Asymptotic Expressions for the Bessel Functions and the Fourier-Bessel Expansions.

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Part I.

Asymptotic Expressions for the Bessel Functions.

From the asymptotic expansion for $K_n(z)$ it follows that, if $-\pi < \operatorname{amp} z < \pi$,

$$\lim_{z \to \infty} K_n(z) \Big/ \sqrt{\left\{\left(\frac{\pi}{2z}\right)e^{-z}\right\}} = 1.$$

This theorem is also true if $\operatorname{amp} z = \pm\pi$; to prove this consider the formula

$$K_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)}\frac{1}{\Gamma(n+\frac{1}{2})}e^{-z}\int_0^\infty e^{-\xi}\xi^{n-\frac{1}{2}}\left(1+\frac{\xi}{2z}\right)^{n-\frac{1}{2}}d\xi.$$

If $R(n+\frac{1}{2}) > 0$, this formula is valid for $z \neq 0$, $-\pi < \operatorname{amp} z < \pi$, since both sides of the equation are holomorphic in this region. Now let $z = xe^{i\theta}$, where x is real and positive, and let the path of integration be deformed into the contour consisting of: (i) the ξ-axis from 0 to $2x-\epsilon$; (ii) a semicircle of centre $2x$ and radius ϵ lying above the ξ-axis; (iii) the ξ-axis from $2x+\epsilon$ to ∞. Then the integral is holomorphic in z at $z = xe^{i\pi}$. If $\theta = -\pi$, the semicircle is taken to lie below the ξ-axis. Since $R(n+\frac{1}{2}) > 0$, the integral round the semi-circle tends to zero with ϵ.

Hence, if $z = xe^{\pm i\pi}$,

$$K_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)}\frac{1}{\Gamma(n+\frac{1}{2})}e^{-z}\{I_1 + e^{\mp i\pi(n-\frac{1}{2})}I_2\},$$

where
$$I_1 = \int_0^{2x} e^{-\xi}\xi^{n-\frac{1}{2}}\left(1-\frac{\xi}{2x}\right)^{n-\frac{1}{2}}d\xi,$$

and
$$I_2 = \int_{2x}^\infty e^{-\xi}\xi^{n-\frac{1}{2}}\left(\frac{\xi}{2x}-1\right)^{n-\frac{1}{2}}d\xi.$$

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Now let M be a large positive quantity less than x; then

$$I_1 = \int_0^M e^{-\xi} \xi^{n-\frac{1}{2}} \left(1 - \frac{\xi}{2x}\right)^{n-\frac{1}{2}} d\xi + V,$$

where $$V = \int_M^{2x} e^{-\xi} \xi^{n-\frac{1}{2}} \left(1 - \frac{\xi}{2x}\right)^{n-\frac{1}{2}} d\xi.$$

Then, if $R(n) = \alpha$,

$$|V| \leqq \int_M^{2x} e^{-\xi} \xi^{\alpha-\frac{1}{2}} \left(1 - \frac{\xi}{2x}\right)^{\alpha-\frac{1}{2}} d\xi.$$

Two cases have to be considered; namely,

$$\alpha \geqq \tfrac{1}{2} \text{ and } -\tfrac{1}{2} < \alpha < \tfrac{1}{2}.$$

Case I. Let $\alpha \geqq \frac{1}{2}$; then, since $\left(1 - \frac{\xi}{2x}\right) < 1$,

$$|V| \leqq \int_M^{2x} e^{-\xi} \xi^{\alpha-\frac{1}{2}} d\xi.$$

Case II. Let $-\frac{1}{2} < \alpha < \frac{1}{2}$; then

$$|V| \leqq \int_M^{x} e^{-\xi} \xi^{\alpha-\frac{1}{2}} \left(1 - \frac{\xi}{2x}\right)^{\alpha-\frac{1}{2}} d\xi + \int_x^{2x} e^{-\xi} \xi^{\alpha-\frac{1}{2}} \left(1 - \frac{\xi}{2x}\right)^{\alpha-\frac{1}{2}} d\xi.$$

Accordingly, by the First Mean Value Theorem

$$|V| \leqq \lambda \int_M^{x} e^{-\xi} \xi^{\alpha-\frac{1}{2}} d\xi + e^{-(1+\theta)x} (1+\theta)^{\alpha-\frac{1}{2}} x^{\alpha-\frac{1}{2}} \int_x^{2x} \left(1 - \frac{\xi}{2x}\right)^{\alpha-\frac{1}{2}} d\xi,$$

where $1 < \lambda < 2^{\frac{1}{2}-\alpha}$ and $0 < \theta < 1$.

