)$. If $\epsilon = 1$, then $D_kE_mQ(m_i)t = D_kE_mtQ(l_i)$, and so $HD_kE_mtK = HWK$, and we choose D_kE_mt as the double coset representative of HWK. If $\epsilon = -1$, then choose $D_kE_mQ(m_i)t^{-1}$ as the double coset representative D of HWK. In either case we supplement our chosen double coset representative D of HWK with a special Schreier representative system for K mod $K \cap D^{-1}HD$ with respect to M. We have now constructed a Schreier coset representative system for G mod H as described above.

Using this Schreier system and the corresponding rewriting process, we may apply the Reidemeister-Schreier method (see [**7**, Section 2.3]) to obtain a presentation for H from our presentation for G. Now H has generators $\{s_{N,x}\}$ and $\{s_{N,t}\}$ where N is a Schreier representative and x is a K-generator. Moreover, $\{s_{N,x}\}$, where N has a fixed (H, K) double coset representative D_k and x ranges over the K-generators, generates the subgroup $D_kKD_k^{-1} \cap H$; $\{s_{N,y}\}$, where N has a fixed (H, M) double coset representative D_kE_m and y ranges over the M-generators, generates the subgroup $D_kE_mME_m^{-1}D_k^{-1} \cap H$. Moreover, if the relators of K are conjugated by those N with a fixed D_k, and then the rewriting process τ is applied, the resulting relators together with the trivial generators $s_{N,x}$ provide a set of defining relators for $D_kKD_k^{-1} \cap H$.

The defining relators for H that arise from rewriting $\{tl_it^{-1} = m_i\}$ enable us to eliminate the generators $s_{N,t}$ where N is not a double coset representative for G mod (H, M); moreover, the remaining relators take the form

$$(3)\quad s_{D_kE_m,t}((D_kE_mt)^*L(D_kE_mt)^{*-1} \cap H)s_{D_kE_m,t}^{-1} = D_kE_mME_m^{-1}D_k^{-1} \cap H.$$

Now (3) describes an amalgamation which takes place between vertices $(D_kE_mt)^*K(D_kE_mt)^{*-1} \cap H$ and $(D_kE_m)K(D_kE_m)^{-1} \cap H$ if $s_{D_kE_m,t}$ is a trivial generator (i.e., $(D_kE_mt)^* \approx D_kE_mt$); otherwise, (3) describes a pair of associated subgroups from these same vertices.

Specifically, if $D_kE_mQ(m_j)$ is a representative, then $s_{D_kE_mQ,t}$ is freely equal to

$$\tau[(D_kE_m)^*Q(m_j)(D_kE_mQ(m_j))^{*-1}] \cdot s_{D_kE_m,t} \cdot \tau[(D_kE_mt)^*Q(l_j)(D_kE_mQ(l_j))^{*-1}],$$

and hence if $Q(m_j) \neq 1$, we may eliminate the generators $s_{D_kE_mQ,t}$; the remaining relators become those in (3) together with the trivial generators in $\{s_{D_kE_m,t}\}$. The amalgamations described in (3) lead to a tree product of vertices $D_kKD_k^{-1} \cap H$ for the following reason (see [**7**, Lemma 1]): Assign as level of the vertex $D_kKD_k^{-1} \cap H$, the number r of t-symbols in D_k; then the unique vertex of level less than r with which $D_kKD_k^{-1} \cap H$ has a subgroup amalgamated is the subgroup $DKD^{-1} \cap H$ where D is obtained from D_k by deleting the last t-symbol and then deleting any K-syllable immediately preceding that.

COROLLARY 1. *The rank of the free part of H as described in Theorem* 1 *is* $[G:(H, M)] - [G:(H, K)] + 1$.

Proof. $(D_kE_mt)^* \approx D_kE_mt$ if and only if either D_kE_mt is a Schreier representative and therefore an (H, K) double coset representative, or $E_m = 1$ and D_k ends in t^{-1}. Thus there exists a one-one correspondence between (H, K) double coset representatives ending in t or t^{-1} and the trivial generators in $\{s_{D_kE_m,t}\}$. But there are $G:(H, K) - 1$ double coset representatives for G mod (H, K) ending in t or t^{-1}; hence the assertion follows.

