

NEMITSKY OPERATORS BETWEEN ORLICZ-SOBOLEV SPACES

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A theorem on Nemitsky operators between Sobolev spaces, contained in a 1973 paper of Marcus and Mizel, is extended to Orlicz-Sobolev spaces.

1. Introduction

Let Ω be a domain in \mathbb{R}_n , and let the function $g(x, t)$ be defined for $x \in \Omega$, $t \in \mathbb{R}_m$. The Nemitsky operator G is defined on functions $u: \Omega \rightarrow \mathbb{R}_m$ by

$$(1.1) \quad (Gu)(x) = g(x, u_1(x), \dots, u_m(x)) \quad \text{for } x \in \Omega.$$

The two papers [6] and [7] of Marcus and Mizel contain theorems giving conditions under which a Nemitsky operator G is a mapping between Sobolev spaces. One of these theorems, Theorem 2.1 in [7], is extended to Orlicz-Sobolev spaces in Hardy [4]. In the present paper, we extend another theorem of the same type, Theorem 2.2 in [6].

Part of the proof of this theorem involves estimates of the terms of a chain rule for Gu . Marcus and Mizel derive a suitable chain rule (see Lemma 2.1 and Theorem 2.1 in [6], and also Equation 3.23 below) under the

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assumptions that (i) g is locally Lipschitz on $\Omega \times \mathbb{R}_m$, and (ii) g possesses a total differential except for a certain null set. (As the complete statement of (ii) is rather long, and moreover is not of direct concern to us, we refer the reader to [6] for details.)

For brevity, we shall refer to conditions (i) and (ii) of Lemma 2.1 in [6] (i.e., (i) and (ii) above) collectively as condition (*).

For g and Ω as described above, Theorem 2.2 in [6] can then be stated as follows:

THEOREM 1.1. (Marcus and Mizel) *Let $\Omega \subset \mathbb{R}_n$ be a bounded domain satisfying the cone condition, and suppose g satisfies the condition (*).*

Let $v \geq 1$, $1 < p < n$, $r = vnp/(v-1)(n-p)$, [$r = \infty$ for $v = 1$], $q = vnp/[n + (v-1)p]$, and $|t| = |t_1| + \dots + |t_n|$.

Suppose that, for every $(x, t) \in \Omega \times \mathbb{R}_m$ where any derivative mentioned below exists,

$$(1.2) \quad \left| \frac{\partial g(x, t)}{\partial x_i} \right| \leq a_1(x) + b_1 |t|^v, \quad i = 1, \dots, n,$$

$$(1.3) \quad \left| \frac{\partial g(x, t)}{\partial t_j} \right| \leq a_2(x) + b_2 |t|^{v-1}, \quad j = 1, \dots, m,$$

where b_1 and b_2 are constants, $a_1 \in L_p(\Omega)$, and $a_2 \in L_r(\Omega)$.

Then

$$(1.4) \quad G : W_{1,q}(\Omega)^m \rightarrow W_{1,p}(\Omega)$$

and

$$(1.5) \quad \|Gu\|_{W_{1,p}(\Omega)} \leq \text{const}(1 + \|u\|_{W_{1,q}(\Omega)^m}^v) \quad \text{for all } u \in W_{1,q}(\Omega)^m,$$

where the constant depends on Ω , a_1 , a_2 , b_1 , b_2 , and

$g(x, 0) = f(x)$, but not on n .

For our extension of this theorem (see Theorem 3.1 below), we also assume that g satisfies condition (*), but make the following changes to the bounds given in (1.2) and (1.3):

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(α) the power function $|t|^\nu$ is replaced by an N -function M , assumed to satisfy a Δ_2 -condition, and to have an increasing derivative;

(β) the Lebesgue spaces L_p, L_q, L_r are replaced by suitable Orlicz spaces $L_P, L_Q,$ and L_R .

We then find that, with the Sobolev spaces replaced by Orlicz-Sobolev spaces, the conclusion (1.4) still holds. We also obtain an estimate analogous to (1.5).

We conclude with two examples. The first, Example 4.1, shows that our Theorem 3.1 contains Theorem 1.1 (for $\nu > 1$ only) as a particular case. The second, Example 4.2, shows that our Theorem 3.1 contains results not included in Theorem 1.1.

2. Preliminaries

2.1. ORLICZ SPACES. We shall use the properties of N -functions and Orlicz spaces as given in Krasnosel'skii and Rutickii [5]. In some places, all of which occur in the list of facts below, our notation differs from that in [5], and sometimes, for later convenience, we have introduced a more specific notation. We shall only need to consider Orlicz spaces defined on bounded domains $\Omega \subset R_n$.

