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ORTHOMORPHISMS OF A COMMUTATIVE W^*-ALGEBRA

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Abstract

If M is a commutative W^*-algebra of operators and if ReM is the Dedekind complete Riesz space of self-adjoint elements of M, then it is shown that the set $\widehat{ReM}$ of densely defined self-adjoint transformations affiliated with ReM is a Dedekind complete, laterally complete Riesz algebra containing ReM as an order dense ideal. The Riesz algebra of densely defined orthomorphisms on ReM is shown to coincide with $\widehat{ReM}$, and via the vector lattice Radon-Nikodym theorem of Luxemburg and Schep, it is shown that the lateral completion of ReM may be identified with the extended order dual of ReM.

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Introduction

An orthomorphism on an Archimedean Riesz space is a densely defined, order bounded and band preserving linear map. The general theory of orthomorphisms has received much recent attention; see for example [5, 6, 8, 16, 20, 21, 22] and the references contained therein. For many Riesz spaces, it is possible to determine the orthomorphisms explicitly as multipliers, in a certain sense. For example, if the Riesz space L is the space $C(X)$ of all real-valued continuous functions on the compact space X, then orthomorphisms on L can be identified with (equivalence classes of) real-valued functions belonging "locally" to L with respect to the filter of dense open subsets of X and acting on L by pointwise multiplication ([6, Theorem 4.3]).

Let now ReM be the self-adjoint part of a commutative von Neumann algebra M of operators acting in a Hilbert space $\mathscr{H}$. It is well known that ReM is a

Dedekind complete Riesz space with respect to the natural (that is, quadratic form) ordering ([11, Chapter 8]). It is also well known [3] that ReM can be identified with the space of all real continuous functions on a compact, hyperstonian space so that the result of [6] cited above completely describes the orthomorphisms of ReM. This description, however, is not entirely satisfactory, as it is purely in terms of the structure space of ReM and therefore not intrinsic. Thus, a principal aim of this paper is to identify the orthomorphisms of ReM as the space $\widetilde{ReM}$ of all (densely-defined) self-adjoint transformations in $\mathscr{H}$ affiliated with ReM (Theorem 5.2). This is achieved by showing that such transformations may be constructed in an elementary fashion from ReM and may be endowed with the structure of a Dedekind complete, laterally complete f-algebra containing ReM as an order dense ideal. There is a further aspect to be mentioned concerning the identification or orthomorphisms of ReM with unbounded self-adjoint transformations. Orthomorphisms in Dedekind complete Riesz spaces were studied by Nakano [15] under the name "dilatators". It is apparent that Nakano was motivated by the theory of self-adjoint transformations; however, no explicit connection was made in [15]. Thus, Theorem 5.2 of the present paper shows exactly the relation of the dilatators of Nakano to self-adjoint transformations in Hilbert space.

The analytic aspects of the theory of orthomorphisms have been shown by Luxemburg and Schep [10] to be very closely related to the classical Radon-Nikodym theorem of classical measure theory and, via their abstract Radon Nikodym theorem, we show that the positive orthomorphisms on ReM may also be identified as the Radon-Nikodym derivatives of the family of all semi-finite normal traces on ReM with respect to a fixed, strictly positive, semi-finite normal trace on ReM (Theorem 6.3). The appropriate setting here is the notion of extended order dual of an Archimedean Riesz space, introduced in [9].

We make some further remarks concerning the order theoretic aspects of this paper. Our development will be from the perspective of the theory of Riesz spaces and accordingly our starting point will be the established lattice structure of ReM as presented in Chapter 8 of the monograph [11]. Thus our results can be viewed as extensions to commutative algebras of unbounded self-adjoint transformations of the results presented in [11]. However, new problems arise in the case of unbounded transformations. The first task is to establish that $\widetilde{ReM}$ has the structure of a commutative algebra. This fact is, of course, well known and was established by Murray and von Neumann [12] in the more general setting of finite von Neumann algebras. In view of the fact that the methods of [12] depend on the notion of a dimension function, we take advantage here of the commutativity of ReM to present a direct and simplified approach to the domain problems necessary to establish the algebraic structure of $\widetilde{ReM}$. The essential idea here is to consider positive elements of $\widehat{ReM}$ as "equivalence classes" of increasing sequences of positive elements of ReM which converge pointwise on some common dense linear subspace of $\mathscr{H}$. This notion is formulated precisely in Lemma 2.4 and Theorem 2.6. Our construction is elementary and not only leads to the desired fact that $\widehat{ReM}$ can be endowed with the structure of a real commutative algebra, but yields a very natural proof of the (well-known) fact that each positive element of $\widehat{ReM}$ has a unique positive square root in $\widehat{ReM}$, and this places in perspective the work of Bernau [1]. Just as in the bounded case, the existence and uniqueness of square roots of positive elements of $\widehat{ReM}$ implies that $\widehat{ReM}$ can be given the structure of a Dedekind complete Riesz space with the identity I as a weak order unit. From this perspective, it can then be seen very clearly that the well-known spectral theorem for self-adjoint transformations is a very special case of the more general lattice spectral theorem of Freudenthal [11, Chapter 6].

The terminology concerning the Riesz space aspects of this paper will be drawn from the monograph [11] of Luxemburg and Zaanen. Convenient references to the theory of orthomorphisms are the paper [10] of Luxemburg and Schep and the lecture notes of Luxemburg [8].

1. Preliminary information

We begin by recalling some terminology concerning linear transformations in Hilbert space. For general properties of these transformations a convenient reference is [13]. Let $\mathscr{H}$ be a complex Hilbert space and let H be a linear transformation with domain $\mathscr{D}(H)$ and range $\mathscr{R}(H)$ linear subspaces of $\mathscr{H}$. The transformation H' is called an extension of H, written $H' \supseteq H$, if and only if $\mathscr{D}(H') \supseteq \mathscr{D}(H)$ and $H'z = Hz$ whenever $z \in \mathscr{D}(H)$. The graph of H is the linear subspace $\{(z, Hz) \colon z \in \mathscr{D}(H)\}$ of the Cartesian product $\mathscr{H} \times \mathscr{H}$, equipped with the usual inner product. H is called closed if the graph of H is a closed subset of $\mathscr{H} \times \mathscr{H}$. If S is a bounded linear operator on $\mathscr{H}$, then S, H are said to commute if $SH \subseteq HS$.

If the linear transformation H is densely defined, then the adjoint map H^* is defined as follows: $\mathscr{D}(H^*)$ is the set of those elements $y \in \mathscr{H}$ for which there exists a (necessarily unique) element $y^* \in \mathscr{H}$ such that

$$(Hz, y) = (z, y^*)$$

for all $z \in \mathscr{D}(H)$, in which case H^*y is defined to be y^*. The transformation H^* so defined is again a linear transformation in $\mathscr{H}$. If H is densely defined, then H is called symmetric if $H \subseteq H^*$ and self-adjoint if $H = H^*$. H is called positive, written $H \geqslant 0$ if and only if H is self-adjoint and $(Hz, z) \geqslant 0$ for each $z \in \mathscr{D}(H)$.

We shall make frequent use of the following elementary results. For the proofs we refer to [17, Sections 114–119] and [7, Chapter XII, Section 1].

LEMMA 1.1. *If H is a densely defined linear transformation in $\mathscr{H}$, then H has a closed linear extension if and only if $\mathscr{D}(H^*)$ is dense in $\mathscr{H}$. In this case, H^{**} is the minimal closed extension of H and the graph of H^{**} is the closure in $\mathscr{H} \times \mathscr{H}$ of the graph of H.*

LEMMA 1.2. *Let H be densely defined, linear and closed.*

(i) *$(I + H^*H)^{-1}$ exists, is everywhere defined and bounded; moreover $\|(I + H^*H)^{-1}\| \leqslant 1$.*

(ii) *The transformation $H(I + H^*H)^{-1}$ is bounded and $\|H(I + H^*H)^{-1}\| \leqslant 1$.*

(iii) *The transformation H^*H is self-adjoint and positive. If H' is the restriction of H to $\mathscr{D}(H^*H)$, then H is the smallest closed linear extension of H'. Consequently, the graph of H' is dense in the graph of H.*

Throughout the paper, M will denote a commutative von Neumann algebra of operators in the Hilbert space $\mathscr{H}$. By ReM is denoted the set of self-adjoint elements of M. With the natural (quadratic form) partial order, ReM is a Dedekind complete Riesz space. The elementary properties of ReM, developed from the point of view of the theory of Riesz spaces, may be found in [11, Chapter 8], or in Chapter XI of [18]. We assume throughout that the identity operator I on $\mathscr{H}$ is an element of ReM. The commutant of M is denoted by M'. If A is a bounded operator on $\mathscr{H}$, then $A \in M$ if and only if A commutes with each unitary operator in M'. We denote by $\widehat{ReM}$, the collection of all self-adjoint transformations in $\mathscr{H}$ which commute with each unitary operator in M'. The elements of ReM are said to be affiliated with M.

2. The square root of a positive self-adjoint transformation

In this section we show that each positive element of $\widehat{ReM}$ can be approximated, in a natural way, by an increasing sequence of positive elements of ReM (Theorem 2.5). We show that this idea then leads to an elementary proof of the existence and uniqueness of the square root of a positive element of $\widehat{ReM}$ (Theorem 2.6.).

We begin with the following observation.

LEMMA 2.1. *Let $0 \leqslant H \in \widehat{ReM}$. The bounded operators $(I + H^2)^{-1}$, $H(I + H^2)^{-1}$ are positive self-adjoint elements of ReM.*

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PROOF. Note that H^2 is self-adjoint and let $z_1, z_2 \in \mathscr{H}$. We have

$$\begin{aligned}\left(H(I+H^2)^{-1}z_1, z_2\right) &= \left((I+H^2)^{-1}z_1, H(I+H^2)^{-1}z_2\right)\\ &\quad + \left(H^2(I+H^2)^{-1}z_1, H(I+H^2)^{-1}z_2\right)\\ &= \left((I+H^2)(I+H^2)^{-1}z_1, H(I+H^2)^{-1}z_2\right)\\ &= \left(z_1, H(I+H^2)^{-1}z_2\right).\end{aligned}$$

Thus $H(I+H^2)^{-1}$ is self-adjoint. Moreover, from

$$\left((I+H^2)^{-1}z, z\right) = \left((I+H^2)^{-1}z, (I+H^2)(I+H^2)^{-1}z\right)$$

for all $z \in \mathscr{H}$, it follows that $(I+H^2)^{-1} \geqslant 0$ since $H^2 \geqslant 0$. A similar calculation yields that $H(I+H^2)^{-1} \geqslant 0$.

To see that $(I+H^2)^{-1} \in ReM$, observe first that

$$\begin{aligned}(I+H^2)(I+H^2)^{-1} = I = I^* &\supseteq (I+H^2)^{-1}(I+H^2)^*\\ &\supseteq (I+H^2)^{-1}(I+H^2).\end{aligned}$$

If now $U \in M'$ is unitary, then

$$U = U(I+H^2)(I+H^2)^{-1} \subseteq (I+H^2)U(I+H^2)^{-1}$$

so that

$$(I+H^2)^{-1}U \subseteq (I+H^2)^{-1}(I+H^2)U(I+H^2)^{-1} \subseteq U(I+H^2)^{-1}.$$

Since $(I+H^2)^{-1}$ is bounded, equality holds and so $(I+H^2)^{-1} \in ReM$. From

$$H(I+H^2)^{-1}U = HU(I+H^2)^{-1} \supseteq UH(I+H^2)^{-1}$$

for each unitary $U \in M'$, it follows that also $H(I+H^2)^{-1} \in ReM$.