The following corollary will be used in the proof of Theorem 4:

COROLLARY 2. *Let G be a treed* HNN *group with finitely many vertices, f.g. free part, and finite amalgamated and associated subgroups. Then any subgroup H of* *f.i. is a treed* HNN *group with finitely many vertices each of which is a conjugate of the intersection of H with some conjugate of a vertex of G; the amalgamated and associated subgroups are conjugates of the intersections of H with certain conjugates of the amalgamated and associated subgroups of G.*

Proof. The proof is by induction on the sum s of the rank of the free part of G and the number of vertices in G. If $s = 2$, the result follows from the subgroup theorem of [**3**] or Theorem 1 above. Otherwise, suppose G is as in (1) where K is now a treed HNN group with smaller s than that of G. Then H is a treed HNN group whose vertices are of the form $cKc^{-1} \cap H = c(K \cap c^{-1}Hc)c^{-1}$, which by inductive hypothesis is a treed HNN group of the desired type. Now an amalgamated or associated subgroup of H has the form $dMd^{-1} \cap H$. Thus H is an HNN group whose base is a tree product with treed HNN groups as vertices and finite amalgamated subgroups, and H itself has finite associated subgroups. It follows as in the argument for the proof of Theorem 1 of [**2**] that H is a treed HNN group of the asserted form. In a similar way, it follows that if $G = (A * B;\ U)$ where B has smaller s than that of G and A is one of the original vertices of G, then H will be a treed HNN group of the desired type.

3. Subgroups with non-trivial center of one-relator groups.

THEOREM 2. *Let G be a group with one defining relator R where R is not a true power, and let H be a f.g. subgroup of G with non-trivial center Z. Then H is free abelian of rank two, or H is a treed* HNN *group with infinite cyclic vertices and Z is contained in the center of the base of H.*

Proof. If R has syllable length one, then G is free, H is infinite cyclic, and the result holds. Assume R has syllable length greater than one; then G can be embedded in an HNN group

$$G_1 = \langle t, K;\ \text{rel}\ K,\ tLt^{-1} = M\rangle$$

where K is a one-relator group whose relator is shorter than R and L, M are free (see e.g., [**4**]). Suppose H is not free abelian of rank two.

Now by Theorem 1, a f.g. subgroup H of G_1 is a treed HNN group

$$H = \langle t_1, \ldots, t_n, S;\ \text{rel}\ S,\ t_1Lt_1^{-1} = M_1, \ldots\rangle$$

where S is a tree product of finitely many vertices $A_1, \ldots, A_r$, each A_i being a subgroup of a conjugate of K; the amalgamated and associated subgroups are free.

If $n \neq 1$, then Z is contained in S; for, $H = \Pi^*(\text{gp}(t_i, S); S)$.

First suppose $Z \not< S^H$. Then $n = 1$. Since some element in Z is not in S^H, and H is f.g., and S/S^H is infinite cyclic, it follows that S^H is f.g. (see Murasugi [**10**]). Therefore $S^H = L_1$ is free and f.g. Consequently, H has the asserted form by [**2**, Theorem 3].

Therefore we may assume $Z < S^H$. We show, in fact, that $Z < S$. If $n \neq 1$, we are finished. Suppose $n = 1$. Then S^H is an infinite stem product (i.e., a tree product in which each vertex has at most two edges incident with it) of vertices $t_1{}^iSt_1{}^{-i}$. If $M_1 \neq S \neq L_1$, then the stem product is proper (i.e., each amalgamated subgroup is a proper subgroup of its containing vertices), and therefore Z is contained in S. If S equals L_1 or M_1, then S is free; S^H is an ascending union of free groups and has a non-trivial center, so that S must be infinite cyclic. If $S = \text{gp}(a) = L_1$, and $M_1 = \text{gp}(a^q)$, then $H = \langle t_1, a;\ t_1at_1{}^{-1} = a^q\rangle$. Since $Z \cap S \neq 1$, $t_1a^rt_1{}^{-1} = a^{qr} = a^r$ for some $r \neq 0$. Hence $q = 1$, and H would be free abelian of rank two. Therefore Z must be contained in S.

Suppose next S consists of a single vertex, $S = gKg^{-1} \cap H$. If $n = 0$, then $H = S$, is a f.g. subgroup with non-trivial center of the group gKg^{-1}; therefore by the inductive hypothesis, H has the desired form. If $n > 0$, and some L_i or M_i equals S, then S is free with non-trivial center, and so must be infinite cyclic. Thus again H has the asserted form with base S.