Let M be an N -function, and let M'_+ denote the right derivative of M . Then

$$(i) \quad uM'_+(u) > M(u) \quad \text{for } u > 0,$$

and

$$(ii) \quad M(\alpha u) \leq \alpha M(u) \quad \text{for } 0 \leq \alpha \leq 1.$$

The inequality is reversed if $\alpha \geq 1$.

M is said to satisfy the Δ_2 -condition if there exists a constant $u_0 \geq 0$ and a constant (≥ 2) , here denoted by K_M , such that

$$(iii) \quad M(2u) \leq K_M M(u) \quad \text{for } u \geq u_0.$$

A condition equivalent to (iii) is that there exists a real-valued

function, here denoted by k_M , and a constant $u_0 \geq 0$, such that, for $l > 1$,

$$(iv) \quad M(lu) \leq k_M(l)M(u) \quad \text{for } u \geq u_0.$$

From (ii), $k_M(l) \geq 1$, and a modification of the proof of (iv) given in [5] shows that k_M may be chosen to have the form $k_M(l) = Cl^\mu$, where $C > 1$ and $\mu > 1$.

Suppose Ψ is a function which is right-continuous, non-decreasing, and positive, for positive values of its argument (for example, M'_+), and for which an inequality of the form (iii) holds. We shall still say that Ψ satisfies the Δ_2 -condition, even if Ψ is not an N -function.

Conditions (iii) and (iv) are still equivalent, but now we do not necessarily have $k_\Psi(l) \geq 1$ for $l > 1$.

If M satisfies the Δ_2 -condition, there exists a constant $u_0 > 0$ such that

$$(v) \quad uM'(u)/M(u) < K_M \quad \text{for all } u \geq u_0.$$

(vi) If the function Q is convex and even for large values of u and if $\lim_{u \rightarrow \infty} Q(u)/u = \infty$, then there exists an N -function M such that

$M(u) = Q(u)$ for large values of u , and in this case Q is said to be the *principal part* of M .

(vii) If M is an N -function, we may characterise the Orlicz space $L_M = L_M(\Omega)$ as the set of all measurable functions u , defined almost everywhere on Ω , such that there exists a constant $\lambda > 0$ such that

$$\int_{\Omega} M[\lambda u(x)] dx < \infty.$$

We shall use the Luxemburg norm, here denoted by $\| \cdot \|_M = \| \cdot \|_{M, \Omega}$.

$$(viii) \quad \text{If } \|u\|_M > 0, \quad \int_{\Omega} M[u(x)/\|u\|_M] dx \leq 1.$$

(ix) Let M_1 and M_2 be N -functions. If there exist constants u_0 and K such that $M_2(u) \leq M_1(Ku)$ for $u \geq u_0$, we write $M_2 < M_1$, and

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in this case there exists a constant C such that

$$\|u\|_{M_2} \leq C\|u\|_{M_1} \quad \text{for } u \in L_{M_1} .$$

(x) Let P, Q and R be N -functions, and suppose that there exist complementary N -functions Φ and $\tilde{\Phi}$ such that

$$\Phi < P^{-1} \circ Q$$

and

$$\tilde{\Phi} < P^{-1} \circ R .$$

Then there exists a constant C such that

$$\|uw\|_P \leq C\|u\|_Q \|w\|_R .$$

(The notation " $<$ " is still meaningful even if $P^{-1} \circ Q$ and $P^{-1} \circ R$ are not N -functions.)

Here, and elsewhere in this paper, we shall use the convention made on page 8 of [5] to define the inverse of an N -function. (The convention is also made that $P^{-1}(v)$ denotes $P^{-1}(|v|)$.)

2.2. ORLICZ-SOBOLEV SPACES. We shall use the definitions and properties of Orlicz-Sobolev spaces as given in Donaldson and Trudinger [2]. See also Adams [1]. We shall only need to consider these spaces defined on bounded domains $\Omega \subset \mathbb{R}_n$ satisfying the cone condition.