Thus in both cases

$$\operatorname*{Lim}_{x \to \infty} |V| \leqq \lambda \int_M^{\infty} e^{-\xi} \xi^{\alpha-\frac{1}{2}} d\xi.$$

But $$\operatorname*{Lim}_{x \to \infty} \int_0^M e^{-\xi} \xi^{n-\frac{1}{2}} \left(1 - \frac{\xi}{2x}\right)^{n-\frac{1}{2}} d\xi = \int_0^M e^{-\xi} \xi^{n-\frac{1}{2}} d\xi.$$

Hence, by making M tend to infinity, it follows that

$$\operatorname*{Lim}_{x \to \infty} I_1 = \int_0^\infty e^{-\xi} \xi^{n-\frac{1}{2}} \, d\xi = \Gamma(n+\tfrac{1}{2}).$$

In the next place

$$| I_2 | \leqq \int_{2x}^\infty e^{-\xi} \xi^{a-\frac{1}{2}} \left(\frac{\xi}{2x} - 1\right)^{a-\frac{1}{2}} d\xi.$$

Here put $\xi = 2x(1+\eta)$; then

$$| I_2 | \leqq e^{-2x} (2x)^{a+\frac{1}{2}} \int_0^\infty e^{-2x\eta} (1+\eta)^{a-\frac{1}{2}} \eta^{a-\frac{1}{2}} \, d\eta$$

$$< e^{-2x} (2x)^{a+\frac{1}{2}} \int_0^\infty e^{-\eta} (1+\eta)^{a-\frac{1}{2}} \eta^{a-\frac{1}{2}} \, d\eta.$$

Hence $\operatorname*{Lim}_{x \to \infty} I_2 = 0.$

Accordingly, if amp $z = \pm \pi$,

$$\operatorname*{Lim}_{z \to \infty} K_n(z) \Big/ \left\{ \sqrt{\left(\frac{\pi}{2z}\right)} e^{-z} \right\} = 1.$$

Since $K_{-n}(z) = K_n(z)$, this is true for all values of n.

The corresponding theorems for the other Bessel Functions can be deduced from this. They are ·

$$\operatorname*{Lim}_{z \to \infty} G_n(z) \Big/ \left\{ \sqrt{\left(\frac{\pi}{2z}\right)} e^{-\frac{1}{2}n\pi i + i(z+\pi/4)} \right\} = 1,$$

where $-\pi/2 \leqq \operatorname{amp} z \leqq 3\pi/2$;

$$\operatorname*{Lim}_{z \to \infty} J_n(z) \Big/ \left\{ \sqrt{\left(\frac{2}{\pi z}\right)} \cos(z - \pi/4 - n\pi/2) \right\} = 1,$$

where $-\pi/2 \leqq \operatorname{amp} z \leqq \pi/2$;

$$\operatorname*{Lim}_{z \to \infty} J_n(z) \Big/ \left\{ i e^{in\pi} \sqrt{\left(\frac{2}{\pi z}\right)} \cos(z + \pi/4 + n\pi/2) \right\} = 1,$$

where $\pi/2 \leqq \operatorname{amp} z \leqq 3\pi/2$;

$$\operatorname*{Lim}_{z \to \infty} I_n(z) \Big/ \left\{ \frac{1}{\sqrt{(2\pi z)}} e^z + e^{-i(n+\frac{1}{2})\pi} \frac{1}{\sqrt{(2\pi z)}} e^{-z} \right\} = 1,$$

where $-\pi \leqq \operatorname{amp} z \leqq 0$;

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$$\lim_{z \to \infty} I_n(z) \Big/ \left\{ \frac{1}{\sqrt{(2\pi z)}} e^z + e^{i(n+\frac{1}{2})\pi} \frac{1}{\sqrt{(2\pi z)}} e^{-z} \right\} = 1,$$

where $0 \leqq \text{amp}\, z \leqq \pi$.