We may therefore assume that S^H is a proper tree product of the vertices $t_i{}^jSt_i{}^{-j}$, and so $Z < L_i \cap M_i$. Since L_i, M_i are free, Z, L_i, M_i must each be infinite cyclic. Therefore S is a f.g. subgroup of gKg^{-1} and the inductive hypothesis applies to S. Since Z is infinite cyclic, it follows that S is a treed HNN group with infinite cyclic vertices each of which contains Z, and each of the associated subgroups contains Z. Therefore S/Z is a treed HNN group with finite cyclic vertices; moreover, L_i/Z goes into M_i/Z under conjugation by t_i. Hence H/Z is an HNN group with finite cyclic vertices, and the associated subgroups of H/Z are finite. Therefore H/Z is a treed HNN group with finite cyclic vertices, and so by the proof of [**2**, Theorem 3] H has the asserted form.

Finally, suppose S does not consist of a single vertex. Then S is a proper tree product and Z is contained in the amalgamated subgroups of S; these are free and therefore infinite cyclic. Moreover, since $Z < L_i \cap M_i$, we have that L_i, M_i are infinite cyclic. Hence each of the vertices A_j of S is f.g. and the inductive hypothesis applies to each A_j. Hence A_j/Z is a treed HNN group with finite cyclic vertices, and the amalgamated and associated subgroups when reduced mod Z yield finite cyclic groups. Thus H/Z is an HNN group whose base is a tree product of treed HNN groups with finite cyclic vertices; the amalgamated and associated subgroups are finite cyclic groups. Hence H/Z is a treed HNN group with finite cyclic vertices, and consequently H has the asserted form (again by the proof of [**2**, Theorem 3]).

COROLLARY 1. *Let H be a subgroup with non-trivial center Z of a torsion-free one-relator group G, H not free abelian of rank two and not locally infinite cyclic. Then Z is infinite cyclic.*

Proof. If H is f.g., then Z is infinite cyclic because Z is in the center of the tree product base of H, which has infinite cyclic vertices.

Suppose H is infinitely generated. Then H is the ascending union of count-

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ably many f.g. subgroups H_i each containing a f.g. subgroup Z_i of Z such that Z is the ascending union of the Z_i. Now by Moldavanski [**9**] or Newman [**11**], no abelian subgroup of G can be a proper ascending union of free abelian groups of rank two. Hence only finitely many H_i can be free abelian of rank two. Thus Z_i must be infinite cyclic, and so Z is infinite cyclic if Z is f.g.

Suppose Z is infinitely generated. Then H/Z is periodic. For otherwise, for some element h of H, $\mathrm{gp}(h, Z_i)$ is free abelian of rank two, and $\mathrm{gp}(h, Z) = \bigcup \mathrm{gp}(h, Z_i)$ which is impossible. Hence, if C_i is the center of H_i, then H_i/C_i is on the one hand periodic, and on the other hand a treed HNN group with finite cyclic vertices. Therefore, H_i/C_i is finite, and so H_i is infinite cyclic. Consequently, H is locally infinite cyclic.

COROLLARY 2. *If H is a f.g. subgroup of the centralizer C of an element x in a torsion-free one-relator group and $H \cap \mathrm{gp}(x) = 1$, then H is a free group.*

Proof. Let $H_1 = \mathrm{gp}(H, x)$, which is the direct product $H \times \mathrm{gp}(x)$. If H_1 is free abelian of rank two, then clearly H is infinite cyclic. If H_1 is not free abelian of rank two, then the center Z of H_1 is infinite cyclic and therefore equals $\mathrm{gp}(x)$. Now since H_1 is a treed HNN group with finitely many cyclic vertices each of which contains Z and each of whose associated subgroups contains Z, it follows that H_1/Z is a treed HNN group with finite cyclic vertices, which is isomorphic to H. Since H is torsion-free, H must be free.