Let Q be an N -function. Then the Sobolev conjugate N -function Q_* of Q is defined by

$$(i) \quad Q_*^{-1}(s) = \int_0^{|s|} Q^{-1}(t) \cdot t^{-1-1/n} dt$$

where it is assumed that, if necessary, $Q(t)$ is redefined for small values of t (obtaining an equivalent N -function) so that

$$(ii) \quad \int_0^1 Q^{-1}(t) \cdot t^{-1-1/n} dt < \infty .$$

From the stronger result $Q \ll Q_*$, proved in Gossez [3], we have

$$(iii) \quad Q < Q_* .$$

The Orlicz-Sobolev space $W^1L_Q(\Omega)$ is defined as the set of all functions in $L_Q(\Omega)$ whose distributional derivatives $\partial_i u$ also belong to $L_Q(\Omega)$. A norm $\|u\|_{1,Q} = \|u\|_{1,Q,\Omega}$ may be defined on $W^1L_Q(\Omega)$ by

$$(iv) \quad \|u\|_{1,Q} = \max\{\|u\|_Q, \|\partial_1 u\|_Q, \dots, \|\partial_n u\|_Q\}.$$

In the case that $\lim_{s \rightarrow \infty} Q_*^{-1}(s) = \infty$, there exists a constant C such that

$$(v) \quad \|u\|_{Q_*,\Omega} \leq C \|u\|_{1,Q,\Omega}.$$

We shall denote the set of vector-valued functions $u = (u_1, \dots, u_m)$, each of whose components belong to $W^1L_Q(\Omega)$, by $W^1L_Q(\Omega)^m$, and use the norm

$$(vi) \quad \|u\|_{1,Q,\Omega^m} = \max_{1 \leq j \leq m} \|u_j\|_{1,Q,\Omega}$$

on this space.

Following [6], let $A(\Omega)$ denote the set of all functions u , measurable on Ω , such that for almost all lines τ parallel to any coordinate axis x_i , $i = 1, \dots, n$, u is locally absolutely continuous on $\tau \cap \Omega$. The Orlicz-Sobolev spaces $W^1L_P(\Omega)$ may be given an alternative characterisation in terms of the class $A(\Omega)$, as follows:

(vii) Let Ω be a bounded domain in R_n with the cone property, and let P be an N -function. A function u defined on Ω is in $W^1L_P(\Omega)$ if and only if there exists $\tilde{u} \in A(\Omega)$ such that

- (a) $\tilde{u} = u$ almost everywhere in Ω ;
- (b) $\partial \tilde{u} / \partial x_i \in L_P(\Omega)$, $(i = 1, \dots, n)$.

Further $\partial \tilde{u} / \partial x_i = \partial_i u$ almost everywhere in Ω .

(vii) is an immediate consequence of Lemma 1.5 in [6] and Theorem 3.1 in Hardy [4].

3. A theorem on Nemitsky operators

We need the three lemmas which follow. The notation introduced in §2 will be used without further comment.

LEMMA 3.1. *Suppose the N-function M satisfies the Δ_2 -condition, so that, for large enough u ,*

$$(i) \quad M(\lambda u) \leq k_M(\lambda)M(u) \quad \text{for } \lambda > 1 .$$

Then

$$(ii) \quad M'_+ \text{ satisfies the } \Delta_2\text{-condition ,}$$

and further, there exists a constant C such that, for large enough $u > 0$,

$$(iii) \quad M'_+(\lambda u) \leq [Ck_M(\lambda)/\lambda]M(u) \quad \text{for } \lambda > 1 .$$

(Thus we may take $k_{M'_+}(\lambda)$ as $C k_M(\lambda)/\lambda$.)

Proof. From 2.1(v), 2.1(iii), and 2.1(i), we have, for sufficiently large $u > 0$,

$$M'_+(2u) < K_M M(2u)/(2u) \leq K_M^2 M(u)/(2u) \leq (K_M^2/2)M'_+(u) ,$$

which proves (ii).

For $\lambda > 1$ and sufficiently large $u > 0$,

$$\lambda \int_0^u M'_+(\lambda t) dt = M(\lambda u) \leq k_M(\lambda)M(u) = k_M(\lambda) \int_0^u M'_+(t) dt$$

so

$$(\lambda u/2)M'_+(\lambda u/2) \leq k_M(\lambda)uM'_+(u) ,$$

and then, for large enough $w = u/2$,

$$M'_+(\lambda w) \leq [2k_M(\lambda)/\lambda]M'_+(2w) \leq 2K_{M'_+}[k_M(\lambda)/\lambda]M'_+(w)$$

on using (ii). This proves (iii).

Note. In our applications of the next lemma, Ψ will be either an N-function M or its derivative M' , where it is supposed that M' exists and is strictly increasing. To avoid trivial complications, we shall make

the convention that $M'(u)$ denotes $M'(|u|)$ if $u < 0$, and that the conventions of 2.1(x) apply to Ψ .

LEMMA 3.2. Let P be an N -function, and let $\Psi(v)$ be an even function which is continuous and strictly increasing for all $v \geq 0$, and such that $\Psi(0) = 0$ and $\Psi(v) \rightarrow \infty$ as $v \rightarrow \infty$. Suppose further that $P \circ \Psi^{-1}$ is an N -function.