Part of the following lemma is of course well known. The other part is probably equally well known, but we include a proof for lack of convenient reference.

LEMMA 2.2. *If* $0 \leqslant H, K \in ReM$, *then* $H^2 \leqslant K^2$ *if and only if* $H \leqslant K$.

PROOF. That $0 \leqslant H \leqslant K$ implies $H^2 \leqslant K^2$ follows from the construction of the square root in ReM (see [11, Section 54]). Assume then that $H^2 \leqslant K^2$. From $(K-H)(K+H) \geqslant 0$, it follows that $((K-H)z, z) \geqslant 0$ whenever $z \in \mathscr{R}((K+H)^{1/2})$. Let $y \in \mathscr{H}$ and suppose that $(K+H)^{1/2}y = 0$; from

$$0 = ((K+H)y, y) = (Ky, y) + (Hy, y) = \|K^{1/2}y\|^2 + \|H^{1/2}y\|^2$$

it follows that $Ky = 0 = Hy$. Let now $x \in \mathscr{H}$ be arbitrary and write $x = y + z$ with z an element of the closure of $\mathscr{R}((K+H)^{1/2})$ and $(K+H)^{1/2}y = 0$. It now follows that $((K-H)x, x) = ((K-H)z, z) \geqslant 0$ and so $K \geqslant H$ as required.

The following simple calculation will be used repeatedly and is therefore presented separately. We observe that the proof is based on the commutativity of ReM.

LEMMA 2.3. *Let* $0 \leqslant H_n \uparrow_n \subseteq ReM$. *If* $z \in \mathscr{H}$, *then* $\lim_{n\to\infty} H_n z$ *exists in* $\mathscr{H}$ *if and only if* $\sup_n \|H_n z\| < \infty$.

PROOF. One implication is clear. Suppose then that $z \in \mathscr{H}$ and $\sup_n \|H_n z\| < \infty$. Since $H_n^2 \uparrow_n$, the sequence $\{\|H_n z\|^2 \colon n = 1, 2, \ldots\}$ is a non-decreasing bounded real sequence. For $n \geqslant m$,

$$\begin{aligned}\|H_n z - H_m z\|^2 &= (H_n z, H_n z) + (H_m z, H_m z) - 2(H_n z, H_m z)\\ &\leqslant \|H_n z\|^2 - \|H_m z\|^2 \to 0 \quad \text{as } m, n \to \infty.\end{aligned}$$

LEMMA 2.4. *Let* $0 \leqslant H_n \uparrow_n \subseteq ReM$. *Suppose that* $\mathscr{D} = \{z \in \mathscr{H} \colon \sup_n \|H_n z\| < \infty\}$ *is dense in* $\mathscr{H}$. *Define the linear transformation* H *on* $\mathscr{D}$ *by setting* $Hz = \lim_{n\to\infty} H_n z$ *for each* $z \in \mathscr{D}$. *The transformation* H *is densely defined, closed and* $H^* \supseteq H$. *Further* $HU \supseteq UH$ *for each unitary* $U \in M'$.

PROOF. That $\lim_{n\to\infty} H_n z$ exists whenever $z \in \mathscr{D}$ follows from the previous lemma. If $z_1, z_2 \in \mathscr{D}$, then

$$(Hz_1, z_2) = \lim_{n\to\infty} (H_n z_1, z_2) = \lim_{n\to\infty} (z_1, H_n z_2) = (z_1, Hz_2).$$

Thus $z_2 \in \mathscr{D}(H^*)$ and $H^* z_2 = Hz_2$.

To see that H is closed, suppose that $\{z_n\}_{n=1}^\infty \subseteq \mathscr{D}(H) = \mathscr{D}$, that $z_n \to z \in \mathscr{H}$ and that $Hz_n \to y$. Let $K > 0$ satisfy $\|Hz_n\| \leqslant K$ for $n = 1, 2, \ldots$ and observe that for $m, n = 1, 2, \ldots$

$$\|H_m z_n\| \leqslant \|Hz_n\| \leqslant K.$$

It follows that $\|H_m z\| \leqslant K$ for $m = 1, 2, \ldots$ and so $z \in \mathscr{D}$. Now H^* is closed and $H^* \supseteq H$ and so $H^* z = y$. It follows also that $Hz = y$ since $z \in \mathscr{D}$ and so H is closed.

Let now $U \in M$ be unitary and let $z \in \mathscr{D}$. From

$$\|H_m Uz\| = \|UH_m z\| = \|H_m z\|$$

for $m = 1, 2, \ldots$, it follows that $Uz \in \mathscr{D}$. Further

$$UHz = U\Big(\lim_{n\to\infty} H_n z\Big) = \lim_{n\to\infty} H_n Uz = HUz$$

and the proof is complete.

We now show that each positive element of $\widehat{ReM}$ can be "approximated" in a natural manner by monotone sequences of elements of ReM.

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THEOREM 2.5. *Let* $0 \leqslant H \in \widehat{ReM}$ *and let* $H_n = H(I + H^2/n)^{-1} \in ReM$ *for* $n = 1, 2, \ldots$. *Then* $0 \leqslant H_n \uparrow_n \subseteq ReM$ *and* $z \in \mathscr{D}(H)$ *if and only if* $\sup_n \|H_n z\| < \infty$. *If* $z \in \mathscr{D}(H)$, *then* $Hz = \lim_{n \to \infty} H_n z$.

PROOF. For $n = 1, 2, \ldots$, let $J_n = (I + H^2/n)^{-1}$. Then $0 \leqslant J_n, H_n \in ReM$ and $\|J_n\| \leqslant 1, \|H_n\| \leqslant \sqrt{n}$ for $n = 1, 2, \ldots$. It is a simple matter to check the following:

(i) $\|H_n z\| \leqslant \|Hz\|$, $n = 1, 2, \ldots$, $z \in \mathscr{D}(H)$.

(ii) $J_m - J_n = (1/m - 1/n) H_n H_m$, $m, n = 1, 2, \ldots$.

(iii) $I - J_n = 1/n H H_n$, $n = 1, 2, \ldots$.

It follows now that $H_m \leqslant H_n$ if $n \geqslant m$. In fact, it follows from (ii) that $J_m \leqslant J_n$ and so also $J_m^2 \leqslant J_n^2$ for $n \geqslant m$. From $(J_m^2 Hz, Hz) \leqslant (J_n^2 Hz, Hz)$ for $n \geqslant m$ and $z \in \mathscr{D}(H)$, it follows that $\|H_m z\| \leqslant \|H_n z\|$ for $n \geqslant m$ and all $z \in \mathscr{H}$. Thus $H_m^2 \leqslant H_n^2$ for $n \geqslant m$ and so $H_m \leqslant H_n$ for $n \geqslant m$ by Lemma 2.2.

We show now that $Hz = \lim_{n \to \infty} H_n z$ whenever $z \in \mathscr{D}(H)$. If $z \in \mathscr{D}(H)$, then $\|H_n z\| \leqslant \|Hz\|$ from (i) above and so, since $0 \leqslant H_n \uparrow_n \subseteq ReM$, it follows from Lemma 2.3 that $\lim_{n \to \infty} H_n z = \lim_{n \to \infty} H J_n z$ exists. By closedness of H, it suffices to show that $J_n z \to z$ whenever $z \in \mathscr{D}(H)$. This however follows from (iii) by noting that for each $z \in \mathscr{D}(H)$ and $n = 1, 2, \ldots$, we have

$$\|(I - J_n) z\| = 1/n \|H H_n z\| \leqslant 1/\sqrt{n} \|Hz\|.$$

Let now $\mathscr{D} = \{z \colon \sup_n \|H_n z\| < \infty\}$ and let H_0 be defined on $\mathscr{D}$ by setting $H_0 z = \lim_{n \to \infty} H_n z$, for $z \in \mathscr{D}$. By Lemma 2.4, H_0 is densely defined, closed and symmetric and from the preceding remarks it is clear that $H_0 \supseteq H$. Thus $H_0^* \subseteq H^* = H$ and from $H_0 \subseteq H_0^*$, it follows that $H_0 = H$ and the proof of the theorem is complete.

We are now in a position to give an elementary proof of the existence and uniqueness of the positive square root of a positive self-adjoint transformation.

THEOREM 2.6. *Let* $0 \leqslant H \in \widehat{ReM}$, *let* $0 \leqslant H_n \uparrow_n \subseteq ReM$ *and suppose that* $z \in \mathscr{D}(H)$ *if and only if* $\sup_n \|H_n z\| < \infty$, *and that* $Hz = \lim_{n \to \infty} H_n z$ *whenever* $z \in \mathscr{D}(H)$. *Define the linear transformation* G *as follows. Let* $\mathscr{D}(G) = \{z \in H\colon \sup_n \|H_n^{1/2} z\| < \infty\}$ *and for* $z \in \mathscr{D}(G)$ *set* $Gz = \lim_{n \to \infty} H_n^{1/2} z$. *Then* $0 \leqslant G \in \widehat{ReM}$ *and* $G^2 = H$. *Moreover, if* $0 \leqslant G_0 \in \widehat{ReM}$ *and* $G_0^2 = G^2$, *then* $G = G_0$.

PROOF. If $z \in \mathscr{D}(H)$, then, for $n = 1, 2, \ldots$,

$$\|H_n^{1/2} z\| \leqslant \|H_n z\|^{1/2} \|z\|^{1/2}.$$

It follows that $\mathscr{D}(H) \subseteq \mathscr{D}(G)$. By Lemma 2.4 G is closed, densely defined, $G \subseteq G^*$ and $GU \supseteq UG$ for each unitary $U \in M'$. That $(Gz, z) \geqslant 0$ for each $z \in \mathscr{D}(G)$ follows simply from the definition of G.