4. Characteristics of groups.

LEMMA 1. *Suppose G is as in* (1) *and R is a subgroup of K such that R has trivial intersection with the conjugates of L and M in K. Let $\{\alpha_i\}$ be a common double coset representative system for K* mod (R, M) *and K* mod (R, L). *Then the subgroup*

$$H = R * \prod_j * \mathrm{gp}\, (\alpha_j t \alpha_j^{-1})$$

is of index $[K:(R, M)] \cdot |M|$. In particular, if $K:R$ and $|M|$ are both finite, then a common double coset representative $\{\alpha_i\}$ exists and H is of finite index in G; if R is free (*or torsion-free*), *then so is H.*

Proof. We show H is a subgroup of the asserted form and index by constructing H using an appropriate Schreier representative system and a corresponding right coset function. For this purpose choose a set of generating symbols for K which is the union of the following three subsets: the symbols $\{\alpha_i\}$, the symbols $\{r_q\}$ where r_q ranges over the elements of R, and the symbols $\{m_j\}$ where m_j ranges over the elements of M; the empty symbol 1 is included among the symbols $\{\alpha_i\}$ as well as $\{m_j\}$. We use the symbols l_j to denote $t^{-1}m_j t$.

As Schreier representatives take the words $\{\alpha_i m_j\}$. A corresponding right coset function is determined by the following assignments: $(\alpha_i m_j k)^* = \alpha_u m_v$ where $\alpha_i m_j k = r_q \alpha_u m_v$, for k any K-symbol; $(\alpha_i m_j t)^* = \alpha_u m_v$ where $\alpha_i l_j =$

$r_q\alpha_u m_v$; and $(\alpha_i m_j t^{-1})^* = \alpha_u m_v$ where $\alpha_i m_j = r_q \alpha_u l_v$. It is not difficult to show that these assignments define a permutation representation of G acting on the chosen representatives $\{\alpha_i m_j\}$, and hence determine a subgroup H of elements of G which leave the representative 1 fixed.

Clearly, $H \cap K = R$; for, the first of the three representative assignments holds when k is any element of K, and so if $(k)^* = \alpha_u m_v = 1$ then $k = r_q$. This enables us to show that the Schreier system $\{\alpha_i m_j\}$ has the required properties to apply Theorem 1. In particular, 1 is the HK double coset representative, and $\{\alpha_i\}$ is a set of representatives for G mod (H, M). Therefore H is a treed HNN group with a single vertex $K \cap H = R$, the amalgamated and associated subgroups are $\alpha_i M \alpha_i^{-1} \cap H = \alpha_i M \alpha_i^{-1} \cap R = 1$; and its free part is generated by $s_{\alpha_i, t} = \alpha_i t(\alpha_i t)^{*-1} = \alpha_i t \alpha_i^{-1}$.

Let G contain a free subgroup F of rank r and finite index j. Then Wall's rational Euler characteristic $\chi(G)$ (mentioned in the introduction) is given by

$$\chi(G) = (1 - r)/j$$

(this is obtained using Wall's formulas quoted and that the Euler characteristic of an infinite cyclic group is 0). In particular, if G is finite, then $\chi(G) = |G|^{-1}$.

LEMMA 2. *Let G be as in* (1). *Suppose that K contains a free subgroup R of finite index, and that M is finite. Then the Wall characteristic of G is given by*

$$\chi(G) = \chi(K) - \chi(M) = \chi(K) - |M|^{-1}.$$

Proof. Applying Lemma 1, we see that H of that Lemma is free and of finite index in G. Moreover,

$$\chi(G) = (1 - \text{rank } H)/[K:(R, M)] \cdot |M|,$$

and rank H = rank $R + [K:(R, M)]$. Therefore

$$\begin{aligned}\chi(G) &= \{1 - (\text{rank } R + [K:(R, M)])\}/[K:(R, M)] \cdot |M| \\ &= (1 - \text{rank } R)/[K:R] - |M|^{-1} \\ &= \chi(G) - \chi(M).\end{aligned}$$

THEOREM 3. *If G is a treed* HNN *group with vertices* $A_1, \ldots, A_r$ *each of which is a finite extension of a free group, finite amalgamated subgroups* $U_1, \ldots, U_{r-1}$, *and pairs of finite associated subgroups* $L_1, M_1, \ldots, L_n, M_n$, *then Wall's characteristic* $\chi(G)$ *is given by*

$$(4) \quad \chi(G) = \chi(A_1) + \ldots + \chi(A_r) - |U_i|^{-1} - \ldots - |U_{r-1}|^{-1} - |M_1|^{-1} - \ldots - |M_n|^{-1}.$$

Proof. The proof of (4) is clearly obtained by using Lemma 2, and Stalling's formula quoted in the introduction.

