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SOME CLASSES OF Θ-COMPACTNESS

BY

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ABSTRACT. Let Λ and Δ denote the classes of ordinal spaces with the order topology and Σ-product spaces of the two point discrete space respectively. Characterizations are given in terms of ultrafilters of clopen sets of those 0-dimensional Hausdorff topological spaces that can be embedded homeomorphically as a closed subspace of a topological product of either spaces from the class Λ or the class Δ. Both classes consist of spaces that are ω_0-bounded. An example is given of a 0-dimensional Hausdorff ω_0-bounded space that cannot be homeomorphically embedded as a closed subset of a product of spaces from either Λ or Δ, answering a question of R. G. Woods.

1. **Introduction.** The notion of E-compactness was introduced by R. Engelking and S. Mrowka in [4]. More recently this notion has been considered in a more general context in the book by J. Porter and R. G. Woods, [7]. Given a topological space E, another space X is said to be **E-compact** if it can be embedded as a closed subset of the product space E^m for some cardinal number m. H. Herrlich in [5] introduces a natural generalization of the notion of E-compactness. If Θ is a class of topological spaces, then a space X is said to be **Θ-compact** if X can be embedded as a closed subset of some topological product of spaces from the class Θ. Given a class of spaces, it is clear that this class is equal to the class of all Θ-compact spaces for some class Θ if and only if the given class is closed under the formation of topological products and closed subspaces. If Θ is a given class of spaces, let **KΘ** denote the class of all Θ-compact spaces.

If m is an infinite cardinal number, then a topological space X is said to be **m-bounded** if every subset of X of cardinality at most m has compact closure in X. For a given cardinal m, it has been shown by R. G. Woods in [9] that the class of m-bounded spaces is a class of Θ-compactness (also referred to in [9] as an extension property) but is not equal to the class of E-compact spaces for any space E. The question has been raised in [9] of whether or not the class of Hausdorff, 0-dimensional (i.e., has a base of clopen sets), ω_0-bounded spaces is equal to $K\Gamma$, where Γ denotes the class of ordinals of uncountable cofinality (in their order topologies). In this paper that question is answered in the negative. We also give characterizations by means of ultrafilters of those spaces in $K\Gamma$ where Γ is as above, and also where Γ is the class of

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Σ-spaces. Some of the results in this paper formed part of the author's doctoral thesis.

We shall confine our attention to completely regular, Hausdorff spaces (all 0-dimensional, Hausdorff spaces are easily seen to be completely regular).

2. **ω_0-bounded spaces and KΓ**. It is well known (see [1]) that every 0-dimensional space X has a 0-dimensional compactification β_0X, maximal in the following sense: If Y is 0-dimensional and $f:X \to Y$ is continuous then there is a continuous map $\beta_0(f):\beta_0 X \to \beta_0 Y$ such that $\beta_0(f)|_X = $ f. The space $\beta_0 X$ can be viewed as the 0-dimensional analogue of the Stone-Cech compactification. It is constructed as the maximal filter (i.e., Stone) space of the Boolean algebra of clopen subsets of X. Thus the points of $\beta_0 X - X$ correspond to free ultrafilters of clopen sets in X in such a way that if p $\in \beta_0 X - X$ and U_p is the corresponding ultrafilter and A is a clopen subset of X, then p $\in cl_{\beta_0 X}$(A) if and only if A $\in U_p$. The following slight modification of Theorem 4.1 of [6] will be needed.

2.1. THEOREM. *Let* Γ *be a class of 0-dimensional spaces and let* X *be a 0-dimensional space. Then* X $\in K\Gamma$ *if and only if given any point* p $\in \beta_0 X - X$, *there is a space* Y $\in \Gamma$ *and a continuous map* $f_p:X \to Y$ *such that* $(\beta_0(f_p))(p) \in \beta_0 Y - Y$.