Then, if Ω is a bounded domain in \mathbb{R}_n ,

(i) $u \in L_P(\Omega)$ implies that $\Psi(u) \in L_{P \circ \Psi^{-1}}(\Omega)$ if and only if Ψ satisfies that Δ_2 -condition;

(ii) in the case that Ψ satisfies the Δ_2 -condition, there exists a constant C such that

$$(3.1) \quad \|\Psi(u)\|_{P \circ \Psi^{-1}} \leq C[1 + k_\Psi(\|u\|_P)],$$

where the function k_Ψ is as in 2.1(iv).

Proof. Suppose first that Ψ satisfies the Δ_2 -condition. Since (3.1) is obvious if $\|u\|_P = 0$, we have only to consider the cases (A) and (B) below.

$$(A) \quad 0 < \|u\|_{P, \Omega} \leq 1.$$

Then

$$\int_{\Omega} P[u(x)] dx \leq \int_{\Omega} P[u(x)/\|u\|_P] dx \leq 1,$$

using 2.1(viii), and so

$$1 \in \{\lambda > 0 : \int_{\Omega} (P \circ \Psi^{-1})[\Psi(u(x))/\lambda] dx \leq 1\},$$

and since this implies that $\|\Psi(u)\|_{P \circ \Psi^{-1}} \leq 1$, (3.1) follows in this case.

$$(B) \quad 1 < \|u\|_{P, \Omega} < \infty.$$

Let u_0 and $k = k_\Psi$ be as in 2.1(iv). Let

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$$\Omega_0 = \{x \in \Omega : |u(x)| < \|u\|_P u_0\}$$

and let

$$\Omega_1 = \Omega \setminus \Omega_0 .$$

For $x \in \Omega_0$, because $\Psi(v)$ increases with v ,

$$\Psi[u(x)] \leq \Psi(\|u\|_P u_0) \leq k(\|u\|_P) \Psi(u_0)$$

and then, (denoting Lebesgue measure by $|\cdot|$)

$$(3.2) \quad \int_{\Omega_0} (P \circ \Psi^{-1}) \{ \Psi[u(x)] / k(\|u\|_P) \} dx \leq |\Omega_0| P(u_0) .$$

For $x \in \Omega_1$, since then $|u(x)| / \|u\|_P \geq u_0$,

$$\Psi[u(x)] = \Psi[\|u\|_P \cdot u(x) / \|u\|_P] \leq k(\|u\|_P) \Psi[u(x) / \|u\|_P]$$

and so

$$(3.3) \quad \begin{aligned} \int_{\Omega_1} (P \circ \Psi^{-1}) \{ \Psi[u(x)] / k(\|u\|_P) \} dx &\leq \int_{\Omega_1} (P \circ \Psi^{-1}) \{ \Psi[u(x) / \|u\|_P] \} dx \\ &\leq \int_{\Omega} P[u(x) / \|u\|_P] dx \leq 1 . \end{aligned}$$

Adding the inequalities (3.2) and (3.3) and using 2.1(ii), we have

$$\int_{\Omega} (P \circ \Psi^{-1}) \{ \Psi[u(x)] / [1 + |\Omega_0| P(u_0)] k(\|u\|_P) \} dx \leq 1 ,$$

which completes the proof of (3.1), from the definition of the Luxemburg norm.

Now suppose that Ψ does not satisfy the Δ_2 -condition. Then there exists a positive increasing unbounded sequence $\{u_j\}$ such that

$$\Psi(2u_j) > 2^{2j} \Psi(u_j) .$$

We may divide Ω into disjoint measurable sets Ω_j such that

$$(3.4) \quad |\Omega_j| = P(u_j) |\Omega| / [2^j P(u_j)], \quad j = 1, 2, \dots$$

We now define $u : \Omega \rightarrow \mathbb{R}$ by

$$u(x) = \begin{cases} 2u_j & \text{for } x \in \Omega_j, j = 1, 2, \dots, \\ 0 & \text{otherwise in } \Omega, \end{cases}$$

and then, on using (3.4), we find that

$$\int_{\Omega} P[u(x)/2] dx = \sum_{j=1}^{\infty} P(u_j) |\Omega_j| < \infty.$$

Therefore $u/2$ and hence $u \in L_P(\Omega)$.

Let $\lambda > 0$, and let m be a positive integer such that $\lambda > 1/2^m$.