We generalize Wall's characteristic as follows:

Definition. Let $\mathscr{C}$ be a class of groups closed under taking subgroups of f.i. Then a *characteristic* χ defined on $\mathscr{C}$ is a real-valued function defined on $\mathscr{C}$ such that if G is in $\mathscr{C}$ and $G:H = j$, then $\chi(H) = j \cdot \chi(G)$.

In addition to the illustrations of characteristics mentioned in the introduction we give the following:

1. Let $\mathscr{C}_1$ be a class of groups with a characteristic χ_1 defined on it. Let $\mathscr{C}$ be the class of all groups which contain a subgroup of f.i. which lies in $\mathscr{C}_1$. If G is in $\mathscr{C}$, and $G:C = p$ where C is in $\mathscr{C}_1$, define $\chi(G) = \chi_1(C)/p$. Clearly if $G:D = q$ where D is in $\mathscr{C}_1$, and $C/C \cap D = c$, $D/C \cap D = d$, then

$$\chi_1(C)/p = \chi_1(C \cap D)/cp = \chi_1(C \cap D)/dq = \chi_1(D)/q,$$

so that $\chi(G)$ is well-defined. Moreover, if $G:H = j$, and $H:E = r$ where E is in $\mathscr{C}_1$, then $\chi(H) = \chi_1(E)/r = j \cdot \chi_1(E)/jr = j \cdot \chi(G)$.

2. Let $\mathscr{C}$ be the class of subgroups of f.i. of a fixed group G. Then a necessary and sufficient condition for a non-zero characteristic to be definable on $\mathscr{C}$ is that isomorphic subgroups of f.i. in G have the same index in G. Indeed, if $H_1 \simeq H_2$, $G:H_1 = j_1$, $G:H_2 = j_2$, and $\chi(G) \neq 0$, then $\chi(H_1) = j_1 \cdot \chi(G) = \chi(H_2) = j_2 \cdot \chi(G)$, so that $j_1 = j_2$. Conversely, define $\chi(G) = 1$, $\chi(H) = j$ when $G:H = j$; then $\chi(G)$ is a well-defined characteristic.

Our last example of a characteristic makes use of Theorem 1 and the subgroup theorem of [**3**].

THEOREM 4. *Suppose $\mathscr{C}_1$ is a class of f.g. groups with a characteristic χ_1 defined on $\mathscr{C}_1$ and such that each group in $\mathscr{C}_1$ contains a torsion-free non-cyclic indecomposable* (*with respect to free product*) *subgroup of finite index. Let $\mathscr{C}$ be the class of treed* HNN *groups with f.g. free part, finitely many vertices each in $\mathscr{C}_1$, and finite amalgamated and associated subgroups. Suppose G is in $\mathscr{C}$, and has a presentation as a treed* HNN *group with vertices $A_1, \ldots, A_r$ in $\mathscr{C}_1$, amalgamated subgroups $U_1, \ldots, U_{r-1}$, and pairs of associated subgroups $L_1, M_1, \ldots, L_n, M_n$. If we set*

$$\chi(G) = \chi(A_1) + \ldots + \chi(A_r) - |U_1|^{-1} - \ldots - |U_{r-1}|^{-1} - |M_1|^{-1} - \ldots - |M_n|^{-1},$$

then χ defines a characteristic on the class $\mathscr{C}$.

Proof. We first observe that the class $\mathscr{C}$ is closed under forming treed HNN groups with vertices from $\mathscr{C}$, using finite amalgamated and associated subgroups (for an argument, see the proof of Theorem 1 of [**2**]).

Next we recall (see [**3**]) that a subgroup H of $(A * B; U)$ is a treed HNN group with vertices $cAc^{-1} \cap H$, $dBd^{-1} \cap H$ where c, d range over double coset representative systems for G mod (H, A) and G mod (H, B), respectively; moreover, the amalgamated and associated subgroups are of the form $eUe^{-1} \cap H$ where e ranges over a double coset representative system for G mod (H, U).