We recall some facts concerning ordinal numbers. An ordinal number κ with the topology induced by its total order is 0-dimensional, locally compact and normal. The cofinality of an ordinal κ is denoted $cf(\kappa)$ and is defined to be the least cardinal number of a cofinal (unbounded) subset of κ. Given an ordinal α, the α-th uncountable cardinal number is denoted ω_α. An ordinal with its induced order topology will be referred to as an ordinal space. Since closed and bounded initial segments in ordinal spaces are compact, if m is a cardinal number then the ordinal space κ is m-bounded if and only if $cf(\kappa) > m$. Furthermore, just as for the ordinal space ω_1 of the first uncountable cardinal we have $\beta(\omega_1) = \beta_0(\omega_1 + 1) = \omega_1$ (the one point compactification of ω_1), it is also true that $\beta(\kappa) = \beta_0(\kappa) = \kappa + 1$ for any ordinal space κ for which $cf(\kappa) > \omega_0$.

Let Γ denote the class of all ordinal spaces of cardinal numbers of uncountable cofinality, i.e., $\Gamma = \{\omega_\alpha : cf(\omega_\alpha) > \omega_0, \alpha$ *is an ordinal*$\}$. A slight modification of Theorem 4 of [2] shows that if κ is an ordinal with $cf(\kappa) = \omega_\alpha$ then as ordinal spaces, κ is ω_α-compact. Recall that a regular cardinal number m is defined to satisfy cf(m) = m. Thus, since $cf(\kappa)$ is a regular cardinal for any ordinal κ, it follows that $K\Gamma = K\Gamma_1 = K\Lambda$ where Γ_1 is the class of all ordinal spaces of uncountable cofinality and Λ is the class of all ordinal spaces of regular uncountable cardinal numbers. We now offer a characterization in terms of clopen ultrafilters of those spaces in $K\Lambda$. Open and closed interval notation will be used for a totally ordered space, and should be clear from the context. Also, a point p $\in \beta_0 X - X$ will be identified with its corresponding ultrafilter of clopen sets.

2.2. PROPOSITION: *Let* X *be a 0-dimensional space, and let* ω_α *be a regular uncountable cardinal viewed as an ordinal space with its induced order topology. The following statements are equivalent.*

(i) X *is* ω_α-*compact (in the sense of E-compact).*

(ii) *For any free ultrafilter of clopen subsets of* X, *say* p, *there is a family* $U = \{U_i : i < \omega_\alpha\} \subseteq p$ *such that* $\cap U = \phi$ *and for all* $j < \alpha$, $U_j \subseteq \cap\{U_i : i < j\}$.

PROOF: i) $\rightarrow$ ii). Suppose that X is ω_α-compact. By Theorem 2.1 there is a continuous map $f : X \rightarrow \omega_\alpha$ such that $(\beta_0(f))(p) = \omega_\alpha$ (where ω_α also denotes the point at infinity in $\omega_\alpha + 1$, the one-point and Stone-Cech compactification of ω_α). Let $U_i = f^{\leftarrow}((i, \omega_\alpha))$ for all $i < \omega_\alpha$. Then $U_i \in p$, for clearly U_i is clopen, and if $(X - U_i) \in p$ then $p \in cl_{\beta_0 X}(X - U_i)$, and hence $(\beta_0(f))(p) \in cl_{\beta_0\omega_\alpha}[f(X - U_i)] \in [0, i]$, which is contrary to hypothesis. Let $U = \{U_i : i < \omega_\alpha\}$. If $x \in X$ then $f(x) \in \omega_\alpha$ and hence there is an $i < \omega_\alpha$ such that $f(x) \leq i$. Thus, $x \in U_i$ and hence, $\cap U = \phi$. It is clear that U satisfies the other condition of (ii).

(ii) $\Rightarrow$ (i). Let p be a free ultrafilter of clopen subsets of X, and let U be the subfamily of p guaranteed by the hypothesis. Let $f : X \rightarrow \omega_\alpha$ be defined as follows: $f(x) = \min \{i < \omega_\alpha : x \in U_i\}$. Then $f^{\leftarrow}((i, j)) = U_i - \cap \{U_t : t < j\}$ if $i < j < \omega_\alpha$. Since U_i is clopen for all $i < \omega_\alpha$, this set is open and hence f is continuous. Also, since $f^{\leftarrow}((i, \omega_\alpha)) = U_i \in p$ for every $i < \omega_\alpha$, and $p \in cl_{\beta_0 X}(U_i)$ we must have $(\beta_0(f))(p) \in cl_{\beta_0\omega_\alpha}[(i, \omega_\alpha)] = (i, \omega_\alpha]$ for every $i < \omega_\alpha$. Thus, $(\beta_0(f))(p) = \omega_\alpha \in \beta_0\omega_\alpha - \omega_\alpha = (\omega_\alpha + 1) - \omega_\alpha$, and it follows from Theorem 2.1 that X is ω_α-compact. □