Then

$$\begin{aligned} \int_{\Omega} (P \circ \Psi^{-1}) \{ \lambda \Psi[u(x)] \} dx &= \sum_{j=1}^{\infty} |\Omega_j| (P \circ \Psi^{-1}) [\lambda \Psi(2u_j)] \\ &> \sum_{j=1}^{\infty} |\Omega_j| (P \circ \Psi^{-1}) [\lambda 2^{2j} \Psi(u_j)] \\ &> \sum_{j=m}^{\infty} |\Omega_j| (P \circ \Psi^{-1}) [2^j \Psi(u_j)] \\ &\geq \sum_{j=m}^{\infty} |\Omega_j| 2^j (P \circ \Psi^{-1}) [\Psi(u_j)] = \infty, \end{aligned}$$

where we have used (3.4), 2.1(ii), and the assumption that $P \circ \Psi^{-1}$ is an N -function. From 2.1(vii), $\Psi(u) \notin L_{P \circ \Psi^{-1}}$, and the proof is complete.

REMARK. Suppose the function Ψ in Lemma 3.2 is also an N -function satisfying the Δ' -condition, i.e., there exist constants $c > 0$ and $u_0 > 0$ such that

$$(3.5) \quad \Psi(uv) \leq c \Psi(u) \Psi(v) \quad \text{for } u, v \geq u_0.$$

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Without loss of generality, we may assume that $u_0 \geq 1$. Then if $1 < \ell \leq u_0 \leq u$,

$$\Psi(\ell u) \leq \Psi(u_0 u) \leq c\Psi(u_0)\Psi(u) \leq c[\Psi(u_0)/\Psi(1)]\Psi(\ell)\Psi(u);$$

and since the case in which $\ell > u_0$ follows directly from (3.5), we have

$$\Psi(\ell u) \leq c_1\Psi(\ell)\Psi(u) \quad \text{for } \ell > 1, u \geq u_0,$$

where $c_1 = c \max\{1, \Psi(u_0)/\Psi(1)\}$. Thus we may take $k_\Psi = c_1\Psi(\ell)$, so that the estimate (3.1) takes the form

$$(3.6) \quad \|\Psi(u)\|_{P \circ \Psi^{-1}} \leq \text{const}[1 + \Psi(\|u\|_P)].$$

LEMMA 3.3. *Suppose M , P , and Q are N -functions satisfying the following conditions:*

(i) *For all positive values of u , $M'(u)$ is strictly increasing and continuous;*

(ii) *For Q_* as in 2.2(i), and for large v ,*

$$(3.7) \quad P^{-1}(v) = M' \circ Q_*^{-1}(v) \circ Q^{-1}(v);$$

(iii) *$R = Q_* \circ (M')^{-1}$ is an N -function.*

Then

(iv) *$Q_* \circ M^{-1}$ is the principal part of an N -function,*

$$(3.8) \quad Q_* \circ M^{-1} \sim P_*,$$

and, if Ω is a bounded domain in R_n , there exists a constant C such that

$$(3.9) \quad \|f_1 f_2\|_{P, \Omega} \leq C \|f_1\|_{R, \Omega} \|f_2\|_{Q, \Omega}$$

whenever $f_1 \in L_{R, \Omega}$ and $f_2 \in L_{Q, \Omega}$.

Proof. From 2.2(i) and (3.7), we have

$$(3.10) \quad (M \circ Q_*^{-1})'(v) = (P_*^{-1})'(v)$$

for large enough v . Using (3.10) and the fact that P_* is an

N -function, we then find that $M \circ Q_*^{-1}(v)$ is concave for large enough v , and that $M \circ Q_*^{-1}(v)/v \rightarrow 0$ as $v \rightarrow \infty$, which shows that $Q_* \circ M^{-1}$ is the principal part of an N -function. On using Lemma 3.2 in [5] as well, we obtain (3.8).

Substituting $v = Q(u)$ in (3.7) we obtain, for some $U_1 \geq 0$,

$$\begin{aligned} (P^{-1} \circ Q)(u) &= u \cdot (M' \circ Q_*^{-1} \circ Q)(u) \quad u \geq U_1 \\ &= u\phi(u), \end{aligned}$$

where ϕ , defined by

$$(3.11) \quad \phi(u) = (M' \circ Q_*^{-1} \circ Q)(u)$$

is continuous and increasing for $u \geq U_1$. For any $U \geq U_1$ and $u > U$,

$$\int_U^u \phi(t)dt \leq \phi(u)(u-U) \leq u\phi(u) - U\phi(U),$$

and so

$$(3.12) \quad (P^{-1} \circ Q)(u) \geq (P^{-1} \circ Q)(U) + \int_U^u \phi(t)dt.$$