To show that $H = G^2$, we show first that $\mathscr{D}(H) = \mathscr{D}(G^2)$. To this end, let $z \in \mathscr{D}(H)$. There is a constant $K > 0$ (depending, of course, on z) with

$$\|H_n z\| \leqslant \|H_m^{1/2} H_n^{1/2} z\| \geqslant \|H_m z\| \leqslant K$$

for $m \geqslant n$. Noting that $\mathscr{D}(H) \subset \mathscr{D}(G)$, it follows that $\sup_m \|H_m^{1/2} Gz\| < \infty$ and this implies that $Gz \in \mathscr{D}(G)$. Thus $\mathscr{D}(H) \subseteq \mathscr{D}(G^2)$. To see the reverse inclusion, let $z \in \mathscr{D}(G)$ and observe that

$$H_n^{1/2} Gz = H_n^{1/2} \lim_{m \to \infty} H_m^{1/2} z = \lim_{m \to \infty} H_m^{1/2} H_n^{1/2} z$$

for each $n = 1, 2, \ldots$. It follows that $H_n^{1/2} z \in \mathscr{D}(G)$ and $GH_n^{1/2} z = H_n^{1/2} Gz$ for $n = 1, 2, \ldots$. Suppose that $z \in \mathscr{D}(G^2)$; from

$$\|H_n^{1/2} H_n^{1/2} z\| \leqslant \|GH_n^{1/2} z\| = \|H_n^{1/2} Gz\|$$

for $n = 1, 2, \ldots$, it follows that $z \in \mathscr{D}(H)$ and so $\mathscr{D}(H) = \mathscr{D}(G^2)$. That G is self-adjoint now follows from the argument of [1, Lemma 3]. We include this argument for the sake of completeness. Since G is closed $G = G^{**}$ and so $GG^* = G^{**}G^*$ is self-adjoint. From $H = G^2 \subseteq GG^*$, it follows that $H = G^2 = GG^*$. Let $z \in \mathscr{D}(G^*)$. Since the graph of the restriction of G to $\mathscr{D}(GG^*)$ is dense in the graph of G^*, there exists a sequence $\{z_n\} \subseteq \mathscr{D}(GG^*) = \mathscr{D}(G^2)$ with $z_n \to z$ and $G^* z_n \to G^* z$. Since $z_n \in \mathscr{D}(G^2)$, follows that $Gz_n \to G^* z$ and by closedness of G, it follows that $z \in \mathscr{D}(G)$ and so $G = G^*$.

Turning to the uniqueness question, suppose that $0 \leqslant G_0 \in \widehat{ReM}$ satisfies $G_0^2 = G^2$. Observe that $G.G(I + G^2)^{-1} \supseteq G(I + G^2)^{-1} G$. From this, it follows that

$$G^2 (I + G^2)^{-2} = \left(G(I + G^2)^{-1}\right)^2 = G_0^2 \left(I + G_0^2\right)^{-2}.$$

By uniqueness of positive square roots in the bounded case, it follows that

$$G(I + G^2)^{-1} = G_0 (I + G^2)^{-1}.$$

Thus, $Gz = G_0 z$ whenever $z \in \mathscr{D}(G^2) = \mathscr{D}(G_0^2)$. That $G = G_0$ now follows from the fact that the graphs of G, G_0 are precisely the respective closures of the restrictions of G, G_0 to $\mathscr{D}(G^2) = \mathscr{D}(G_0^2)$. By this, the proof of the theorem is complete.

Let $H \in \widehat{ReM}$. As usual, we define $|H|$ to be $(H^2)^{1/2}$. For later reference, we remark that $\mathscr{D}(H) = \mathscr{D}(|H|)$. This is shown, for example, in Lemma 13 of [1]. It is now convenient to state the well known polar decomposition lemma. As the proof is standard (see, for example, Theorem 22 of [1] or Lemma 4.4.1 of [12]), we omit details of proof, although we do make the explicit remark that the decomposition depends only on the existence and uniqueness assertion of the preceding theorem.

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THEOREM 2.7. *Let $H \in \widehat{ReM}$. There exists a unitary operator U such that $H = U|H|$. U is uniquely determined by the requirement that $Ux = x$ whenever $Hx = 0$. The operator U is an element of M.*

We observe finally that to each self-adjoint transformation in $\mathscr{H}$, there corresponds naturally an abelian ring of operators to which H is affiliated. This allows comparison of the construction of Theorem 2.6 with that of Bernau [1] and Wouk [19]. As usual, $\mathscr{L}(\mathscr{H})$ denotes the linear space of all bounded linear operators on $\mathscr{H}$.

THEOREM 2.8. *Let H be a self-adjoint transformation on the Hilbert space $\mathscr{H}$. Let $N = \{S \in \mathscr{L}(\mathscr{H}): SH \subseteq HS\}$ and let $M(H)$ be the commutant of N. $M(H)$ is an abelian von Neumann algebra and $H \in \widehat{ReM}(H)$.*

PROOF. It is easily seen that N is a subalgebra of $\mathscr{L}(\mathscr{H})$ containing I and if $S \in N$ then also $S^* \in N$. From this it follows that $M(H)$ is a von Neumann algebra. To show that $M(H)$ is abelian, observe first that the bounded self-adjoint operators $(I + H^2)^{-1}$, $H(I + H^2)^{-1}$ belong to $N \cap M(H)$. In fact, from

$$H(I + H^2)^{-1} = \left(H(I + H^2)^{-1}\right)^* \subseteq (I + H^2)^{-1}H$$

it follows that both $(I + H^2)^{-1}$, $H(I + H^2)^{-1}$ belong to N. If $S \in N$, then $SH^2 \subseteq H^2S$ and from

$$S(H^2 + I) = SH^2 + S \subseteq H^2S + S = (H^2 + I)S$$

it follows that $(I + H^2)^{-1}S = S(I + H^2)^{-1}$. Further, from $SH \subseteq HS$, it follows immediately that

$$SH(I + H^2)^{-1} = HS(I + H^2)^{-1} = H(I + H^2)^{-1}S.$$

Thus $(I + H^2)^{-1}$ and $H(I + H^2)^{-1}$ are elements of $M(H)$.

Let now $S \in M(H)$. Since S commutes with $(I + H^2)^{-1}$ and $H(I + H^2)^{-1}$, it again follows that

$$HS(I + H^2)^{-1} = H(I + H^2)^{-1}S = SH(I + H^2)^{-1}.$$

From this it follows that $HS = SH$ on $\mathscr{D}(H^2)$. Let $z \in \mathscr{D}(H)$. Since the graph of the restriction of H to $\mathscr{D}(H^2)$ is dense in the graph of H, there exists a sequence $\{z_n\} \subseteq \mathscr{D}(H^2)$ such that $z_n \to z$ and $Hz_n \to H_z$. Thus $Sz_n \in \mathscr{D}(H)$ and $Sz_n \to Sz$ and $HSz_n \to SHz$. By closedness of H, $Sz \in \mathscr{D}(H)$ and $HSz = SHz$. Therefore $SH \subseteq HS$. It follows now that $S \in N$ and so $M(H)$ is abelian. At the same time, it follows that $H \in \widehat{ReM}(H)$ and the proof is complete.

3. The algebraic structure of $\widetilde{ReM}$

The main result of this section is that $\widetilde{ReM}$ is a real commutative algebra, provided the algebraic operations are suitably defined. The domain problems which arise quite naturally were resolved in [12] in a setting which is essentially more general than that considered here. We remark however, that even in the commutative case, the situation does not appear to be completely trivial and, in line with the objectives stated in the Introduction, we present here proofs of the fact that if $A, B \in \widetilde{ReM}$, then the transformations $A + B$, AB have closed extensions $[A + B]$, $[AB]$ to elements of $\widetilde{ReM}$ and that the algebraic operations so defined have all the properties that are expected.

We begin with an adaptation of Lemma 16.4.1 of [12].

LEMMA 3.1. *Let H be a linear, densely defined transformation in $\mathscr{H}$ such that $HU \supseteq UH$ for each unitary operator $U \in M'$. If $H \subseteq H^*$, then H^{**} is a maximal symmetric, self-adjoint extension of H to an element of $\widetilde{ReM}$. Any extension of H to an element of $\widetilde{ReM}$ coincides with H^{**}, and is therefore unique.*

PROOF. Let H satisfy the condition of the lemma and suppose that $H \subseteq H^*$. Since H is densely defined, it follows that H^{**} is defined and is the minimal closed extension of H. Since the graph of H^{**} is the closure of the graph of H, it follows from the closedness of H^{**} that $H^{**}U \supseteq UH^{**}$ for each unitary $U \in M'$. Without loss of generality, assume then that H is closed and let V be the Cayley transform of H. Since H is closed, the subspaces $\mathscr{D}(V) = \mathscr{R}(H - iI)$ and $\mathscr{R}(V) = \mathscr{R}(H - iI)$ are closed subspaces of $\mathscr{H}$. It is not difficult to see that V extends to a partial isometry in M. From $VV^* = V^*V$, it then follows that $\mathscr{R}(H + iI) = \mathscr{R}(H - iI)$. To show that H is self-adjoint (and hence maximal symmetric), it suffices to show that $\mathscr{R}(H + iI) = \mathscr{R}(H - iI) = \mathscr{H}$. If now $z \in \mathscr{H}$ satisfies

$$((H + iI)x, z) = 0 = ((H - iI)x, z)$$

for all $x \in \mathscr{D}(H)$, then $(x, z) = 0$ for all $x \in \mathscr{D}(H)$ and so $z = 0$ since $\mathscr{D}(H)$ is dense in $\mathscr{H}$, and the first assertion of the lemma is established.

Suppose now that H is densely defined and $HU \subseteq UH$ for all unitary $U \in M'$. Suppose that $H_0 \in \widetilde{ReM}$ is an extension of H. Then $H^{**} \subseteq H_0$ since H^{**} is the minimal closed extension of H. From $H \subseteq H_0$, it follows that $H^* \supseteq H_0^*H$, so that H is symmetric and so, from the first part of the lemma, it follows that H^{**} is self-adjoint. Since H^{**} is, in particular, maximal symmetric it follows that, in fact, $H^{**} = H_0$, and by this the lemma is proved.

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The following result may be viewed as the basis of the general spectral theorem for self-adjoint transformations. A similar result may be found in [17, page 316], and in [18]. It will provide the main tool in the discussion of the domain problems referred to above.

LEMMA 3.2. *Let $H \in \widetilde{ReM}$. There exists a sequence $\{E_n\}_{n=1}^{\infty}$ of projections on ReM with the following properties.*

(i) $E_n(\mathscr{H}) \subseteq \mathscr{D}(H)$ *for* $n = 1, 2, \ldots$.

(ii) $E_n \uparrow_n I$.

(iii) *The restriction of H to $E_n(\mathscr{H})$ is an element of ReM for $n = 1, 2, \ldots$.*

(iv) *If $z \in \mathscr{H}$, then $z \in \mathscr{D}(H)$ if and only if $\sup_n \|HE_n z\| < \infty$ in which case $\lim_{n\to\infty} HE_n z$ exists and is equal to Hz.*

PROOF. Via the polar decomposition and the fact that $\mathscr{D}(H) = \mathscr{D}(|H|)$ for each $H \in \widetilde{ReM}$, it suffices to consider only the case that $H \geqslant 0$.

For $n, m = 1, 2, \ldots$, let $H_n = H(I + H^2/n)^{-1}$, let $E_{m,n} \in ReM$ be the component of I in the band generated by $(nI - H_m)^+$ and let $E_n = \inf_m E_{m,n}$. Observe that, from

$$0 \leqslant E_n(nI - H_m)^+ = E_n E_{m,n}(nI - H_m) = E_n(nI - H_m)$$

it follows that $H_m E_n \leqslant nE_n$ for all m, n. In particular, if $z \in \mathscr{H}$ and if $E_n z = z$, then $\|H_m z\| \leqslant n\|z\|$ for $m = 1, 2, \ldots$. It follows from Theorem 2.5 that $E_n(\mathscr{H}) \subseteq \mathscr{D}(H)$, and from the closed graph theorem, it follows that the restriction of H to $E_n(\mathscr{H})$ is bounded; moreover, it is clear that this restriction is an element of ReM.