Part II.

The Fourier-Bessel Expansions.

These theorems make it possible to establish the validity of the Fourier-Bessel Expansions by means of Coutour Integration. For simplicity, the expansion

$$f(r) = \sum_{s=1}^{\infty} A_s J_0(\lambda_s r), \quad \text{.......................(1)}$$

where λ_1, λ_2, ... are the positive zeros of $J_0(x)$ will first be considered.

Here

$$A_s = 2\, \frac{\int_0^1 x f(x)\, J_0(\lambda_s x)\, dx}{\{J_0'(\lambda_s)\}^2},$$

so that if S_n is the sum of the first n terms of the series on the right-hand side of (1),

$$S_n = 2 \int_0^1 x f(x) \sum_{s=1}^{n} \frac{J_0(\lambda_s x)\, J_0(\lambda_s r)}{\{J_0'(\lambda_s)\}^2} dx.$$

Let the integral

$$\int \frac{\zeta\, G_0(\zeta)\, J_0(\zeta x)\, J_0(\zeta r)}{J_0(\zeta)}\, d\zeta$$

be taken round the contour consisting of the ξ-axis from $-M$ to M, indented at $\zeta = 0$ and at the zeros of $J_0(\zeta)$, and the lines $\xi = M$, $\eta = N$, and $\xi = -M$, where M and N are positive and M is chosen to lie between the zeros λ_ν and $\lambda_{\nu+1}$. The integrand is holomorphic within the contour, so that the value of the integral is zero.

The integral round the small semi-circle at $\zeta = 0$ tends to zero with the radius. Also, since

$$G_0(\zeta)\, J_0'(\zeta) - J_0(\zeta)\, G_0'(\zeta) = 1/\zeta,$$

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it follows that $\lambda_s G_0(\lambda_s) = 1/J_0'(\lambda_s)$; hence the sum of the integrals round the small semi-circles at the zeros of $J_0(\zeta)$ tends to

$$-2\pi i \sum_{s=1}^{\nu} \frac{J_0(\lambda_s x) J_0(\lambda_s r)}{\{J_0'(\lambda_s)\}^2}$$

as the radii tend to zero.

Again, along the ξ-axis the integrand is uniform and odd apart from the term in $G_0(\zeta)$ which involves $\log \zeta$; this latter term gives rise to an integral

$$i\pi \int_0^M \xi J_0(\xi x) J_0(\xi r)\, d\xi,$$

while the remaining integrals from $-M$ to 0 and 0 to M cancel each other.

But

$$\int_0^M \xi J_0(\xi x) J_0(\xi r)\, d\xi$$
$$= \frac{M}{x^2 - r^2}\{r J_0(Mx) J_0'(Mr) - x J_0(Mr) J_0'(Mx)\}$$
$$= \frac{M}{x^2 - r^2}\{-r J_0(Mx) J_1(Mr) + x J_0(Mr) J_1(Mx)\}.$$

In the right-hand side of this equation replace the Bessel Functions by their asymptotic expansions; then

$$\int_0^M \xi J_0(\xi x) J_0(\xi r)\, d\xi$$
$$= \frac{2}{\pi\sqrt{(xr)}} \frac{1}{x^2 - r^2} \left\{ \begin{array}{l} -x\cos(Mr - \pi/4)\cos(Mx + \pi/4) \\ +r\cos(Mr + \pi/4)\cos(Mx - \pi/4) \end{array} \right\} + \frac{P}{M}$$
$$= \frac{1}{\pi\sqrt{(xr)}} \left\{ \frac{\sin\{M(x-r)\}}{x-r} - \frac{\cos\{M(x+r)\}}{x+r} \right\} + \frac{P}{M},$$

where P is finite for all values of M.

