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H(ϕ) SPACES

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ABSTRACT. Let ψ be a non-decreasing continuous subadditive function defined on $[0, \infty)$ and satisfy $\psi(x) = 0$ if and only if $x = 0$. The space $H(\psi)$ is defined as the set of analytic functions in the unit disk which satisfy

$$\sup_{0<r<1} \int_0^{2\pi} \psi(|f(re^{i\theta})|)\, d\theta < \infty,$$

and the space $H^+(\psi)$ is the space of all $f \in H(\psi)$ for which

$$\sup_{0<r<1} \int_0^{2\pi} \psi(|f(re^{i\theta})|)\, d\theta = \int_0^{2\pi} \psi(|f(\theta)|)\, d\theta$$

where $f(\theta) = \lim_{r \uparrow 1} f(re^{i\theta})$ almost everywhere.

In this paper we study the $H(\psi)$ spaces and characterize the continuous linear functionals on $H^+(\psi)$.

Introduction. Let ϕ be a real-valued function defined on $[0, \infty)$ satisfying the following:

1. ϕ is increasing,
2. $\phi(x + y) \leq \phi(x) + \phi(y)$ for all x, y in $[0, \infty)$,
3. $\phi(x) = 0$ if and only if $x = 0$ and
4. ϕ is continuous at zero (from the right).

Such a function is called a modulus function; some examples of modulus functions are x^P, $0 < p \leq 1$, $\log(1 + x)$, in fact if ϕ is modulus then so is $\dfrac{\phi}{1+\phi}$.

Let $H(\Delta)$ denote the space of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let

$$H(\phi) = \left\{ f : f \in H(\Delta) \text{ and } \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(|f(re^{i\theta})|) d\phi < \infty \right\}$$

where ϕ is a modulus function. We define a distance function on $H(\phi)$ by

$$|f - g|_\phi = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(|f(re^{i\theta}) - g(re^{i\theta})|)\, d\theta.$$

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If we take $\phi(x) = x^P$, $0 < p \le 1$, then $H(\phi) = H^P$ and if $\phi(x) = \log(1 + x)$, then $H(\phi) = N$, see [6] for definition. We will use $|f|_\phi$ to denote

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta.$$

DEFINITION. A modulus function ϕ is called strongly modulus if it satisfies:

1) $\displaystyle\int_1^\infty \frac{\phi(x)}{x^2}\, dx < \infty,$

2) $\displaystyle\lim_{x \to \infty} \frac{\phi(x)}{\log x} > 0$ and

3) $\phi(|f|)$ is subharmonic for all $f \in H(\Delta)$.

Examples of strongly modulus functions are x^P, $0 < p < 1$, and $\log(1 + x)$. We define

$$H^+(\phi) = \left\{f \in H(\phi) : \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(|f(re^{i\theta})|)d\theta = \frac{1}{2\pi} \int_0^{2\pi} \phi(|f(e^{i\theta})|)d\theta\right\}$$

where

$$f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}).$$

In this paper we study some basic properties of $H(\phi)$ spaces and give an example of a modulus function $\hat{\phi}$ such that $H(\hat{\phi}) \underset{\neq}{\subset} H^P$ for all p, $0 < p < 1$ but $H^1 \underset{\neq}{\subset} H(\hat{\phi})$. We also characterize the continuous linear functionals on $H^+(\phi)$ for ϕ strongly modulus, a result which could be considered a generalization of the one given in [5] and [6].

1. Basic properties of H(ϕ):

LEMMA 1. *If ϕ is a modulus function, then* $H^1 \subset H(\phi)$.

PROOF. $\phi(x) \le \phi([x] + 1) \le ([x] + 1)\phi(1)$, so if $x > 1$, then $\phi(x) \le 2x\phi(1)$, and if $x \le 1$, then $\phi(x) \le \phi(1)$.

Now let $f \in H^1$, and for any $0 < r < 1$, let

$$A_r = \{\theta : |f(re^{i\theta})| \le 1\},$$
$$B_r = \{\theta : |f(re^{i\theta})| > 1\}.$$

Then

$$\begin{aligned}\int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta &= \int_{A_r} \phi(|f(re^{i\theta})|)d\theta + \int_{B_r} \phi|f(re^{i\theta})|d\theta \\ &< 2\pi\phi(1) + 2\phi(1) \int_{B_r} |f(re^{i\theta})|\, d\theta \\ &\le 2\pi\phi(1) + 2\phi(1) \int_0^{2\pi} |f(re^{i\theta})|d\theta < \infty,\end{aligned}$$

since $f \in H^1$. Hence $f \in H(\phi)$.