To show that $E_n \uparrow_n I$, set $F = \sup_n E_n$ and note that

$$(nI - H_m) - E_{m,n}(nI - H_m) = -(nI - H_m)^- \leqslant 0$$

so that $H_m(I - E_{m,n}) \geqslant n(I - E_{m,n})$ for all m, n. If $z \in \mathscr{H}$, then from

$$\|H_m z\| \geqslant \|H_m(I - E_{m,n})z\| \geqslant n\|(I - E_{m,n})z\|$$

for all m, n, it follows from $E_{m,n} \downarrow_m E_n$ that

$$\lim_{m\to\infty} \|H_m z\| \geqslant n\|(I - E_n)z\|$$

for every n. Since $\|(I - E_n)z\| \to \|(I - F)z\|$ as $n \to \infty$, it follows that $(I - F)z \neq 0$ implies $z \notin \mathscr{D}(H)$. Since $\mathscr{D}(H)$ is dense, it follows that $I - F = 0$. The proof of assertions (i), (ii), (iii) is complete.

To see that (iv) holds, observe that $\mathscr{D} = \{z\colon \sup_n \|HE_n z\| < \infty\} \supseteq \bigcup_{n=1}^{\infty} E_n(\mathscr{H})$ and so $\mathscr{D}$ is dense in $\mathscr{H}$. Since $0 \leqslant HE_n \uparrow_n \subseteq ReM$, there exists, by Lemma 2.5 and Lemma 3.1, and element $H' \in \widetilde{ReM}$ such that $\mathscr{D}(H') = \mathscr{D}$ and $H'z = \lim HE_n z$ whenever $z \in \mathscr{D}$. It is clear that H', H have equal restrictions to

$\bigcup_{n=1}^{\infty} E_n(\mathscr{H})$ and it follows from Lemma 3.1 that $H' = H$. By this (iv) follows and the proof of the lemma is complete.

We come now to the main result of this section.

THEOREM 3.3. *Let n be a natural number and let $A_1, \ldots, A_n$ be elements of $\widehat{ReM}$. Then $A_1 + \cdots + A_n$, $A_1, \ldots, A_n$ have unique extensions, denoted $[A_1 + \cdots + A_n]$, $[A_1 \cdots A_n]$ respectively, to elements of $\widehat{ReM}$. If $1 \leqslant k \leqslant n$, then*

(i) $[[A_1 + \cdots + A_k] + [A_{k+1} + \cdots + A_n]] = [A_1 + \cdots + A_n]$,

(ii) $[[A_1 \cdots A_k].[A_{k+1} \cdots A_n]] = [A_1 \cdots A_n]$.

With the algebraic operations so defined, $\widehat{ReM}$ is a real commutative algebra. Further, if A, B, C are elements of $\widehat{ReM}$, then

(iii) $[A[B + C]] = [[AB] + [AC]]$.

PROOF. Let $A_1, \ldots, A_n$ be elements of $\widehat{ReM}$ and for $1 \leqslant j \leqslant n$, let $\{E_i^{(j)}\}_{i=1}^{\infty}$ be a sequence of projections in ReM which satisfy the assertions (i)–(iv) of Lemma 3.2 for A_j. From the commutativity of M, it follows that

$$A_1 \cdots A_n E_i^{(1)} \cdots E_i^{(n)} = A_1 E_i^{(1)} \cdots A_n E_i^{(n)}$$

holds for $i = 1, 2, \ldots$ so that $E_i^{(1)} \cdots E_i^{(n)}(\mathscr{H}) \subseteq \mathscr{D}(A_1 \cdots A_n)$. Since $E_i^{(1)} \cdots E_i^{(n)} = E_i^{(1)} \wedge \cdots \wedge E_i^{(n)} \uparrow_i I$, it follows that $\mathscr{D}(A_1 \cdots A_n)$ is dense in $\mathscr{H}$. For each $x, y \in \mathscr{H}$ and for each $i_1, \ldots, i_n = 1, 2, \ldots$ observe that

$$\left(A_1 E_{i_1}^{(1)} \cdots A_n E_{i_n}^{(n)} x, y\right) = \left(x, A_1 E_{i_1}^{(1)} \cdots A_n E_{i_n}^{(n)} y\right)$$

and so successive applications of part (iv) of Lemma 3.2 yield that $(A_1 \cdots A_n x, y) = (x, A_1 \cdots A_n y)$ for each $x, y \in \mathscr{D}(A_1 \cdots A_n)$ and so $A_1 \cdots A_n \subseteq (A_1 A_n)^*$. A similar argument shows that $A_1 \cdots A_n U \supseteq U A_1 \cdots A_n$ for each unitary $U \in M'$ and so $(A_1 \cdots A_n)^{**}$ provides the unique closed extension of $A_1 \cdots A_n$ to an element of $\widehat{ReM}$ and we denote this extension by $[A_1 \cdots A_n]$. Moreover, if C denotes the restriction of $A_1 \cdots A_n$ to $\bigcup_{i=1}^{\infty} E_i^{(1)} \cdots E_i^{(n)}(\mathscr{H})$, then it is easily verified that $CU \subseteq UC$ for each unitary $U \in M'$, that C is densely defined and that $C \subseteq C^*$. Consequently, since $C \subseteq [A_1 \cdots A_n]$, it follows that $C^{**} = [A_1 \cdots A_n]$ by Lemma 3.1. If now $1 \leqslant k \leqslant n$ and σ is any permutation of $\{1, 2, \ldots, m\}$, then C coincides with the restrictions to $\bigcup_{i=1}^{\infty} E_i^{(1)} \cdots E_i^{(n)}(\mathscr{H})$ of both $[[A_1 \cdots A_n][A_{k+1} \cdots A_n]]$ and $[A_{\sigma(1)} \cdots A_{\sigma(n)}]$. Again by the uniqueness part of Lemma 3.1, it follows that

$$[A_1 \cdots A_n] = \left[[A_1 \cdots A_k][A_{k+1} \cdots A_n]\right] = \left[A_{\sigma(1)} \cdots A_{\sigma(n)}\right].$$

The remaining assertions of the theorem are proved in a similar manner and accordingly the proofs are omitted.

4. The Riesz space structure of $\widehat{ReM}$

We show now that the partial order on ReM may be extended to a partial ordering of $\widehat{ReM}$ and that $\widehat{ReM}$ is itself a Dedekind complete Riesz space. Recall that $A \in \widehat{ReM}$ is called positive, written $A \geqslant 0$, if $(Ax, x) \geqslant 0$ whenever $x \in \mathscr{D}(A)$. We show first that the cone properties are satisfied so that $\widehat{ReM}$ is an ordered vector space.

LEMMA 4.1. (i) *If* $A, B \in \widehat{ReM}, A \geqslant 0, B \geqslant 0$ *then* $[A + B] \geqslant 0$.
(ii) *If* $A \in \widehat{ReM}$, *then* $aA \geqslant 0$ *for each nonnegative real number* a.
(iii) *If* $A \in \widehat{ReM}, A \geqslant 0, -A \geqslant 0$ *then* $A = 0$.

PROOF. (i) is a immediate consequence of the fact that the graph of $[A + B]$ is the closure of the graph of $A + B$.

(ii) is obvious.

(iii) $A \geqslant 0$, $-A \geqslant 0$ implies $(Az, z) = 0$ for all $z \in \mathscr{D}(A)$. It follows that $A^{1/2} = 0$ and so $A = 0$.

THEOREM 4.2. *Let* $0 \leqslant A, B \in \widehat{ReM}$.
(i) $[AB] \geqslant 0$.
(ii) $A \geqslant B$ *implies* $A^2 \geqslant B^2$.
(iii) $A \geqslant B$ *if and only if* $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ *and* $(Az, z) \geqslant (Bz, z)$ *for each* $z \in \mathscr{D}(A)$.
(iv) $A \geqslant B$ *implies* $A^{1/2} \geqslant B^{1/2}$.

PROOF. (i) Follows immediately from the fact that $[AB] = [B^{1/2}AB^{1/2}]$.

(ii) By (i), it follows that

$$[A[A - B]] = \left[A^2 - [AB]\right] \geqslant 0 \quad \text{and} \quad [[A - B]B] = \left[[AB] - B^2\right] \geqslant 0.$$

Hence

$$0 \leqslant \left[\left[A^2 - [A - B]\right] + \left[[AB] - B^2\right]\right] = [A^2 - B^2].$$

(iii) Suppose $A \geqslant B$; then $A^2 \geqslant B^2$ by (ii). For $n = 1, 2, \ldots,$ set $B_n = B(I + B^2/n)^{-1}$. Then $B_n \uparrow_n$ and $z \in \mathscr{D}(B)$ if and only if $\sup_n \|B_n z\| < \infty$. Observe that if $z \in \mathscr{D}(A^2) \cap \mathscr{D}(B^2)$ and $n = 1, 2, \ldots,$ then

$$(B_n z, B_n z) \leqslant (Bz, Bz) \leqslant (Az, Az).$$

If A_0 is the restriction of A to $\mathscr{D}(A^2) \cap \mathscr{D}(B^2)$, then it is easily seen from the proof of Theorem 3.3(i) that A_0 is densely defined. Moreover, it is not difficult to see that $A_0 U \supseteq U A_0$ for every unitary $U \in M'$ and consequently from Lemma

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3.1, it follows that $A_0^{**} = A$. From Lemma 1.1, the graph of A_0 is dense in the graph of A so that

$$(B_n z, B_n z) \leqslant (Az, Az)$$

for $n = 1, 2, \ldots$, whenever $z \in \mathscr{D}(A)$. It follows immediately that $\mathscr{D}(A) \subseteq \mathscr{D}(B)$. From $[A - B] \geqslant 0$ it follows that $(Az, z) \geqslant (Bz, z)$ for all $z \in \mathscr{D}(A) \cap \mathscr{D}(B)$ thus for all $z \in \mathscr{D}(A)$.

Conversely, if $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $(Az, z) \geqslant (Bz, z)$ for all $z \in \mathscr{D}(A)$, then $[A - B] \geqslant 0$ follows from the fact that the graph of $A - B$ is dense in the graph of $[A - B]$.

(iv) Suppose that $A \geqslant B \geqslant 0$ and let $\{P_n\}_{n=1}^{\infty} \subseteq ReM$ be a sequence of projections for A as in Lemma 3.2 with $P_n(\mathscr{H}) \subseteq \mathscr{D}(A)$ for $n = 1, 2, \ldots$. Now $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and so $P_n(\mathscr{H}) \subseteq \mathscr{D}(B)$ for $n = 1, 2, \ldots$. From $B \geqslant BP_n$, it follows that $B^2 \geqslant (BP_n)^2$ for $n = 1, 2, \ldots$, and so $\mathscr{D}(B) \subseteq \{z \in \mathscr{H}: \sup_n \|BP_n z\| < \infty\}$. From $BP_n \uparrow_n$, and Lemmas 2.5, 3.1, it follows that $z \in \mathscr{D}(B)$ if and only if $\sup_n \|BP_n z\| < \infty$, in which case $Bz = \lim_{n \to \infty} BP_n z$. Now observe that from $A \geqslant B$ it follows that $AP_n \geqslant BP_n$ and $(AP_n)^{1/2} \geqslant (BP_n)^{1/2}$ for $n = 1, 2, \ldots$. It follows now from the construction of $A^{1/2}$, $B^{1/2}$ that $A^{1/2} \geqslant B^{1/2}$ and the proof of the theorem is complete.