Note that the family U in the statement of Proposition 2.2 need not be a base for the ultrafilter p. For example, if D_{ω_1} denotes the discrete space of cardinality ω_1 and if $X = \cup \{cl_{\beta D_{\omega_1}}(A) : A \subset D_{\omega_1}, |A| < \omega_1\}$, then X is ω_1-compact (in the sense of E-compact) but no free ultrafilter of clopen subsets of X has a well-ordered base (ordered by reverse inclusion).

The following theorem combines the result of Proposition 2.2 for all regular uncountable cardinals.

2.3. THEOREM: *Let* X *be a 0-dimensional space. The following statements are equivalent.*

(i) $X \in k\Lambda$.

(ii) *Every free ultrafilter of clopen subsets of* X *has a non-empty well-ordered subfamily with empty intersection and with the order type of a regular uncountable cardinal (where the ordering is by reverse inclusion).*

Question 5.5 of [9] asks whether or not $K\Lambda$ as defined above is equal to the class of all 0-dimensional ω_0-bounded spaces. We show that the answer is no. The space that provides the answer happens to be a Σ-product of the two point discrete space $\{0, 1\}$ (henceforth denoted **2**). Σ-products were introduced by H. Corson in [3] and have been studied extensively, with particular interest in Corson compact and Eberlein compact spaces.

2.4. DEFINITION: *Let* κ *be an infinite cardinal number. The* Σ*-product space* Σ_κ *is defined as follows:* $\Sigma_\kappa = \{f \in \mathbf{2}^\kappa : |\{i < \kappa : f(i) = 1\}| < \kappa\}$.

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It is shown in [8] that $\beta_0\Sigma_\kappa = \beta\Sigma_\kappa = \mathbf{2}^\kappa$ for any uncountable cardinal κ. The following lemma gives conditions on cardinals m and κ equivalent to Σ_κ being m-bounded.

2.5. Lemma: *Let* m *and* κ *be infinite cardinal numbers. Then* Σ_κ is *m-bounded if and only if* m $< cf(\kappa)$.

Proof: Necessity. Suppose Σ_κ is m-bounded and $cf(\kappa) \leq m$. Let $\{A_i : i < cf(\kappa)\}$ be a family of subsets of κ such that for all i, $|A_i| < \kappa$ and $A_i \subset A_j$ if $i < j$ and $\cup_i A_i = \kappa$. For each $i < cf(\kappa)$ define the point p_i in Σ_κ as follows: for $t < \kappa$ let $p_i(t) = 1$ if $t \in A_i$, $p_i(t) = 0$ *otherwise*. Then for each i, $p_i \in \Sigma_\kappa$. Let $p \in \mathbf{2}^\kappa$ be such that for all $t < \kappa$, $p(t) = 1$. Clearly p is in the closure in Σ_κ of the set $\{p_i : i < cf(\kappa)\}$. Since p is not in Σ_κ, it follows that Σ_κ is not $cf(\kappa)$-bounded, and hence it is not m-bounded. This contradiction shows that $m < cf(\kappa)$.

Sufficiency. Suppose $m < cf(\kappa)$ and $P = \{p_i : i < m\} \subseteq \Sigma_\kappa$. For each $i < m$, define A_i to be $A_i = p_i^{\leftarrow}(\{1\})$. Then each A_i has cardinality less than κ and hence $A = \cup_{i<m} A_i$ has cardinality less than κ since $m < cf(\kappa)$. Thus, $cl_{\Sigma_\kappa} P \subseteq \mathbf{2}^A \times [\Pi(\{0\}_t : t \in \kappa - A)] \subset \Sigma_\kappa$. As the set in the middle is compact, it follows that Σ_κ is m-bounded.