THEOREM 1. *If* $\lim_{x\to\infty} \frac{\phi(x)}{x} = \alpha > 0$ *then* $H(\phi) = H^1$.

PROOF. The condition implies that there exists $M > 0$ such that $\phi(x) > \alpha x$ in $[M, \infty)$.

Let $f \in H(\phi)$ and consider for $0 < r < 1$, $A_r = \{\theta : |f(re^{i\theta})| < M\}$, $B_r = \{\theta : |f(re^{i\theta})| \geq M\}$, then

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq \int_{A_r} M d\theta + \frac{1}{\alpha}\int_{B_r} \phi\Big(|f(re^{i\theta})|\Big) d\theta$$
$$\leq 2\pi M + \frac{1}{\alpha}|f|_\phi,$$

hence $f \in H^1$. Using Lemma 1 we conclude that $H(\phi) = H^1$.

REMARK: It is clear that if ϕ is bounded then $H(\phi) = H(\Delta)$.

LEMMA 2. *Let* ϕ *be a modulus, then* $\frac{1}{1-z} \in H(\phi)$ *if and only if* $\int_1^\infty \frac{\phi(x)}{x^2} dx < \infty$.

PROOF. Suppose $1/(1-z) \in H(\phi)$ and let $z = re^{i\theta}$, then

$$|1-z|^2 = (1 - r\cos\theta)^2 + r^2\sin^2\theta$$
$$= 1 - 2r\cos\theta + r^2 = 1 - 2r\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots\right) + r^2$$
$$\leq (1-r)^2 + \theta^2, \qquad 0 \leq \theta \leq \pi.$$

Let δ be a (small) positive number and let r_0 be such that $0 < 1 - r_0 < \delta$, then for $z = re^{i\theta}$ with $\pi \geq \theta > \delta$ and $r > r_0$ we have

$$|1 - z|^2 \leq 2\theta^2, \qquad \text{hence } \phi\left|\frac{1}{1-z}\right| \geq \phi\left(\frac{1}{2\theta}\right),$$

but

$$\frac{1}{1-z} \in H(\phi), \text{ so } \int_\delta^\pi \phi\left(\frac{1}{2\theta}\right) d\theta < M \qquad \text{for all } \delta > 0.$$

Set $x = \frac{1}{2\theta}$, then

$$\int_1^{1/2\delta} \frac{\phi(x)}{x^2} dx < M \quad \text{for all } \delta > 0$$

so

$$\int_1^\infty \frac{\phi(x)}{x^2} dx < \infty.$$

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Conversely: suppose that $\int_1^\infty \phi(x)/x^2\,dx < \infty$, as it was shown in [5] $|1 - z|^2 \geq \theta^2/4$ for $\frac{1}{2} \leq r < 1$ and $0 < \theta \leq \delta$ (where δ is sufficiently small). To show that $1/1 - z \in H(\phi)$, it is enough to show that $\int_0^\delta \phi|1/(1 - re^{i\theta})|\,d\theta \leq M$ for all $r > \frac{1}{2}$. But for $0 < \theta < \delta$ we have

$$\int_0^\delta \phi\left|\frac{1}{1 - re^{i\theta}}\right| d\theta \leq \int_0^\delta \phi\left(\frac{2}{\theta}\right) d\theta \leq 2\int_{2/\delta}^\infty \frac{\phi(x)}{x^2}\,dx < M$$

hence $1/1 - z \in H(\phi)$.

From Lemmas 1 and 2 we get

THEOREM 2. *If ϕ is a modulus and $\int_1^\infty \frac{\phi(x)}{x^2}\,dx < \infty$, then $H^1 \underset{\neq}{\subset} H(\phi)$.*

REMARK. We believe that $\int_1^\infty \phi(x)/x^2\,dx < \infty$ is a necessary and sufficient condition for $H^1 \underset{\neq}{\subset} H(\phi)$.

We now give an example of a modulus function $\hat{\phi}$ such that $H(\hat{\phi}) \neq H^1$ and $H(\hat{\phi}) \underset{\neq}{\subset} H^p$ for all $0 < p < 1$.