LEMMA 4.3. *If* $A \in \widehat{ReM}$, *then* $A \leqslant |A|$ *and* $-A \leqslant |A|$. *Equivalently* $A^+ \geqslant 0$, $A^+ \geqslant A$.

PROOF. Let $A = V|A|$ be the polar decomposition of A and let $\{P_n\}_{n=1}^{\infty} \subseteq ReM$ be a sequence of projections as in Lemma 3.2 with $P_n(\mathscr{H}) \subseteq \mathscr{D}(|A|) = \mathscr{D}(A)$ for $n = 1, 2, \ldots$. If $z \in \mathscr{D}(A)$, then for $n = 1, 2, \ldots$,

$$|(V|A|P_n z, z)| \leqslant \|V\| \, \|(|A|P_n)^{1/2} z\|^2 \leqslant (|A|P_n z, z)$$

and so

$$|(Az, z)| \leqslant (|A|z, z)$$

for each $z \in \mathscr{D}(A)$, and the result follows.

THEOREM 4.4. *Let* $A, B \in \widehat{ReM}$ *satisfy* $B \geqslant A$, $B \geqslant 0$. *Then* $B \geqslant A^+$. *Equivalently* $C \geqslant A$, $C \geqslant -A$, $C \in \widehat{ReM}$ *imply* $C \geqslant |A|$. *In particular*, $\widehat{ReM}$ *is a Riesz space and* $A^+ = A \vee 0$, $|A| = A \vee (-A)$ *hold for each* $A \in \widehat{ReM}$.

The details of the proof are now exactly as in the bounded case ([11, Theorem 54.2]) and are accordingly omitted.

THEOREM 4.5. *The Riesz space $\widehat{ReM}$ is Dedekind complete and contains ReM as an order dense ideal. The identity operator I is a weak unit for $\widehat{ReM}$.*

PROOF. We show first that ReM is an order dense ideal in $\widehat{ReM}$. If $T \in ReM$, $S \in \widehat{ReM}$ and $0 \leqslant |S| \leqslant T$, then $\mathscr{H} = \mathscr{D}(T) \subseteq \mathscr{D}(|S|) = \mathscr{D}(S)$ and so the closed graph theorem implies that $S \in ReM$. Thus ReM is an order ideal in $\widehat{ReM}$. Let now $0 \leqslant S \in \widehat{ReM}$ be given and let $\{P_n\}_{n=1}^{\infty} \subseteq ReM$ be a sequence of projections in ReM satisfying the conditions of Lemma 3.2 for S. Suppose that $A \in \widehat{ReM}$ satisfies $A \geqslant SP_n$, for $n = 1, 2, \ldots$. It follows that $\|Az\| \geqslant \|SP_n z\|$ for all $z \in \mathscr{D}(A)$; consequently $\mathscr{D}(A) \subseteq \mathscr{D}(S)$ and $(Az, z) \geqslant (Sz, z)$ for $z \in \mathscr{D}(A)$, by part (iv) of Lemma 3.2. It follows that $S = \sup_n SP_n$ holds in $\widehat{ReM}$. Thus ReM is order dense in $\widehat{ReM}$ since $SP_n \in ReM$ for $n = 1, 2, \ldots$.

We prove now that $\widehat{ReM}$ is Dedekind σ-complete. Thus, let $0 \leqslant S_n \in \widehat{ReM}$ and suppose that $0 \leqslant S_n \leqslant S$ holds for $n = 1, 2, \ldots$. In the notation of the preceding paragraph, to show that $\sup_n S_n$ exists in $\widehat{ReM}$, it suffices to show that $\sup_n S_n \wedge SP_n$ exists in $\widehat{ReM}$. Since $S_n \wedge SP_n \in ReM$, it suffices to assume that $S_n \in ReM$ for $n = 1, 2, \ldots$. Let $\mathscr{D} = \{z \in \mathscr{H}: \sup_n \|S_n z\| < \infty\}$. From $0 \leqslant S_n \leqslant S$, it follows that $\mathscr{D}$ is dense. Define S' on $\mathscr{D}$ via $S'z = \lim_{n \to \infty} S_n^* z$, $z \in \mathscr{D}$. Via Lemmas 2.4, 3.2, $S' \in \widehat{ReM}$ and it is a simple matter to show that $S' = \sup_n S_n$ holds in $\widehat{ReM}$. Thus $\widehat{ReM}$ is Dedekind σ-complete.

From the Dedekind σ-completeness of $\widehat{ReM}$, it follows that if $0 \leqslant S \in \widehat{ReM}$, then $\sup_n S \wedge nI$ exists and satisfies $\sup_n S \wedge {}_n S \wedge nI \leqslant S$. To see that equality holds, let $\{P_n\}_{n=1}^{\infty} \subseteq ReM$ be the sequence of projections for S given by Lemma 3.2 and observe that for each $n = 1, 2, \ldots$, there exists a natural number $m(n)$ with $SP_n \leqslant m(n)I \wedge S$. Thus,

$$S = \sup_n SP_n \leqslant \sup_n nI \wedge S \leqslant S$$

and so I is a weak order unit for $\widehat{ReM}$.

Let now $\mathscr{P}$: $\widehat{ReM} \to \widehat{ReM}$ be a band projection and let $E = \mathscr{P}(I)$. Then E being a component of I is a projection in ReM and we show that $\mathscr{P}(A) = AE$ for each $A \in \widehat{ReM}$. In fact, if $0 \leqslant S \in \widehat{ReM}$ and $\{P_n\}_{n=1}^{\infty}$ is the sequence of projections for S given by Lemma 3.2, then $SE = \sup_n SP_n E$. Consequently

$$\mathscr{P}(S) = \sup_n \mathscr{P}(SP_n) = \sup_n SP_n E = SE$$

since $\mathscr{P}(A) = AL$ for each $A \in ReM$.

We now show that if $E \in ReM$ is a projection, then the principal band generated by E in $\widehat{ReM}$ consists precisely of those $S \in \widehat{ReM}$ for which $SE = S$. It will then follow that the Boolean algebra of all projection bands in $\widehat{ReM}$ is Dedekind complete. From this it will then follow that $\widehat{ReM}$ is Dedekind complete by an appeal to Theorem 42.9 of [11].

Let $E \in ReM$ be a projection and let $0 \leqslant S \in \widehat{ReM}$ be in the principal band generated by E in $\widehat{ReM}$. Let $\{P_n\} \subseteq ReM$ be a sequence of projections satisfying Lemma 3.2 for S. From $0 \leqslant SP_n \leqslant S$, it follows that $SP_n \wedge (I - E) = 0$ and from $SP_n \in ReM$, it follows that $SP_n(I - E) = 0$ for $n = 1, 2, \ldots$ From part (iv) of Lemma 3.2, it follows that $S(I - E) = 0$ and by this the proof of the theorem is complete.

LEMMA 4.6. *Let $0 \leqslant A_\tau \uparrow_\tau \subseteq \widehat{ReM}$. Let $z \in \mathscr{H}$ and suppose that $z \in \mathscr{D}(A_\tau)$ for each index τ. The net $\{A_\tau z\}$ converges in $\mathscr{H}$ if and only if $\sup_\tau \|A_\tau z\| < \infty$. If $\mathscr{D} = \{z\colon z \in \mathscr{D}(A_\tau)$ for each τ and $\sup_\tau \|A_\tau z\| < \infty\}$ is dense, then the transformation A defined on $\mathscr{D}$ by setting $Az = \lim A_\tau z$ for $z \in \mathscr{D}$ is an element of $\widehat{ReM}$. Moreover, $A_\tau \uparrow_\tau A$ holds in $\widehat{ReM}$.*

PROOF. Let $z \in \mathscr{D}$. If $A_{\tau'} \geqslant_\tau$, then

$$\|A_{\tau'}z - A_\tau z\|^2 \leqslant \|A_{\tau'}z\|^2 - \|A_\tau z\|^2.$$

From this it follows that the net $\{A_\tau z\} \subseteq \mathscr{H}$ is Cauchy and hence convergent.

Let now $\mathscr{D}$ be dense and for $z \in \mathscr{D}$, put $Az = \lim_\tau A_\tau z$. That $AU \supseteq UA$ for each unitary $U \in M'$ and that $A \subseteq A^*$ are easily established as in Lemma 2.4. We show that A is closed, whence an appeal to Lemma 3.1 will show that $A \in \widehat{ReM}$.

Suppose that $z_n \in \mathscr{D}$, that $z_n \to z$ and that $Az_n \to y$. In particular, there exists $K \geqslant 0$ with $\|Az_n\| \leqslant K$ for $n = 1, 2, \ldots$. For each index τ, let $\{A_{\tau,n}\}_{n=1}^\infty \subseteq ReM$ satisfy $0 \leqslant A_{\tau,n} \leqslant A_\tau$ and $z \in \mathscr{D}(A_\tau)$ if and only if $\sup_n \|A_{\tau,n} z\| < \infty$, in which case $A_\tau z = \lim_{n\to\infty} A_{\tau,n} z$. Observe, then, that for $n, n = 1, 2, \ldots$ and for each index τ, it follows that

$$\|A_{\tau,m} z_n\| \leqslant \|A_\tau z_n\| \leqslant \|Az_n\| \leqslant K.$$

From this it follows that $\|A_{\tau,m} z\| \leqslant K$ for all τ, m so that $z \in \mathscr{D}(A_\tau)$ for each τ and, from $\|A_\tau z\| \leqslant K$, it then follows that $z \in \mathscr{D}$. As $A \subseteq A^*$, and A^* is closed it follows that $Az = A^*z = \lim_{n\to\infty} A^* z_n = \lim_{n\to\infty} Az_n$. It follows that A is closed. It is now a simple matter to see that $A = \sup_\tau A_\tau$ in $\widehat{ReM}$.

THEOREM 4.7. *Let $\{A_\sigma\}, \{B_\tau\} \subseteq ReM$. If $0 \leqslant A_\sigma \uparrow_\sigma A \in \widehat{ReM}$ and $0 \leqslant B_\tau \uparrow_\tau B \in ReM$, then $[A_\sigma B_\tau] \uparrow_{\sigma,\tau} [AB]$. In particular, if $\{A_\tau\}, \{B_\tau\}$ are indexed by the same set $\{\tau\}$, then $[A_\tau B_\tau] \uparrow_\tau [AB]$.*

PROOF. Using the fact that ReM is order dense in $\widehat{ReM}$, we may without loss of generality assume that $\{A_\tau\}, \{B_\tau\} \subseteq ReM$. Observe that $0 \leqslant A_\sigma B_\tau \uparrow_{\sigma,\tau} \leqslant [AB]$. Indeed, this follows from the fact that

$$\left(A_\sigma B_\tau^{1/2} x,\, B_\tau^{1/2} x\right) \leqslant \left(AB^{1/2} x,\, B^{1/2} x\right), \qquad x \in \mathscr{D}(B^{1/2} A B^{1/2}),$$

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combined with the usual graph closure argument. By Dedekind completeness of $\widehat{ReM}$, there exists $0 \leqslant C \in \widehat{ReM}$ such that $A_\sigma B_\tau \uparrow_{\sigma,\tau} C \leqslant [AB]$ and so $\mathscr{D}([AB]) \subseteq \mathscr{D}(C)$. If $z \in \mathscr{D}(AB)$, then

$$ABz = \lim_\tau \lim_\sigma A_\sigma B_\tau z = \lim_{\tau,\sigma} A_\sigma B_\tau z = Cz.$$

It follows then that $[AB] = C$.