It follows from Corollary 2 of Theorem 1 that $\mathscr{C}$ is closed under taking subgroups of f.i. We now show that if $G:H = j$, then for each presentation of G as a treed HNN group in $\mathscr{C}$, H has a presentation as a treed HNN group in $\mathscr{C}$ for which $\chi(H) = j \cdot \chi(G)$. Indeed, suppose that this assertion holds for A, B in $\mathscr{C}$, and consider $G = (A * B;\ U)$, U finite. Now $cAc^{-1}:cAc^{-1} \cap H = j_c$ is the number of H cosets in HcA. Hence $cAc^{-1} \cap H$ has a treed HNN presentation in $\mathscr{C}$ such that $\chi(cAc^{-1} \cap H) = j_c \cdot \chi(cAc^{-1}) = j_c \cdot \chi(A)$. Similarly, if $j_d = dBd^{-1}:dBd^{-1} \cap H$, and $j_e = eUe^{-1}:eUe^{-1} \cap H$, then

$$\begin{aligned}\chi(H) &= \sum_c j_c \cdot \chi(A) + \sum_d j_d \cdot \chi(B) - \sum_e j_e \cdot |U|^{-1} \\ &= j[\chi(A) + \chi(B) - |U|^{-1}] \\ &= j \cdot \chi(G).\end{aligned}$$

Similarly, if the assertion of the preceding paragraph holds for K in $\mathscr{C}$, and G is as in (1) with M finite, and $G:H = j$, then H is a treed HNN group with vertices $fKf^{-1} \cap H$ where f ranges over a representative system for G mod (H, K); moreover the amalgamated and associated subgroups are of the form $gMg^{-1} \cap H$ where g ranges over a coset representative system for G mod (H, M). If $j_f = fKf^{-1}:fKf^{-1} \cap H$, and $j_g = gMg^{-1}:gMg^{-1} \cap H$, then

$$\begin{aligned}\chi(H) &= \sum_f j_f \cdot \chi(K) - \sum_g j_g \cdot |M|^{-1} \\ &= j \cdot [\chi(K) - |M|^{-1}] \\ &= j \cdot \chi(G).\end{aligned}$$

Finally, we show that χ is well-defined on the class $\mathscr{C}$. Clearly, the only ambiguity in the definition of $\chi(G)$ is that G may be presentable in several ways as a treed HNN group in $\mathscr{C}$. Now an element G_1 of $\mathscr{C}_1$ cannot be written in a non-trivial way as a treed HNN group with finite amalgamated and associated subgroups; for otherwise, G_1 would have two or infinitely many ends (see Stallings [**13**]), so that any torsion free subgroup of finite index would have two or infinitely many ends and would therefore be infinite cyclic or a proper free product (see Stallings [**13**]), contrary to hypothesis. Hence χ is well-defined on the elements of $\mathscr{C}_1$. Consider any torsion-free group T in $\mathscr{C}$. Now T has a unique representation as a treed HNN group in $\mathscr{C}$, namely, as a free product of a free group and groups from $\mathscr{C}_1$. Using the uniqueness of representation of a f.g. group as a free product of indecomposable groups, it follows that $\chi(T)$ is well-defined. Lastly, a group G in $\mathscr{C}$ has a torsion free subgroup T of f.i., say p (by Stallings [**14**] and Lemma 1 above), and so $\chi(G) = \chi(T)/p$, so that $\chi(G)$ is well-defined.

COROLLARY. *Let G be as described in Theorem 4, and $G:H = j$. Suppose that H has a presentation as a treed HNN group with vertices $B_1, \ldots, B_s$, amalgamated subgroups $V_1, \ldots, V_{s-1}$, and pairs of associated subgroups $P_1, Q_1, \ldots, P_m, Q_m$. Then*

$$\chi_1(B_1) + \ldots + \chi_1(B_s) = j(\chi_1(A_1) + \ldots + \chi_1(A_r)),$$

and

$$|V_1|^{-1} + \ldots + |V_{s-1}|^{-1} + |Q_1|^{-1} + \ldots + |Q_m|^{-1} = j(|U_1|^{-1} + \ldots + |U_{r-1}|^{-1} + |M_1|^{-1} + \ldots + |M_n|^{-1}).$$

Proof. Since χ_1 can be replaced by $\chi_2 = 2\chi_1$ and the assertion of Theorem 4 will still hold, the result follows.

As an illustration of Theorem 4, let $\mathscr{C}_1$ be the class of Fuchsian groups described in the introduction, and let χ_1 be the characteristic mentioned there. Then it is well-known that each group G in $\mathscr{C}_1$ is a finite extension of a torsion-free non-cyclic indecomposable subgroup G_1 of the form

$$G_1 = \langle a_1, b_1, \ldots, a_g, b_g; \prod[a_i, b_i]\rangle, \quad g \geqq 0.$$

The resulting class $\mathscr{C}$ includes Kleinian function groups (see [**6**]).

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