As we shall see, the Σ-product spaces provide the answer to question 5.5 of [9].

2.6. Definition: *If* m *is an infinite cardinal number, then a space* X *is called* ***strongly-m-bounded*** *if every union of at most* m *compact subsets of* X *has compact closure in* X.

A straightforward argument shows that for a given cardinal number m, the class of strongly-m-bounded spaces is a class of Θ-compactness. It is also clear that every member of the class Λ defined above is strongly-ω_0-bounded (in fact, ω_α is strongly-m-bounded if and only if $cf(\omega_\alpha) > m$). Thus, every member of $K\Lambda$ is strongly-ω_0-bounded. However, Σ_{ω_1} has the following dense, σ-compact subset $A : A = \cup_{i<\omega_0} A_i$ where $A_i = \{p \in \mathbf{2}^{\omega_1} : |\{t : p(t) = 1\}| \leq i\}$. Since Σ_{ω_1} is not compact, it cannot be strongly-ω_0-bounded as it has a dense σ-compact subset. Thus, Σ_{ω_1} is ω_0-bounded (by Lemma 2.5) but is not in the class $K\Lambda$ as it is not strongly-ω_0-bounded.

We now provide a characterization of those spaces that are Σ_κ-compact (in the sense of E-compact) that is analagous to Proposition 2.2. This characterization will show that the Σ-product spaces considered here "generate" a larger compactness class of ω_0-bounded spaces than the ordinal spaces.

2.7. Proposition: *Let* κ *be an infinite cardinal of uncountable cofinality, and let* X *be a 0-dimensional space. The following are equivalent.*

(i) X *is* Σ_κ*-compact (in the sense of E-compact).*

(ii) *If* p *is a free ultrafilter of clopen subsets of* X, *there is a subfamily* $U = \{U_i : i < \kappa\} \subset p$ *such that for every* $x \in X$, $|\{i : x \in U_i\}| < \kappa$.

Proof: (i) $\rightarrow$ (ii). Suppose that X is Σ_κ-compact and p is a free ultrafilter of clopen subsets of X. By Theorem 2.1 there is a continuous map $f: X \rightarrow \Sigma_\kappa$ such that

$(\beta_0(f))(p) \in \beta_0\Sigma_\kappa - \Sigma_\kappa$. Thus if $q = (\beta_0(f))(p)$ then the set $A = q^{\leftarrow}(\{1\}) \subset \kappa$ has cardinality κ. For each $i \in A$ let $U_i = f^{\leftarrow}(\pi_i^{\leftarrow}(\{1\}))$ (where π_i denotes the projection map to the i-th factor of $\mathbf{2}^\kappa$). Since $\{1\}$ is a clopen subset of $\{0, 1\}$, U_i is a clopen subset of X. Furthermore, $U_i \in p$, for if $(X - U_i) \in p$, then $p \in cl_{\beta_0 X}(X - U_i)$ and hence $(\beta_0(f))(p) \in cl_{2^\kappa}(f(X - U_i)) \subset \pi_i^{\leftarrow}(\{0\})$. It would follow that q(i) = 0 which is false if $i \in A$. Thus, $U_i \in p$. *Let* $U = \{U_i : i \in A\}$, and let $x \in X$. It is clear that $x \in U_i$ if and only if $(f(x))(i) = 1$. Since $f(x) \in \Sigma_\kappa$, it follows that $|\{i \in A : x \in U_i\}| = |\{i \in A : \pi_i(f(x)) = 1\}| < \kappa$. Hence, U satisfies condition (ii).

(ii) $\rightarrow$ (i). Suppose that X satisfies condition (ii). Let p be a free ultrafilter of clopen subsets of X. Let U be the subfamily of p guaranteed by hypothesis. Define a function $f : X \rightarrow \Sigma_\kappa$ as follows: if $i < \kappa$, let $\pi_i(f(x)) = 1$ if $x \in U_i$ and $\pi_i(f(x)) = 0$ if $x \in U_i$. Clearly f is continuous. Also, for each $i < \kappa$, $\pi_i((\beta_0(f))(p)) = 1$. Thus, $(\beta_0(f))(p) \in \beta_0\Sigma_\kappa - \Sigma_\kappa$ and it then follows from Theorem 2.1 that X is Σ_κ-compact.