From (3.8) and (3.11), for sufficiently large values of the argument,

$$P^{-1} \circ R = (M' \circ Q_*^{-1} \circ R) \circ Q_*^{-1} \circ R = [M' \circ Q_*^{-1} \circ Q_* \circ (M')^{-1}] \circ Q_*^{-1} \circ Q_* \circ (M')^{-1}$$

and therefore there exists $U_2 \geq 0$ such that

$$(P^{-1} \circ R)(u) = u\phi^{-1}(u) \quad \text{for } u \geq U_2.$$

Therefore, for any $u \geq U_2$, we have

$$(3.13) \quad (P^{-1} \circ R)(u) \geq (P^{-1} \circ R)(U) + \int_U^u \phi^{-1}(s)ds.$$

From [5], pages 16 and 17, U may be chosen large enough so that there exists an N -function Φ such that

$$(3.14) \quad \Phi(u) = (P^{-1} \circ Q)(U) + \int_U^u \phi(t)dt, \quad u \geq U.$$

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It is evident from the construction referred to above that the complementary function $\tilde{\Phi}$ to Φ has the representation

$$\tilde{\Phi}(u) = K + \int_{U_3}^u \phi^{-1}(s) ds, \quad u \geq U_3,$$

where $K > 0$ and $U_3 \geq 0$ are constants. Then if $u_0 = \max\{U_1, U_2, U_3\}$ and $0 < \beta \leq \min\{1, P^{-1}[R(u_0)]/K\}$, we have, for $u \geq u_0$,

$$\begin{aligned} \tilde{\Phi}(\beta u) &\leq \beta \tilde{\Phi}(u) \leq (P^{-1} \circ R)(u_0) + \beta \int_{u_0}^u \phi^{-1}(s) ds \\ (3.15) \qquad \qquad \qquad &\leq (P^{-1} \circ R)(u). \end{aligned}$$

By (3.12), (3.14), and (3.15), $\Phi < P^{-1} \circ Q$ and $\tilde{\Phi} < P^{-1} \circ R$, so that (3.9) follows from 2.1(x).

We can now obtain our extension of Theorem 1.1. We shall assume that the function k_M of 2.1(iv) is taken to be of the form

$$(3.16) \qquad \qquad \qquad k_M(\lambda) = C\lambda^\mu, \quad (\lambda > 1)$$

where $C > 1$ and $\mu > 1$ are constants.

THEOREM 3.1. *Let Ω be a bounded domain in R_n satisfying the cone condition, and suppose that the function g defined on $\Omega \times R_m$ satisfies the condition (*) of Theorem 1.1. Let M and P be N -functions such that*

(α) *M satisfies the Δ_2 -condition and $M'(u)$ is strictly increasing and continuous for $u > 0$;*

(β) *$P_*^{-1}(s) \rightarrow \infty$ as $s \rightarrow \infty$.*

Suppose that the N -functions Q and R are as in Lemma 3.3, so that

$$(3.17) \qquad \qquad \qquad P_* \sim Q_* \circ M^{-1};$$

$$(3.18) \qquad \qquad \qquad R = Q_* \circ (M')^{-1},$$

and suppose that $Q_* \circ M^{-1}$, as well as $Q_* \circ (M')^{-1}$, is an N -function.

Further, suppose that for every $(x, t) \in \Omega \times \mathbb{R}_m$ where any derivative mentioned below exists,

$$(3.19) \quad \left| \partial g(x, t) / \partial x_i \right| \leq a_1(x) + b_1 M(|t|) ; \quad i = 1, \dots, n ,$$

$$(3.20) \quad \left| \partial g(x, t) / \partial t_j \right| \leq a_2(x) + b_2 M'(|t|) ; \quad j = 1, \dots, m ,$$

where $a_1 \in L_P(\Omega)$, $a_2 \in L_R(\Omega)$, and b_1 and b_2 are positive constants.

Then

$$(3.21) \quad G: W^1 L_Q(\Omega)^m \rightarrow W^1 L_P(\Omega)$$

and

$$(3.22) \quad \|Gu\|_{1, P, \Omega} \leq \text{const}[1 + k_M(\|u\|_{1, Q, \Omega^m})].$$

Proof. Since the proof (except for the derivation of the inequalities (3.24), (3.25), and (3.26)) follows that of Theorem 2.2 in Marcus and Mizel [6], it will only be given in outline.