THEOREM 4.8. (i) *If* $0 \leqslant A_\tau \uparrow_\tau \subseteq \widehat{ReM}$, *then* $\sup_\tau A_\tau$ *exists in* $\widehat{ReM}$ *if and only if* $\{z \in \mathscr{H}: \sup_\tau(A_\tau z, z) < \infty\}$ *is dense in* $\mathscr{H}$.

(ii) *If* $0 \leqslant A_\tau \uparrow_\tau A$ *holds in* $\widehat{ReM}$ *then* $\sup_\tau(A_\tau z, z) = (Az, z)$ *for every* $z \in \mathscr{H}$.

(iii) *If* $0 \leqslant A_\tau \uparrow_\tau A$ *holds in* $\widehat{ReM}$, *then* $0 \leqslant A_\tau^{1/2} \uparrow_\tau A^{1/2}$.

PROOF. Let $\mathscr{D} = \{z \in \mathscr{H}: \sup_\tau(A_\tau z, z) < \infty\}$. Observe that $\mathscr{D} = \{z \in \mathscr{H}: \sup_\tau \|A_\tau^{1/2} z\| < \infty\}$. Since $\mathscr{D}$ is dense by assumption, it follows from Lemma 4.6 that $B = \sup_\tau A_\tau^{1/2}$ exists in $\widehat{ReM}$. From Theorem 4.7, it follows that $B^2 = \sup_\tau A_\tau$ holds in ReM. This proves the "if" assertion of (i). If $\sup_\tau A_\tau$ exists in $\widehat{ReM}$ then it is clear from Theorem 4.2 that $\mathscr{D}(\sup_\tau A_\tau) \subseteq \{z \in \mathscr{H}: \sup_\tau(A_\tau z, \tau) < \infty\}$. Thus (i) is proved.

If $0 \leqslant A_\tau \uparrow_\tau A$, then the proof of (i) shows that $\sup_\tau A_\tau^{1/2} = B$ exists in $\widehat{ReM}$ and satisfies $B^2 = A$. By the uniqueness of positive square roots, it follows that $B = A^{1/2}$ and so (iii) is proved.

Finally (ii) follows from (iii) and Lemma 4.6.

LEMMA 4.9. (i) *If* $A, B \in \widehat{ReM}$, *then* $A \wedge B = 0$ *holds if and only if* $[AB] = 0$, *and also if and only if* $A^2 \wedge B^2 = 0$.

(ii) *If* $A, B \in \widehat{ReM}$, *then* $A \wedge B = 0$ *if and only if* $R(A) \perp R(B)$.

(iii) *If* $A, B, C \in \widehat{ReM}$, *then* $[AC] \wedge [BC] = 0$ *if* $A \wedge B = 0$.

PROOF. Part (i) may be deduced simply from Lemma 55.3(i) of [11] by using the order density of ReM in $\widehat{ReM}$ and Theorem 4.7. Part (iii) is then a simple consequence of (i) and the commutativity of $\widehat{ReM}$.

To show that (ii) holds, suppose that $A \wedge B = 0$ so that $[AB] = 0$. Observe that $(x, ABy) = 0$ whenever $x \in \mathscr{D}(A)$ and $y \in \mathscr{D}(AB)$. Since the graph of the restriction of B to $\mathscr{D}(AB)$ is dense in the graph of B, it follows that $(Ax, By) = 0$ whenever $x \in \mathscr{D}(A)$, $y \in \mathscr{D}(B)$ and so $R(A) \perp R(B)$.

Conversely, if $R(A) \perp R(B)$, then $(x, ABy) = 0$ whenever $x \in \mathscr{D}(A)$ and $y \in \mathscr{D}(AB)$. Consequently $AB = 0$ and so $[AB] = 0$ and the result follows.

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LEMMA 4.10. (i) *Let $A \in \widehat{ReM}$ and let E be the component of I in the principal band generated by A in $\widehat{ReM}$. Then E is the orthogonal projection on $\overline{R(A)}$.*

(ii) *If $A, B \in \widehat{ReM}$, then A, B generate the same principal band in $\widehat{ReM}$ if and only if $\overline{R(A)} = \overline{R(B)}$.*

The proof is almost identical to that of Lemma 55.5 of [11], and is accordingly omitted.

Before proceeding to the next lemma, we recall the notion of direct product of transformations [13, Chapter 5]. Let $\{\mathscr{H}_i\}_{i\in\mathscr{I}}$ be a family of Hilbert spaces, and for each index i, let A_i be a linear transformation in $\mathscr{H}_i$. Let $\mathscr{H}$ be the product Hilbert space $\Pi_i \times \mathscr{H}_i$ and in $\mathscr{H}$, define a transformation A as follows. $\mathscr{D}(A)$ consists of all elements $z = \{z_i\}_{i\in\mathscr{I}} \in \mathscr{H}$ such that $z_i \in \mathscr{D}(A_i)$ for each index i and for which $\{A_i z_i\}_{i\in\mathscr{I}} \in \mathscr{H}$, that is, $\Sigma_i \|A_i z_i\|^2 < \infty$. For such an element z, let Az be $\{A_i z_i\}_{i\in\mathscr{I}}$. The mapping A so defined is linear and is called the direct product of the family $\{A_i\}_{i\in\mathscr{I}}$ and is written $A = \Pi_i \times A_i$. For basic properties of direct products, the reader is referred to [13]. We note explicitly from [13] that if each A_i is self-adjoint then $A = \Pi_i \times A_i$ is again self-adjoint and is the only self-adjoint transformation in $\mathscr{H}$ which is reduced to A_i by $\mathscr{H}_i$ for each index $i \in \mathscr{I}$.

Suppose now that $\{E_i\}_{i\in\mathscr{I}}$ is a family of mutually disjoint projections of ReM for which $\Sigma_{i\in\mathscr{I}} E_i = I$. We identify $\mathscr{H}$ with $\Pi_i \times E_i(\mathscr{H})$ in an obvious manner. For each $A \in ReM$, it then follows $A = \Pi_i \times A_i$ where $A_i = AE_i$ is the part of A in $E_i(\mathscr{H})$ for each i. If M' is the commutant of M, then the commutant of ME_i is just $M'E_i$. This is shown, for example, in [2, Chapter 1]. It follows that if $A_i \in \widehat{Re(ME_i)}$ for each index i, then $A = \Pi_i \times A_i \in \widehat{ReM}$.

LEMMA 4.11. *Let $\{E_i\}_{i\in\mathscr{I}}$ be a family of mutually disjoint projections of ReM which satisfies $\Sigma_{i\in\mathscr{I}} E_i = I$. For $i \in \mathscr{I}$ and $A \in \widehat{ReM}$, set $A_i = AE_i$ and let $\{\mathscr{F}\}$ denote the family of finite subsets $\mathscr{F} \subseteq \mathscr{I}$.*

(i) *If $0 \leqslant A \in \widehat{ReM}$, then*

$$A = \Pi_i \times A_i = \sup_{\{\mathscr{F}\}} \Pi_{i\in\mathscr{F}} \times A_i.$$

(ii) *If $0 \leqslant A, B \in \widehat{ReM}$, then*

$$[AB] = \sup_{i\in\mathscr{I}} [A_i B_i].$$

PROOF. (i) The quality $A = \Pi_i \times A_i$ has been noted above. Note that $E_i E_j = 0$ for $i \neq j$ implies $A_i \wedge A_j = 0$ and consequently $A_i \vee A_j = A_i + A_j$ for $i \neq j$ and so $\sup_{i\in\mathscr{F}} A_i = \Pi_{i\in\mathscr{F}} \times A_i$ for each finite subset $\mathscr{F} \subseteq \mathscr{I}$. Since $\widehat{ReM}$ is Dedekind complete, there exists $A' \in \widehat{ReM}$ with $0 \leqslant A' \leqslant A$ for which $A' = \sup_{\mathscr{F}} \Pi_{i \in \mathscr{F}} \times A_i$. From $A' \leqslant A$, it follows that $A'E_i \leqslant AE_i$ for each index i. However, it is clear that $A_i \leqslant A'$ and so $A_i = A_i E_i \leqslant A'E_i$ and consequently $A_i = A'E_i$ for each index i. It follows that $A = A'$ and (i) is proved.

(ii) In view of (i), it suffices to note that $[AB]E_i = [AE_i BE_i] = [A_i B_i]$ for each index $i \in \mathscr{I}$.

THEOREM 4.12. *Let $\{A_i\}_{i \in \mathscr{I}}$ be any system of mutually disjoint positive elements of ReM. Then $A = \bigvee_{i \in \mathscr{I}} A_i$ exists in $\widehat{ReM}$. In other words, the Riesz space $\widehat{ReM}$ is laterally complete.*

PROOF. It suffices to let $A = \Pi_{i \in \mathscr{I}} \times A_i$ and use part (i) of the preceding lemma.

A Riesz space L is called an f-algebra if L is an algebra such that $uv \geqslant 0$ whenever $u, v \in L^+$ and such that $(uw) \wedge v = (wu) \wedge v = 0$ for all $u, v, w \in L^+$ satisfying $u \wedge v = 0$. We may summarize the results of this section as follows.

THEOREM 4.13. *$\widehat{ReM}$ is a Dedekind complete, laterally complete f-algebra containing ReM as an order dense ideal. The identity operator I is a weak order unit for $\widehat{ReM}$.*

5. Orthomorphisms of *ReM*

We recall briefly the notion of orthomorphism. It is convenient to refer to [10] for a brief survey of the basic properties of orthomorphisms on Riesz spaces.

Let L be an Archimedean Riesz space. A linear mapping defined on an order dense ideal $\mathscr{D}_\theta \subset L$ into L is called a positive orthomorphism whenever $\theta(u) \geqslant 0$ for all $0 \leqslant u \in \mathscr{D}_\theta$ and $u \wedge v = 0$ with $0 \leqslant u \in \mathscr{D}_\theta$ and $0 \leqslant v \in L$ implies $\theta(u) \wedge v = 0$. The linear map θ from an order dense ideal $\mathscr{D}_\theta \subset L$ of L into L is called an orthomorphism whenever θ can be written in the form $\theta = \theta_1 - \theta_2$, where θ_1, θ_2 are positive orthomorphisms satisfying $\mathscr{D}_\theta \subseteq \mathscr{D}_{\theta_1} \cap \mathscr{D}_{\theta_2}$. The set of all orthomorphisms is denoted by $\mathrm{Orth}^\infty(L)$. Those orthomorphisms with domain L are denoted by $\mathrm{Orth}(L)$. It is shown in [10] that each orthomorphism on the Archimedean Riesz space L is order continuous. From this it follows that, for each $\theta \in \mathrm{Orth}^\infty(L)$, there exists a unique maximal domain to which θ can be extended. Two orthomorphisms are considered to be the same whenever they agree on an order dense ideal (and thus agree on their common maximal domain of definition).

