COMMUTATIVE NON-ASSOCIATIVE NUMBER THEORY

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Introduction

Trevor Evans in (8) introduced postulates for a non-associative number theory similar to, but less general than, those of A. Robinson (9). Evans' number theory is also non-commutative under addition and multiplication, but an alternative equality axiom also suggested by Robinson leads to a number theory which is commutative under addition and still non-associative except in the special case:

$$a+(b+a) = (a+b)+a. \qquad (1)$$

In this paper we prove that the theorems in Evans' paper hold for the commutative system and provide some interpretations based mainly on work of Etherington (1-7).

Evans' and Robinson's systems

Evans' primitive terms are the set of non-associative numbers (" numbers " in the axioms below) and the binary operation of addition. His axioms are as follows:

(i) 1 is a number,

(ii) to every pair of numbers a, b there corresponds a third called the sum of a and b and written $a+b$,

(iii) there are no numbers a, b such that $a+b = 1$,

(iv) if the numbers a, b and c, d are such that $a+b = c+d$, then $a = c$ and $b = d$,

(v) if a set of numbers contains 1, and if whenever it contains numbers a, b then it contains $a+b$, then the set contains all numbers.

Robinson's axioms are equivalent except that he allows for a number of elements that are (like 1) irreducible in the sense of (iii). Any realisation of his axioms he calls a *forest*. (This suggests one of the interpretations to be given later.) The set of irreducible elements is called the *base* of the forest and the cardinal number of this set the *order* of the forest.

Evans' system is a forest of order 1 (which Robinson calls a simple forest), the restriction to one irreducible element is also included in the induction axiom (v).

Robinson is able to prove a much more general induction axiom:

If the base of a forest F belongs to a subset F' of F, and if $a \in F'$, $b \in F'$ implies $(a+b) \in F'$ then $F' = F$, as well as the theorem that forests of equal order are isomorphic.

The altered system

We now look at Evans' theorems within what Robinson calls a simple commutative forest. Axioms for this can be Evans' with (iv) replaced by:

(iv)$'$ if a, b and c, d are numbers, then $a+b = c+d$ if and only if $a = c$ and $b = d$ or $a = d$ and $b = c$.

This immediately leads to the commutative law.

Theorem A. *$a+b = b+a$ for all numbers a, b.*

Also we obtain the cancellation laws for addition, which in Evans' system were encompassed in postulate (iv).

Theorem B. *If $a+b = a+c$ then $b = c$.*

If $a+b = c+b$ then $a = c$.

Taking Evans' definition of multiplication †:

$$\text{(i) } a.1 = a, \quad \text{(ii) } a.(b+c) = a.b+a.c,$$

a number of his theorems can be rewritten with only minor alterations to the proofs. Using the first of these as well as Theorem A it can be shown that (1) is the only form of the associative law that holds.

The numbering of the following theorems is exactly as in (**8**).

Theorem 1. *For all numbers a, b, $a \neq a+b$.*

Theorem 2. *$1.a = a$ for all numbers a.*

Theorem 3. *$(ab)c = a(bc)$ for all numbers a, b, c.*

The proofs of our counterparts of Evans' Theorems 4 and 5 rely on the notion of length, as does the proof of Theorem 5 in (**8**).

The length of a number n is the positive integer obtained from n by regarding $+$ in the expression for n as the addition of ordinary arithmetic. We denote the length of n by $|n|$.

Theorem 4. *If $xa = ya$ then $x = y$.*

Proof. If $xa = ya$ clearly $|x| = |y|$.

If $a = 1$ the result holds; if a is not 1, $a = a_1+b_1$ for some numbers a_1 and b_1 and so $xa_1+xb_1 = ya_1+yb_1$ and either $xa_1 = ya_1$ and $xb_1 = yb_1$ or $xa_1 = yb_1$ and $xb_1 = ya_1$.

† Robinson also uses this definition and has identical Theorems 2 and 3.

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In the second case, since $|x| = |y|$ we must have $|a_1| = |b_1| < |a|$ and in either case we have, for some $a_2, b_2, xa_2 = yb_2$ with $|a_2| = |b_2| < |a|$.

If $a_2 = 1$ the theorem holds, if not let $a_2 = c_2 + d_2$ and $b_2 = e_2 + f_2$ so that $xc_2 + xd_2 = ye_2 + yf_2$ and so either $xc_2 = ye_2$ and $xd_2 = yf_2$ or $xc_2 = yf_2$ and $xd_2 = yf_2$.

Clearly in either case we have for some numbers a_3 and b_3, $xa_3 = yb_3$, with $|a_3| = |b_3| < |a_2| = |b_2| < |a|$.

The number a must consist of a finite summation of 1's so continuation of this process must lead to $x.1 = y.1$, i.e. to $x = y$.

Theorem 5. *If $ax = ay$ then $x = y$.*

Proof. This is similar to that of Theorem 4, this time x and y are broken up into components when $x \neq 1$.

The number theory results in (**8**) such as the fundamental theorem of arithmetic and Fermat's last theorem can now be reproduced using identical definitions and only minor alterations in the proofs.

Also the results proved by Minc in (**10**) which extend those of Evans go through. In particular:

Minc's Theorem 1. *$ab = ba$ if and only if a and b are powers of the same number.*

Thus this number theory which is commutative and generally not associative under addition is associative and generally not commutative under multiplication.

Interpretations

In (**2**) Etherington allows $a+b$ (ab in his notation) to stand for the offspring of two individuals or populations. He states that this operation is in general not associative, but can always be taken to be non-commutative. There is more on the arithmetic of such systems in (**4**) and (**6**).

A well-known interpretation of non-commutative non-associative number theory, the arithmetic of bifurcating root trees (Etherington (**1**), (**3**) and (**5**) and Robinson (**9**)) represents Evans' system. The arithmetic becomes commutative if we consider " trees in a breeze " (roughly, trees with all branches pointing in the same direction).

A directly arithmetical example can be made up as follows: Define

$$a \circ b = 2^{\min(a,b)}.3^{\max(a,b)}.$$

Clearly the operator $\circ$ is commutative but not associative and the set of numbers generated by the definition from 1 satisfies rules (i), (ii), (iii), (iv)$'$ and (v) (where in each expression $\circ$ replaces $+$).

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