Now let N tend to infinity; then, if $|x+r| < 2$, the integral along $\eta = N$ tends to zero. Also, if the Bessel Functions in the integrals along $\xi = \pm M$ be replaced by their asymptotic expansions, these integrals have the values $I_1 + Q/M$ and $I_2 + R/M$, where Q and R are finite* and

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* This will be clear if N and M tend to infinity together in such a way that the line joining the origin to the point $M + iN$ makes a finite angle with the imaginary axis; for instance, if $M = N$.

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$$I_1 = -\frac{1}{\surd(xr)}$$

$$\int_0^\infty \frac{e^{i(M+i\eta-\pi/4)} \cos\{x(M+i\eta)-\pi/4\} \cos\{r\,(M+i\eta)-\pi/4\}}{\cos(M+i\eta-\pi/4)}\,d\eta,$$

$$I_2 = \frac{1}{\surd(xr)}$$

$$\int_0^\infty \frac{e^{i(-M+i\eta+\pi/4)} \cos\{x(-M+i\eta)+\pi/4\} \cos\{r(-M+i\eta)+\pi/4\}}{\cos(-M+i\eta+\pi/4)}\,d\eta.$$

Again

$$\left|\frac{e^{i(\pm M+i\eta\mp\pi/4)}}{\cos(\pm M+i\eta\mp\pi/4}\right| = \left|2\frac{e^{-2\eta}}{1+e^{\pm 2i(M-\pi/4)-2\eta}}\right|$$

$$\leqq 2Ke^{-2\eta},$$

where K is a finite positive constant.

But

$$I_1 = -\frac{1}{2\surd(xr)}$$

$$\int_0^\infty \frac{e^{i(M+i\eta-\pi/4)}[\sin\{(x+r)(M+i\eta)\} + \cos\{(x-r)(M+i\eta)\}]}{\cos(M+i\eta-\pi/4)}\,d\eta$$

$$= \frac{1}{2\surd(xr)}[\sin\{(x+r)M\} \times V_1 + \cos\{(x+r)M\} \times V_2$$

$$+ \cos\{(x-r)M\} \times V_3 + \sin\{(x-r)M\} \times V_4],$$

where

$$|V_1| \leqq 2K\int_0^\infty e^{-2\eta}\cosh\{(x+r)\eta\}\,d\eta$$

$$\leqq K\left(\frac{1}{2+x+r} + \frac{1}{2-x-r}\right),$$

provided that $|x+r| < 2$, and similarly for V_2, V_3, and V_4. I_2 can also be expressed as the sum of four similar expressions.

Accordingly, if $0 \leqq r < 1$, and since $0 \leqq x \leqq 1$,

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$$\sum_{s=1}^{\nu} \frac{J_0(\lambda_s x)\, J_0(\lambda_s r)}{\{J_0(\lambda_s)\}^2} = \frac{1}{2\pi \sqrt{(xr)}} \left\{ \frac{\sin\{M(x-r)\}}{x-r} - \frac{\cos\{M(x+r)\}}{x+r} \right\}$$

$$+ \frac{P}{2M} + \sum_{s=1}^{16} \frac{\sin}{\cos} \{(x \pm r) M\} W_s + \frac{Q'}{M} + \frac{R'}{M},$$

where W_s, Q' and R' are finite.