Using 2.2(vii) (instead of Lemma 1.5 in [6]) we find that $f(x) = g(x, 0) \in W^1 L_P(\Omega)$, and that, if $u \in W^1 L_Q(\Omega)^m$ and $v = Gu$, the chain rule

$$(3.23) \quad \partial_i v = \partial g(\cdot, u) / \partial x_i + \sum_{j=1}^m \partial g(\cdot, u) / \partial t_j \partial_i u_j , \quad i = 1, \dots, n ,$$

holds almost everywhere in Ω . Further, there exists $\tilde{u}_j \in A(\Omega)$ such that $\tilde{u}_j = u_j$ almost everywhere in Ω , $j = 1, \dots, m$. Let $\tilde{v} = G\tilde{u}$.

Observing that (3.17) and (β) implies that $Q_*^{-1}(s) \rightarrow \infty$ as $s \rightarrow \infty$, and using Minkowski's inequality, 2.1(ix), 2.2(iii), 2.2(v), Lemma 3.2, and (3.16), we have

$$(3.24) \quad \begin{aligned} \|M(|\tilde{v}|)\|_P &\leq \text{const}\|M(|\tilde{u}|)\|_{P_*} \leq \text{const}\|M(|\tilde{u}|)\|_{Q_* \circ M^{-1}} \\ &\leq \text{const}[1 + k_M(\|\tilde{u}\|_{Q_*})] \leq \text{const}[1 + k_M(\|\tilde{u}\|_{1, Q, \Omega^m})] . \end{aligned}$$

Similarly we obtain (using (3.18) and Lemmas 3.1 and 3.3 as well):

$$\begin{aligned}
 \sum_{j=1}^m \|\alpha_j \partial \tilde{u}_j / \partial x_i\|_P &\leq \text{const} \sum_{j=1}^m \|\alpha_j\|_R \|\partial u_j / \partial x_i\|_Q \\
 (3.25) \qquad \qquad \qquad &\leq \text{const} \|\tilde{u}\|_{1,Q,\Omega^m},
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{j=1}^m \|M'(|\tilde{u}|) \partial \tilde{u}_j / \partial x_i\|_P &\leq \text{const} \sum_{j=1}^m \|M'(|\tilde{u}|)\|_R \|\partial u_j / \partial x_i\|_Q \\
 &\leq \text{const} [1 + k_M(\|\tilde{u}\|_{Q_*})] \|\tilde{u}\|_{1,Q,\Omega^m} \\
 &\leq \text{const} [1 + k_M(\|\tilde{u}\|_{1,Q,\Omega^m})] \|\tilde{u}\|_{1,Q,\Omega^m} \\
 (3.26) \qquad \qquad \qquad &\leq \text{const} [\|\tilde{u}\|_{1,Q,\Omega^m} + k_M(\|\tilde{u}\|_{1,Q,\Omega^m})].
 \end{aligned}$$

Combining (3.19), (3.20), (3.23), (3.24), (3.25) and (3.26) then gives

$$\begin{aligned}
 \|\partial v / \partial x_i\|_P &\leq \text{const} [1 + \|\tilde{u}\|_{1,Q,\Omega^m} + k_M(\|\tilde{u}\|_{1,Q,\Omega^m})] \\
 (3.27) \qquad \qquad \qquad &\leq \text{const} [1 + k_M(\|\tilde{u}\|_{1,Q,\Omega^m})].
 \end{aligned}$$

Finally, from the argument in [6] and (3.24), we obtain

$$(3.28) \qquad \qquad \qquad \|\tilde{v} - f\|_P \leq \text{const} [1 + k_M(\|\tilde{u}\|_{1,Q,\Omega^m})].$$

(3.27) and (3.28) give the estimate (3.22).

4. Examples

We shall now give two sets of N -functions M , P , Q , and R satisfying the hypotheses of Theorem 3.1.

EXAMPLE 4.1. Let $v > 1$, $1 < p < n$, $q = np/[n+(v-1)p]$, and $r = np/(v-1)(n-p)$, as in Theorem 1.1. Then $1 < q < n$, $1/r + 1/q = 1/p$, and if we let $p_* = np/(n-p)$ and $q_* = nq/(n-q)$, we have

$q_* = \nu p_* = (\nu-1)r$. Let $M(u) = |u|^\nu$, $Q(u) = q_*^q |u|^q$, and $P(u) = p_*^p |u|^p$. From 2.2(i), $Q_*(u) = |u|^{q_*}$, $P_*(u) = |u|^{p_*} = |u|^{q_*/\nu} = Q_* \circ M^{-1}(u)$, and $P_*^{-1}(s) \rightarrow \infty$ as $s \rightarrow \infty$. Let $R(u) = \nu^{-q_*/(\nu-1)} |u|^{q_*(\nu-1)} = \nu^{-r} |u|^r = Q_* \circ (M')^{-1}(u)$.