Let $p_n = n/n + 1$, $n = 1, 2, \ldots$. Define $\hat{\phi}(x)$ on $[0, \infty)$ by

$$\hat{\phi}(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 4 \\ x^{p_n}, & 2^{n(n+1)} \leq x \leq 2^{(n+1)(n+2)}; \quad n \text{ even} \\ y_n^{(x)}, & 2^{n(n+1)} \leq x \leq 2^{(n+1)(n+2)}; \quad n \text{ odd} \end{cases}$$

where $y_n(x)$ represents the line segments joining the points $(2^{n(n+1)}, 2^{(n-1)(n+1)})$, $(2^{(n+1)(n+2)}, 2^{n(n+2)})$. Using elementary computations one can show that $\hat{\phi}$ is a modulus. To show that $H(\hat{\phi}) \underset{\neq}{\subset} H^p$ for all p, $0 < p < 1$, choose n such that

$$p < \frac{n}{n+1} = p_n, \qquad \text{then } H^{p_n} \underset{\neq}{\subset} H^p$$

and since $x^{p_n} < \hat{\phi}(x)$ for all $x > 2^{(n+1)(n+2)}$ one can obtain by an argument similar to the one given in Lemma 1, that $H(\hat{\phi}) \subset H^{p_n} \underset{\neq}{\subset} H^p$. Consider now $\int_1^\infty \phi(x)/x^2\,dx$, it is clear that

$$\int_1^\infty \frac{\hat{\phi}(x)}{x^2}\,dx = \sum_{n=0}^\infty \int_{I_n} \frac{\hat{\phi}(x)}{x^2}\,dx,$$

where $I_0 = [0, 2^2]$, $I_n = [2^{n(n+1)}, 2^{(n+1)(n+2)}]$, but

$$\int_{I_n} \frac{\hat{\phi}(x)}{x^2}\,dx \leq \int_{I_n} \frac{x^{p_n}}{x^2}\,dx \leq (n+1)\left[\frac{1}{2^n} - \frac{1}{2^{n+2}}\right],$$

hence $\int_1^\infty \frac{\hat{\phi}(x)}{x^2}\,dx < \infty$, so by Theorem 2 we get $H^1 \underset{\neq}{\subset} H(\hat{\phi})$.

LEMMA 3. *Let* ϕ *be strictly increasing modulus function such that* $\phi(|f|)$ *is subharmonic for all* $f \in H(\Delta)$, *then* $|f(z)| \leq \phi^{-1}(4|f|_\phi/1 - r)$ *for all* $z = re^{i\theta} \in \Delta$.

PROOF. Since $\phi(|f|)$ is subharmonic in $|z| < \rho$ and continuous in $|z| \leq \rho$ where $0 < \rho < 1$, and since plurisubharmonic is a subharmonic in one variable, then by Lemma 1 in [3] we get

$$\phi(|f(z)|) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z|^2}{|\rho e^{\theta} - z|^2}\, \phi(|f(\rho e^{i\theta})|)\, d\theta.$$

Hence if $z = re^{i\theta}$, at $\rho = (1 + r)/2$ we obtain

$$\phi(|f(z)|) \leq \frac{4|f|_\phi}{1 - r},$$

hence

$$|f(z)| \leq \phi^{-1}\left(\frac{4|f|_\phi}{1 - r}\right).$$

LEMMA 4. *If* ϕ *is a modulus function which satisfies* $\underline{\lim}_{x\to\infty} (\phi(x))/(\log x) > 0$ *and* $\phi(|f|)$ *is subharmonic for every* $f \in H(\Delta)$, *then* $\lim_{r\to 1} f(re^{i\theta})$ *exists almost everywhere and* $|f|_\phi = \lim_{r\to 1^-} 1/2\pi \int_0^{\pi} \phi(|f(re^{i})|)\, d\theta$.

PROOF. An argument similar to the one given in Lemma 1 yields that

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \log^+(|f(re^{i\theta})|)\, d\theta < \infty,$$

hence $f \in N$, so f has a radial limit a.e.[2]. Now $\phi(|f|)$ is subharmonic for each $f \in H(\Delta)$ so

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta$$

is an increasing function of r, $r \in [0, 1)$ so

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta,$$

and that proves the lemma.

If ϕ is modulus such that $\phi(|f|)$ is subharmonic for all $f \in H(\Delta)$ then $H^+(\phi)$ becomes the subspace of $H(\phi)$ which consists of all f such that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \phi(|f(e^{i\theta})|)\, d\theta$$

For $\phi(x) = x^p$, $H^+(\phi) = H^P$ [2] and for $\phi(x) = \log(1 + x)$, $H(\phi) = N$ and $H^+(\phi) = N^+$ [6].