The set $\mathrm{Orth}^\infty(L)$, with scalar multiplication and addition defined pointwise, is a linear space. Moreover, with the lattice operations sup and inf defined pointwise, $\mathrm{Orth}^\infty(L)$ is an Archimedean Riesz space which is laterally complete (that is, every mutually disjoint family of positive elements in $\mathrm{Orth}^\infty(L)$ has supremum) and which is Dedekind complete if L is Dedekind complete. For these results and their proofs, the reader is referred to the lecture notes of Luxemburg [8] and to the forthcoming paper [5]. If $\theta \in \mathrm{Orth}^\infty(L)$, and if $\mathscr{D}$ is an order dense ideal in L, then it can be shown that $\{u \in \mathscr{D}_\theta: \theta(u) \in \mathscr{D}\}$ is again an order dense ideal. For a relatively simple proof of this result, the reader is referred to [6, Lemma 1.3]. From this it follows that composition provides $\mathrm{Orth}^\infty(L)$ with a multiplicative structure under which $\mathrm{Orth}^\infty(L)$ is an f-algebra.

It is shown in Theorem 1.1 of [10] that if L is Dedekind complete, then each orthomorphism on L commutes on its domain with each band projection of L into L. We wish to note the following result for future reference.

LEMMA 5.1. *Let L be Dedekind complete and let $\theta_1, \theta_2 \in \mathrm{Orth}^\infty(L)$. Suppose that $u \in \mathscr{D}_{\theta_1} \cap \mathscr{D}_{\theta_2}$ and that $\theta_1(u) = \theta_2(u)$. Then θ_1 coincides with θ_2 on $\mathscr{D}_{\theta_1} \cap \mathscr{D}_{\theta_2} \cap B(u)$, where $B(u)$ denotes the principal band generated by u in L.*

PROOF. For each band projection P on L, observe that

$$\theta_1(Pu) = P\theta_1(u) = P\theta_2(u) = \theta_2(Pu).$$

The lemma now follows from the spectral theorem of Freudenthal and the order continuity of θ_1, θ_2.

We return now to the commutative W^*-algebra M. Note first that each mapping of the form $A \mapsto SA$, $A, S \in ReM$ is evidently a member of $\mathrm{Orth}(ReM)$. Conversely, if $\theta \in \mathrm{Orth}(ReM)$, then θ coincides with the orthomorphism $A \mapsto (I).A$ on the strong unit I. Consequently, by Lemma 5.1, θ is precisely the orthomorphism $A \mapsto \theta(I)A$. In an obvious way then, ReM may be identified with $\mathrm{Orth}(ReM)$. We now show that $\mathrm{Orth}^\infty(ReM)$ may be identified with $\widetilde{ReM}$.

THEOREM 5.2. *If $S \in \widetilde{ReM}$, then $\mathscr{D} = \{A \in ReM: SA \in ReM\}$ is an order dense ideal in ReM. The linear mapping $\theta(S)$: $A \mapsto SA$, $A \in \mathscr{D}$, is an orthomorphism on ReM with domain $\mathscr{D}$. Conversely, if θ is an orthomorphism on ReM with domain $\mathscr{D}_\theta$ an order dense ideal of ReM, then there exists a unique element $S(\theta) \in \widetilde{ReM}$ such that $\theta(A) = S(\theta).A$ for all $A \in \mathscr{D}_\theta$. The mapping $S \to \theta(S)$, $S \in \widetilde{ReM}$, is an f-algebra isomorphism of $\widetilde{ReM}$ onto $\mathrm{Orth}^\infty(ReM)$.*

PROOF. The first statement of the theorem follows from the introductory remarks preceding Lemma 5.1. That the map $\theta(S)$: $A \to SA$, $A \in \mathscr{D}$, is an orthomorphism is an immediate consequence of Lemma 4.9(i).

Let now θ *be an orthomorphism on* ReM with domain $\mathscr{D}_\theta$ an order dense ideal of ReM. Let $\{S_i\}_{i\in\mathscr{I}} \subseteq \mathscr{D}_\theta$ be a maximal disjoint system of positive elements. If E_i denotes the orthogonal projection onto the closure of the range of S_i, then it follows from the order density of $\mathscr{D}_\theta$ and the maximality of the system $\{S_i\}_{i\in\mathscr{I}}$ that $\sum_{i\in\mathscr{I}} E_i = I$. Now, for each index i, S_iE_i is invertible on the Hilbert space $E_i(\mathscr{H})$ and it is not difficult to verify that the transformation T_i in $E_i(\mathscr{H})$, which is inverse to the restriction of S_iE_i to $E_i(\mathscr{H})$, is an element of $Re(ME_i)$, where ME_i denotes the restriction of M to $E_i(\mathscr{H})$ (see [7]). Let θ_i be the restriction of θ to $\mathscr{D}_\theta \cap Re(ME_i)$; then θ_i is again an orthomorphism on $Re(ME_i)$. Consider the orthomorphism Φ_i: $AE_i \to [T_i\theta_i(S_i)]AE_i$ for $AE_i \in \mathscr{D}_{\theta_i}$. It is clear that $\theta_i(S_iE_i) = \Phi_i(S_iE_i)$ and so by order continuity, θ_i agrees with Φ_i on the ideal generated in $Re(ME_i)$ by S_iE_i. Now define $S(\theta) \in \widehat{ReM}$ to be $\Pi_i \times [T_i\theta_i(S_iE_i)]$. For each $A \in \mathscr{D}_\theta$,

$$\begin{aligned}\theta(A) &= \theta(\Pi_i \times AE_i) = \Pi_i \times \theta_i(AE_i)\\ &= \Pi_i \times [T_i\theta_i(S_iE_i)].AE_i = S(\theta).A,\end{aligned}$$

where we have used the order continuity of θ and Lemma 4.11.

It is now clear that the map $S \to \theta(S)$ is an injective linear map of $\widehat{ReM}$ onto $\mathrm{Orth}^\infty(ReM)$. If now $S_1, S_2 \in \widehat{ReM}$ and $S_1 \wedge S_2 = 0$, then for each $A \in \mathscr{D}_{\theta(S_1)} \cap \mathscr{D}_{\theta(S_2)}$ it follows that $\theta(S_1) \wedge \theta(S_2)(A) = 0$, for $\theta(S_1) \wedge \theta(S_2)(A) = \theta(S_1)(A) \wedge \theta(S_2)(A) = S_1A \wedge S_2A$. However $S_1 \wedge S_2 = 0$ implies $S_1S_2 = 0$ and also $S_1A.S_2A = 0$ from which it follows that also $S_1A \wedge S_2A = 0$. It follows that $\widehat{ReM}$ is Riesz isomorphic to $\mathrm{Orth}^\infty(ReM)$ and the proof of the theorem is complete, as it is clear that the isomorphism is an f-algebra isomorphism.

THEOREM 5.3. *If* $\theta \in \mathrm{Orth}^\infty(\widehat{ReM})$, *there exists a uniquely determined element* $S(\theta) \in \widehat{ReM}$ *such that* $\theta(T) = [S(\theta)T]$ *for all* $T \in \mathscr{D}_\theta$. *The Riesz spaces* $\mathrm{Orth}^\infty(\widehat{ReM})$, $\mathrm{Orth}(\widehat{ReM})$ *and* $\widehat{ReM}$ *are isomorphic as f-algebras.*

PROOF. If $\theta \in \mathrm{Orth}^\infty(\widehat{ReM})$, then

$$\begin{aligned}\mathscr{D} &= \{T \in ReM \cap \mathscr{D}_\theta: \theta(T) \in ReM\}\\ &= \{T \in \mathscr{D}_\theta: \theta(T) \in ReM\} \cap ReM\end{aligned}$$

is an order dense ideal in ReM. Consequently, from Theorem 5.2 above, there exists a unique element $S(\theta) \in \widehat{ReM}$ such that $\theta(T) = [S(\theta)T]$ for all $T \in \mathscr{D}'$, and hence, by order continuity, for all $T \in \mathscr{D}_\theta$. It follows immediately that the maximal domain of θ is $\widehat{ReM}$ so that, in fact, $\mathrm{Orth}^{\infty}(\widehat{ReM})$ and $\mathrm{Orth}(\widehat{ReM})$ coincide. The remaining assertion of the theorem follows quite simply and its proof is omitted.

6. The extended order dual of *ReM*

We recall briefly the notion of the extended order dual of an Archimedean Riesz space L as defined by Luxemberg and Masterson [9]. Elements of the extended order dual $\Gamma(L)$ of L are equivalence classes of normal integrals defined on order dense ideals of L, with two densely defined normal integrals determining the same equivalence class if and only if they are equal on an order dense ideal. It is shown in [9] that each equivalence class $[\phi] \in \Gamma(L)$ may be identified with a maximal representative ϕ whose domain $\mathscr{D}_{\phi}$ is the largest order dense ideal of L to which any (and hence all) representatives of $[\phi]$ may be finitely extended by normality. With respect to the ordering given by setting $\phi_1 \leqslant \phi_2$ if and only if $\mathscr{D}_{\phi_2} \subseteq \mathscr{D}_{\phi_1}$ and $\phi_1(f) \leqslant \phi_2(f)$ for all $f \in \mathscr{D}_{\phi_2}$, $\Gamma(L)$ is a Dedekind complete, laterally complete Riesz space.

It is also shown in [9] that if $\Gamma(L)$ separates the points of L, then L can be identified with an order dense Riesz subspace of $\Gamma(\Gamma(L))$ and that $\Gamma(\Gamma(L))$ is the lateral completion of L. Returning now to the case that L is the self-adjoint part ReM of a commutative von Neumann algebra M, it follows from Theorem 4.12 above that the lateral completion of the Dedekind complete Riesz space is just $\widehat{ReM}$. In this section, we wish to show that the lateral completion of ReM may also be identified with $\Gamma(ReM)$ via the Radon-Nikodym theorem of Luxemburg and Schep [10]. Since $\Gamma(ReM)$ is Riesz isomorphic to $\Gamma(\widehat{ReM})$, by [9, Theorem 2.6], we will show that $\widehat{ReM}$ is Riesz isomorphic to $\Gamma(\widehat{ReM})$.

Let $0 \leqslant \phi \in \Gamma(\widehat{ReM})$ and for $0 \leqslant T \in \widehat{ReM}$, define

$$\bar{\phi}(T) = \sup\{\phi(S): 0 \leqslant S \leqslant T \text{ and } S \in \mathscr{D}_{\phi}\}.$$

The function $\bar{\phi}$ is additive, positively homogeneous and monotone and is the minimal such extension of ϕ to the positive cone of ReM with values in the extended real number system. The function $\bar{\phi}$ is normal in the sense that $0 \leqslant S_\tau \uparrow_\tau S$ in ReM implies $\bar{\phi}(S) = \sup_\tau \bar{\phi}(S_\tau)$. In a related but somewhat different setting, the mappings $\bar{\phi}$ were considered by J. Dixmier [3] ("pseudo-mesure normale et essentielle").