It may seem reasonable to consider a slightly more general approach to Σ-product spaces. If $\omega_0 < \lambda \leq \kappa$ then we define the space ${}_\lambda\Sigma_\kappa = \{f \in \mathbf{2}^\kappa : |\{i : f(i) = 1\}| < \lambda\}$. It follows from Proposition 2.7 that ${}_\lambda\Sigma_\kappa$ is Σ_λ-compact, (in fact is a closed subspace of $(\Sigma_\lambda)^\kappa$. Thus, such a generalization does not enlarge the class of compactness generated by Σ-product spaces.

We combine the result of Proposition 2.7 for all cardinals of uncountable cofinality to obtain the following theorem. We will denote by Δ the class of the spaces Σ_κ for which κ has uncountable cofinality.

2.8. THEOREM: *Let* X *be a 0-dimensional space. The following are equivalent.*

(i) $X \in K\Delta$.

(ii) *If* p *is a free ultrafilter of clopen subsets of* X, *then there is a cardinal* κ *of uncountable cofinality and a subfamily* $U \subset p$ *such that* $|U| = \kappa$ *and for each* $x \in X$, $|\{i : x \in U_i\}| < \kappa$.

If $\omega_\alpha \in \Lambda$, and we let U be the family of all cofinal clopen intervals in ω_α, then U satisfies condition (ii) of Theorem 2.8, and it follows that $K\Lambda \subset K\Delta$. This containment is strict since members of $K\Lambda$ are strongly-ω_0-bounded whereas members of $K\Delta$ may not be (for instance no $\Sigma_\kappa \in \Delta$ is strongly-ω_0-bounded). Even though $K\Delta$ strictly contains $K\Lambda$, $K\Delta$ still does not contain all (strongly-) 0-dimensional ω_0-bounded spaces.

2.9. EXAMPLE: Let X be the subspace of the product of ordinal spaces $(\omega_1 + 1) \times (\omega_2 + 1)$ formed by removing the point at infinity (i.e., the point at the "upper right hand corner", (ω_1, ω_2)). The point at infinity completes the Stone-Cech comactification of X. It is also clear that X is 0-dimensional and given any clopen subset of X, either it or its complement is compact. X is strongly-ω_0-bounded, and has a base of clopen sets of size ω_2. If the ultrafilter of clopen sets that corresponds to the point at infinity of X is denoted by p, then any member of p contains an "upper right rectangle" in X (i.e., a set of the form $(A \times B) \cap X$, where A and B are cofinal intervals in $\omega_1 + 1$ and $\omega_2 + 1$ respectively).

If U is subfamily of p of cardinality ω_1, then there will be a point on the "right edge" of X that will be in all members of U (since ω_1- many cofinal intervals on the right edge of X have non-empty intersection in X). Furthermore, if $|U|$ has cofinality ω_1 (with sets of U possibly repeated) a similar argument shows that some point (in fact some cofinal interval) on the right hand edge is in $|U|$-many members of U. Thus, $|U|$ cannot serve as κ in condition (ii) of Theorem 2.8.

If $|U| = \omega_2$, then some cofinal interval on the "top edge" of X must be contained in ω_2-many members of U (the top edge of X is a copy of the ordinal space ω_1 and each member of U meets the top edge of X in a cofinal interval). Thus, ω_2 cannot serve as κ in condition (ii) of Theorem 2.8. A similar argument shows that since the weight of X is ω_2, if $|U|$ has cofinality ω_2 then $|U|$ cannot serve as κ in condition (ii) of Theorem 2.8.

Finally, if the cofinality of $|U|$ is greater than ω_2, then since the weight of X is ω_2, some member of U must be repeated $|U|$-many times (assuming that members of p are "upper-right rectangles" in X). Thus, $|U|$ cannot serve as κ in condition (ii) of Theorem 2.8. This completes the example.

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