THEOREM 3. *If ϕ is strictly increasing modulus function which satisfies $\phi(|f|)$ is subharmonic for every $f \in H(\Delta)$ and* $\lim_{x\to\infty} (\phi(x))/(\log x)) > 0$, *then $(H^+(\phi), |\,|_\phi)$ is an F-space over $\mathbb{C}$.*

PROOF. To show that the space is complete. Let $\{f_n\}$ be a Cauchy sequence in $H(\phi)$, then by Lemma 3 we have for any compact set K

$$|f_n(z) - f_m(z)| \leq \phi^{-1}\left(\frac{4|f_n - f_m|_\phi}{1-r}\right)$$

for all $z \in K \subset \{w \in \mathbb{C} : |w| < r\}$, this shows that $\{f_n\}$ is a Cauchy sequence in $H(\Delta)$, hence it converges uniformly on compact subsets of Δ to a function $f \in H(\Delta)$. We need to show that $f \in H(\phi)$, and f_n converges to f in $H(\phi)$. Since f_n converges uniformly on compact sets, then $\phi(|f_n|)$ converges uniformly on compact sets to $\phi(|f|)$. Since for all $r < 1$ we have $\{z \in \mathbb{C} : |z| = r\}$ is compact in Δ so

$$\int_0^{2\pi} \phi(|f(re^{i\theta})|)\, d\theta = \lim_{n\to\infty} \int_0^{2\pi} \phi(|f_n(re^{i\theta})|)\, d\theta \leq \lim_{n\to\infty} \int_0^{2\pi} \phi(|f_n(e^{i\theta})|)\, d\theta \leq M,$$

the inequality before the last is because of Lemma 4 and the last one is because $\{f_n\}$ is a Cauchy sequence in $H(\phi)$. Now the rest of the proof is similar to the one given in [6] for N^+, one only needs to use properties of ϕ among which is the fact that $\phi(|\alpha x|) \leq ([|\alpha|] + 1)\phi(x)$ where $[|\alpha|]$ is the largest integer in $|\alpha|$.

2. **Continuous linear functions on $\mathbf{H^+(\phi)}$.** We now study the space of continuous linear complex valued functionals on $H^+(\phi)$ which we will denote by $(H^+(\phi))^*$. The spaces $(H^p)^*$, $0 < p < 1$ were studied in [1, 5] and $(N^+)^*$ in [6].

THEOREM 4. *Let ϕ be a strongly modulus function. Then $T \in (H^+(\phi))^*$ if and only if there exists $g \in H(\Delta)$ such that*

$$T(f) = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{r}{\rho} e^{i\theta}\right) g(\rho e^{-i\theta})\, d\theta$$

where $0 < r < \rho < 1$.

PROOF. Let $T \in (H^+(\phi))^*$ and let $b_k = T(z^k)$, $k = 0, 1, 2, \ldots$. Now $\{z^k : k = 0, 1, 2, \ldots\}$ is a bounded set in $H^+(\phi)$ and T is a continuous linear functional on F-space, so T is bounded [4] and $T(z^k)$ $(k = 0, 1, 2, \ldots)$ is a bounded set, so the function $g(z) = \sum_{k=0}^\infty b_k z^k$ is analytic in Δ. Let $f(z) = \sum_{n=0}^\infty a_n z^n \in H^+(\phi)$, then for $r \in (0, 1)$ put $f_r(z) = f(rz)$, f_r converges to f in $H^+(\phi)$, the proof is exactly as in [6].

Now,

$$T(f_r) = T\left(\lim_{n\to\infty} \sum_{k=0}^{n} a_k r^k z^k\right) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k b_k r^k = \sum_{k=0}^{\infty} a_k b_k r^k$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(r\rho^{-1} e^{i\theta}) g(\rho e^{-i\theta})\, d\theta, \qquad 0 < r < \rho < 1.$$

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But $f_r \to f$ in $H^+(\phi)$ as $r \to 1$, hence

$$T(f) = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} f(r\rho^{-1}e^{i\theta})\, g(\rho e^{-i\theta})\, d\theta.$$

Conversely, suppose that

$$T(\mathrm{f}) = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} f(r\rho^{-1}e^{i\theta}) g(\rho e^{i\theta})\, d\theta$$

exists for all $f \in H^+(\phi)$.

For each $r \in (0, 1)$ let

$$T_r(f) = \frac{1}{2\pi}\int_0^{2\pi} f(r\rho^{-1}e^{i\theta}) g(\rho e^{i\theta})\, d\theta.$$

Clearly $T_r \in (H^+(\phi))^*$ for T_r is linear and if f_n converges to f in $H^+(\phi)$, then by Lemma 3 f_n converges to f uniformly on compact subsets of Δ, hence $T_r(f_n)$ converges to $T_r(f)$. But $\lim_{r\to 1} T_r(f)$ exists for all $f \in H^+(\phi)$, hence by the uniform boundedness principle [4] it follows that $T(f) = \lim_{r\to 1} T_r(f)$ is continuous.

REMARK. Although the topologies on $H^+(\phi)$ and N^+ are different in general we do have the following:

COROLLARY. *If* $T \in (N^+)^*$, then $T \in (H^+(\phi))^*$.

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