If $0 \leqslant \phi \in \Gamma(ReM)$, define $N(\phi) = \{T: T \in \mathscr{D}_{\phi} \text{ and } \phi(|T|) = 0\}$. Similarly, let $N(\bar{\phi}) = \{T \in ReM: \bar{\phi}(|T|) = 0\}$. Then $N(\bar{\phi}) = N(\phi)$ and $N(\phi)$ is a band in the Dedekind complete Riesz space Φ_{ϕ}. It follows that $\mathscr{D}_{\phi} = N(\phi) \oplus C(\phi)$ where $C(\phi) = \{T \in \mathscr{D}_{\phi}: T \perp N(\phi)\}$ is called the carrier band of ϕ. It is shown in Theorem 1.4 of [7] that if $0 \leqslant \phi, \psi \in \Gamma(\widehat{ReM})$, then $\phi \wedge \psi = 0$ if and only if $C(\phi) \perp C(\psi)$.

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THEOREM 6.1. *Let* $0 \leqslant \phi, \psi \in \Gamma(ReM)$. *The following statements are equivalent.*
(i) $N(\psi) \subseteq N(\phi)$.
(ii) *There exists* $0 \leqslant S \in \widehat{ReM}$ *such that*

$$\bar{\phi}(T) = \bar{\psi}([ST])$$

for all $0 \leqslant T \in \widehat{ReM}$.

PROOF. (ii) $\Rightarrow$ (i). Suppose that (ii) is satisfied and that $\psi(T) = 0$ for some $0 \leqslant T \in \widehat{ReM}$. Let $\{S_n\} \subseteq ReM$ satisfy $0 \leqslant S_n \uparrow_n S$ in $\widehat{ReM}$. It is easily verified that $[S_nT] \leqslant \|S_n\|T$ for each n and so $\psi([S_nT]) = 0$ for $n = 1, 2, \ldots$. By normality, $\bar{\phi}(T) = \lim_{n\to\infty} \bar{\psi}([S_nT]) = 0$. It follows that $N(\psi) \subseteq N(\phi)$.

(i) $\Rightarrow$ (ii). By restriction to the carrier of ψ, we may assume that ψ is strictly positive. Let $\{E_i\}$ be a maximal family of mutually disjoint projections in ReM such that $\bar{\phi}(E_i)$, $\bar{\psi}(E_i) < \infty$ for each index i. Since ϕ and ψ are densely defined, maximality implies that $\sum_i E_i\mathrm{i} = I$. Now, for each index i, the restrictions of ϕ, ψ to $(ReM)E_i$ are normal integrals. Since ψ is strictly positive and since $(\widehat{ReM})E_i = E_i.\widehat{ReM}$, it follows from Theorem 3.3 of [8] and Theorem 5.2 above that there exists $S_i \in \widehat{ReM}$ such that $\phi(E_iT) = \psi([S_iE_iT])$ for each index i and each $0 \leqslant T \in ReM$. By normality $\bar{\phi}(E_iT) = \bar{\psi}([S_iE_iT])$ holds for each index i and each $0 \leqslant T \in \widehat{ReM}$, since ReM is order dense in $\widehat{ReM}$. Let now $S = \Pi_i \times S_iE_i$; note that $T = \sup_i E_iT$, and that $[ST] = \sup_i [S_iE_iT]$ for each $0 \leqslant T \in \widehat{ReM}$. It now follows from normality that $\bar{\phi}(T) = \bar{\psi}([ST])$ for each $0 \leqslant T \in \widehat{ReM}$.

It is to be remarked that, using the notion of center-valued trace, the preceding theorem yields the specialization of the Dye Radon-Nikodym theorem for finite von Neumann algebras given by Nakamura and Takeda [14]. We omit the details.

If x is an element of $\mathscr{H}$, denote by ω_x the (normal) vector state $T \mapsto (Tx, x)$ for $T \in ReM$ and by Ω_x the element of $\Gamma(\widehat{ReM})$ determined by ω_x. It is not difficult to see that if $0 \leqslant T \in \widehat{ReM}$, then T lies in the domain of Ω_x if and only if $x \in \mathscr{D}(T^{1/2})$, in which case $\Omega_x(T) = \|T^{1/2}x\|^2$. If now $0 \leqslant \phi$ is a normal integral on ReM and if Φ is the element of $\Gamma(\widehat{ReM})$ determined by ϕ, then the component of the identity I in the carrier band of Φ coincides with the component of I in the carrier band of ϕ in ReM and will be called the carrier of ϕ (or Φ). It is not difficult to see that the carrier of Ω_x is the orthogonal projection onto the closure in $\mathscr{H}$ of the linear subspace $\{M'x\}$ and this projection is denoted $E_x^{M'}$.

THEOREM 6.2. (i) *If* $0 \leqslant \psi$ *is a normal integral on* ReM, *if* $x \in \mathscr{H}$ *and if the carrier of* ψ *is majorized by the carrier of the vector state* ω_x, *then there exists* $0 \leqslant S \in \widehat{ReM}$ *with* $x \in \mathscr{D}(S^{1/2})$ *such that*

$$\psi(T) = (TS^{1/2}x, S^{1/2}x)$$

for all $T \in ReM$.

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(ii) *If* $0 \leqslant \psi \in \Gamma(\widehat{ReM})$, *then there exists* $x \in H$ *with* $\bar{\psi} = \bar{\Omega}_x$ *if and only if* $\bar{\psi}(I) < \infty$.

(iii) *If* $0 \leqslant \psi \in \Gamma(\widehat{ReM})$, *then there exists a family* $\{x_i\}$ *of vector in* $\mathscr{H}$ *such that the family of projections* $\{E_{x_i}^{M'}\} \subseteq ReM$ *are mutually disjoint and such that* $\bar{\psi} = \sum_i \bar{\Omega}_{x_i}$, ψ *is strictly positive if and only if* $\sum_i E_{x_i}^{M'} = I$. *In particular, there exists a strictly positive element* $0 \leqslant \psi \in \Gamma(\widehat{ReM})$.

Proof. (i) If Ψ denotes the element of $\Gamma(\widehat{ReM})$ determined by ψ, then part (i) follows immediately by applying the preceding theorem to Ψ and Ω_x, then specializing to ReM.

(ii) If $\bar{\psi} = \bar{\Omega}_x$, then it is trivial that $\bar{\psi}(I) < \infty$. Conversely, if $\bar{\psi}(I) < \infty$, then the restriction of ψ to ReM is a normal integral on ReM. From [2, page 19], it follows that the carrier of a normal integral on ReM is majorized by the carrier of a vector state and so part (ii) is a consequence of (i).

(iii) Let $\{E_i\} \subseteq ReM$ be a maximal system of mutually disjoint projections of ReM such that $\bar{\psi}(E_i) < \infty$. Maximality implies that $\sum_i E_i = I$. Part (ii) above implies the existence of elements $\{x_i\} \subseteq \mathscr{H}$, with $E_i x_i = x_i$, such that the restrictions ψ_i of ψ to $E_i(ReM)$ is just ω_{x_i}. It is clear that $E_{x_i}^{M'} \leqslant E_i$ and so the family $\{E_{x_i}^{M'}\}$ is mutually disjoint. Moreover, since $\psi = \sum_i \psi_i$, it is clear that ψ is strictly positive if and only if each ψ_i is strictly positive and this is the case if and only if $E_{x_i}^{M'} = E_i$ for each index i.

Finally, to show that there exists a strictly positive element $0 \leqslant \psi \in \Gamma(\widehat{ReM})$, let $\{x_i\}$ be a maximal system of vectors in $\mathscr{H}$ such that the family of projections $\{E_{x_i}^{M'}\}$ is mutually disjoint. By maximality, $\sum_i E_{x_i}^{M'} = I$ and consequently the functional ψ defined by setting $\psi = \sum_i \Omega_{x_i}$ is strictly positive.

We remark that part (ii) of the preceding Theorem 6.2 is (essentially) a result of R. Pallu de la Barrière [10, Theorem 5.1].

Theorem 6.3. *Let* $0 \leqslant \psi \in \Gamma(\widehat{ReM})$ *be strictly positive and denote by* θ *the Riesz isomorphism of* $\widehat{ReM}$ *onto* $\mathrm{Orth}^{\infty}(\widehat{ReM})$ *defined via Theorem* 5.3 *above. The mapping* $S \mapsto \psi \circ \theta(S)$, $S \in \widehat{ReM}$ *defines a Riesz isomorphism of* $\widehat{ReM}$ *onto* $\Gamma(\widehat{ReM})$.

Proof. Observe first that if $S \in \widehat{ReM}$, then $\{T \in \widehat{ReM}: \theta(S)(T) \in \mathscr{D}_{\psi}\}$ is again an order dense ideal in $\widehat{ReM}$. Normality of $\psi \circ \theta(S)$ on it domain is a simple consequence of the normality of ψ and of the normality of the orthomorphism $\theta(S)$. Thus $\psi \circ \theta(S)$ defines an element of $\Gamma(\widehat{ReM})$ which we continue to denote by $\psi \circ \theta(S)$. It is clear that the map $S \mapsto \psi \circ \theta(S)$, $S \in \widehat{ReM}$, is linear and positive. We show now that it is injective.

To this end, suppose that $\psi \circ \theta(S) = 0$ for some $S \in \widehat{ReM}$. It follows that $\psi([ST]) = 0$ for all elements T belonging to some order dense ideal $\mathscr{D} \subseteq \widehat{ReM}$ and, consequently, by normality, $\psi([ST]) = 0$ for all $T \in \widehat{ReM}$. Let now E^+, E^- be the components of I in the bands generated in $\widehat{ReM}$ by S^+, S^- respectively and set $T = E^+ - E^-$. It follows that $0 = \psi([S(E^+ - E^-)]) = \psi(S^+) + \psi(S^-)$ and since ψ is strictly positive it follows that $S^+ = S^- = 0$.

That $\psi \circ \theta$ maps $\widehat{ReM}$ onto $\Gamma(\widehat{ReM})$ is an immediate consequence of Theorem 6.1 above. To show that $\psi \circ \theta$ is a Riesz isomorphism, it now suffices to show that if $S_1, S_2 \in \widehat{ReM}$ satisfy $S_1 \wedge S_2 = 0$, then also $\psi \circ \theta(S_1) \wedge \psi \circ \theta(S_2) = 0$. Now , it is clear that if $0 \leqslant S \in \widehat{ReM}$, then the carrier of $\psi \circ \theta(S)$ is contained in the band generated by S in $\widehat{ReM}$. Consequently from $S_1 \wedge S_2 = 0$, it follows that $\psi \circ \theta(S_1)$, $\psi \circ \theta(S_2)$ have disjoint carriers in $\widehat{ReM}$. From an earlier remark, it now follows that $\psi \circ \theta(S_1) \wedge \psi \circ \theta(S_2) = 0$ and the proof is complete.

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