For $M(u)$ as above, $k_M(\ell) = \ell^\nu$, and we obtain Theorem 1.1 (for $\nu > 1$ only) as a particular case of Theorem 3.1.

EXAMPLE 4.2. Let ν, p, r, q and Q be as in Example 4.1. For $u_0 > 1$, let

$$(4.1) \quad M(u) = |u|^\nu (\ln|u| + 1) \quad \text{for} \quad |u| \geq u_0.$$

Let

$$\alpha(u) = uM'(u)/M(u) = \nu + 1/(\ln u + 1)$$

and let α_0 denote $\alpha(u_0)$. We complete the definition of M by defining

$$(4.2) \quad M(u) = M(u_0) (|u|/u_0)^{\alpha_0} \quad \text{for} \quad 0 \leq |u| \leq u_0.$$

From the arguments in [5], pages 16 and 17, M is a differentiable N -function which also satisfies the Δ_2 -condition. Since $\alpha(u) \downarrow \nu$ as $u \rightarrow \infty$ and $\nu/q_* = 1/p_* < 1$, we may also assume that u_0 is chosen large enough so that $M \circ Q_*^{-1}(v)$ is concave for $0 \leq v \leq u_0^{q_*}$. Note that this last relation is not disturbed if u_0 is further enlarged.

For sufficiently large positive v , we have

$$(4.3) \quad M \circ Q_*^{-1}(v) = v^{\nu/q_*} (\ln v^{1/q_*} + 1) = v^{1/p_*} (\ln v + q_*)/q_*$$

and then

$$(M \circ Q_*^{-1})''(v) = v^{(1/p_*)-2} [-(1-1/p_*)(\ln v + q_* + p_*) + 1]/p_* q_*,$$

which is negative for sufficiently large v . Thus we may enlarge u_0 so that $M \circ Q_*^{-1}$ is concave for all positive values of the argument, i.e.,

(i) $Q_* \circ M^{-1}$ is convex.

The limit

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$$(ii) \quad \lim_{u \rightarrow 0^+} Q_* \circ M^{-1}(u)/u = 0$$

follows from (4.2) and the fact that q_*/α_0 is greater than 1. The limit

$$(iii) \quad \lim_{u \rightarrow \infty} Q_* \circ M^{-1}(u)/u = \infty$$

follows from the equivalent limit

$$\lim_{v \rightarrow \infty} M \circ Q_*^{-1}(v)/v = 0,$$

obtained from (4.3). From (i), (ii), and (iii), it follows that $Q_* \circ M^{-1}$ is an N -function.

Similar arguments, using the fact that $q_*/(v-1) = r > p_* > 1$, show that $R = Q_* \circ (M')^{-1}$ is an N -function.

For large positive values of the argument, we define P by

$$\begin{aligned} P^{-1}(v) &= M' \circ [Q_*^{-1}(v)] \circ Q^{-1}(v) \\ &= v^{(v-1)/q_*} (\ln v)^{1/q_*} + (v+1)v^{1/q/q_*} \\ (4.4) \quad &= v^{1/p} [\ln v + (v+1)p_*] / (p_* q_*) . \end{aligned}$$

We now observe that the second derivative of P^{-1} is negative for large enough positive values of the argument, that

$$\lim_{u \rightarrow \infty} u P'(u)/P(u) = \lim_{v \rightarrow \infty} P^{-1}(v)/v(P^{-1})'(v) = p,$$

and that $1 < p < n$. From these facts it follows that we may choose u_1 so that $P(u)$ is defined and convex for $u \geq u_1$, and that

$$\rho = u_1 P'(u_1)/P(u_1)$$

satisfies $1 < \rho < n$. We can now complete the definition of P as an N -function by using (4.4) to define $P(u)$ for $|u| \geq u_1$, and setting

$$(4.5) \quad P(u) = P(u_1) (|u|/u_1)^\rho \quad \text{for } 0 \leq |u| \leq u_1 .$$

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Finally, because $1 < p < n$, we find from (4.5) that

$$\int_0^1 P^{-1}(t) \cdot t^{-1-1/n} dt < \infty$$

and because $1 < p < n$, (4.4) and an obvious comparison argument shows that

$$\lim_{s \rightarrow \infty} \int_0^s P^{-1}(t) \cdot t^{-1-1/n} dt = \infty.$$

Thus M , P , Q and R satisfy all the conditions on them which occur in Theorem 3.1.

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