Now multiply this equation by $2xf(x)$, integrate from 0 to 1, and let ν tend to infinity: then

$$\operatorname*{Lim}_{\nu \to \infty} 2 \int_0^1 xf(x) \sum_{s=1}^{\nu} \frac{J_0(\lambda_s x)\, J_0(\lambda_s r)}{\{J_0'(\lambda_s)\}^2}$$

$$= \frac{1}{\pi} \operatorname*{Lim}_{M \to \infty} \int_0^1 xf(x) \frac{1}{\sqrt{(xr)}} \frac{\sin\{M(x-r)\}}{x-r}\, dx$$

$$- \frac{1}{\pi} \operatorname*{Lim}_{M \to \infty} \int_0^1 xf(x) \frac{1}{\sqrt{(xr)}} \frac{\cos\{M(x+r)\}}{x+r}\, dx$$

$$+ \sum_{s=1}^{16} \operatorname*{Lim}_{M \to \infty} \int_0^1 xf(x)\, W_s \frac{\sin}{\cos} \left\{ (x \pm r) M \right\} dx.$$

By the theory of Dirichlet Integrals,

$$\operatorname*{Lim}_{M \to \infty} \int_a^b \phi(x) \frac{\sin}{\cos} \left(Mx \right) dx = 0,$$

provided that, for $a \leqq x \leqq b$, $\phi(x)$ is nnite and continuous, except for a finite number of finite discontinuities, and has only a finite number of maxima and minima; while, subject to the same conditions,

$$\operatorname*{Lim}_{M \to \infty} \int_a^b \phi(x) \frac{\sin\{M(x-r)\}}{x-r}\, dx = \frac{\pi}{2} \{\phi(r+0) + \phi(r-0)\}$$

for $a < r < b$.

Accordingly, if $f(x)$ satisfies these conditions for $0 \leqq x \leqq 1$,

$$\sum_{s=1}^{\infty} A_s J_0(\lambda_s r) = \tfrac{1}{2}\{f(r+0) + f(r-0)\}, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

provided that $0 < r < 1$.

When $r = 1$, $J_0(\lambda_s r) = 0$, and therefore the sum of the series is zero.

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When $r=0$, it follows from the theory of Dirichlet Integrals that the sum is $\frac{1}{2}f(+0)$.

Similarly for the series $\Sigma A_s J_n(\lambda_s r)$, where λ_s is a positive zero of $J_n(x)$, consider the integral of $\zeta G_n(\zeta) J_n(\zeta x) J_n(\zeta r) / J_n(\zeta)$ * and when λ_s is a positive zero of $A x J_n'(x) + B J_n(x)$ employ the integral of

$$\frac{\zeta \{A \zeta G_n'(\zeta) + B G_n(\zeta)\} J_n(\zeta x) J_n(\zeta r)}{A \zeta J_n'(\zeta) + B J_n(\zeta)}.$$

In the former case

$$A_s = 2 \int_0^1 x f(x) J_n(\lambda_s x) \, dx \, / \, \{J_n'(\lambda_s)\}^2,$$

and in the latter case

$$A_s = \frac{A^2 \lambda_s^2}{\{B^2 + A^2(\lambda_s^2 - n^2)\} J_n^2(\lambda_s)} \int_0^1 x f(x) J_n(\lambda_s x) \, dx.$$

If in (2) r, x, and λ are replaced by r/a, x/a, and $a\lambda$, and $\phi(r)$ is written in place of $f(r/a)$, then the equation may be written

$$\sum_{s=1}^{\infty} J_0(\lambda_s r) \frac{2 \int_0^a x \phi(x) J_0(\lambda_s x) \, dx}{\{a J_0'(\lambda_s a)\}^2} = \tfrac{1}{2}\{\phi(r+0) + \phi(r-0)\},$$

provided that $0 < r < a$, where λ_s is a positive zero of $J_0(\lambda a)$. These transformations can also be applied to the other expansions.

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* When n is not an integer the integral along the ξ-axis becomes

$$P \int_0^M \frac{\xi \pi \left\{ J_{-n}(\xi) - e^{-in\pi} J_n(\xi) \right\} J_n(\xi x) J_n(\xi r)}{2 \sin n\pi \, J_n(\xi)} d\xi$$

$$- P \int_0^M \frac{\xi \pi \left\{ J_{-n}(\xi) - e^{in\pi} J_n(\xi) \right\} J_n(\xi x) J_n(\xi r)}{2 \sin n\pi \, J_n(\xi)} d\xi$$

$$= i\pi \int_0^M \xi J_n(\xi x) J_n(\xi r) \, d\xi.$$

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