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VANISHING OF HOCHSCHILD COHOMOLOGIES FOR LOCAL RINGS WITH EMBEDDING DIMENSION TWO

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ABSTRACT. Let $S = k[[x, y]]$ be a formal power series ring in two variables x, y over a field k and I an (x, y)-primary ideal of S. We show that S/I is selfinjective if $H^i(S/I, S/I \otimes_k S/I) = 0$ for $i = 1$ and 2.

1. **Introduction.** Let A be a finite dimensional algebra over a field k and denote by $H^i(A, A \otimes_k A)$ the i-th Hochschild cohomology group of A with coefficient module $A \otimes_k A$ for $i \geq 1$. Is A selfinjective if $H^i(A, A \otimes_k A) = 0$ for all $i \geq 1$? This question was posed by Tachikawa [7] as a consequence of the conjecture of Nakayama [6] and has been recently considered by several authors (see, for instance, [5], [8], [1], [9], [10], [2] and [3]).

In this note we will prove the following

THEOREM. *Let A be a commutative artinian local ring with maximal ideal m. Assume that A contains the residue field $k = A/m$ and that* $\dim_k m/m^2 \leq 2$. *Then A is selfinjective if $H^i(A, A \otimes_k A) = 0$ for $i = 1$ and 2.*

REMARK. Let $S = k[[x, y]]$ be a formal power series ring in two variables x, y over a field k and I an (x, y)-primary ideal of S with $I \subset (x, y)^2$. Then $\dim_k k \otimes_S I = 1 + \dim_k \operatorname{Hom}_S(k, S/I)$ so that S/I is selfinjective if and only if it is a complete intersection.

2. **Serial local rings.** In this section, we recall a few well known results on serial local rings.

Throughout this section Λ is a commutative artinian serial local ring with maximal ideal n. We assume that Λ contains the residue field $k = \Lambda/n$.

LEMMA 1. *Let M be a finitely generated Λ-module. Then the following statements hold:*

(1) $\dim_k \operatorname{Hom}_\Lambda(k, M) = \dim_k k \otimes_\Lambda M$, *and*

(2) $\operatorname{Hom}_k(M, k) \simeq M$.

PROOF. Every finitely generated Λ-module M decomposes into a finite direct sum of uniserial Λ-modules, and for uniserial Λ-modules L_1 and L_2, $\dim_k L_1 = \dim_k L_2$ implies $L_1 \simeq L_2$.

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LEMMA 2. *Let $L(i)$ denote the uniserial Λ-module of length i (for $0 \leq i \leq \dim_k \Lambda$). Then for any i_1 and i_2 we have isomorphisms*

$$\mathrm{Hom}_\Lambda\big(L(i_1), L(i_2)\big) \simeq L(i_1) \otimes_\Lambda L(i_2) \simeq L(i_0)$$

where $i_0 = \min\{i_1, i_2\}$.

PROOF. By Lemma 1(2) the first isomorphism follows. The last isomorphism is obvious.

In the following, we will use Lemma 1 very frequently, so we will use it without any references.

3. **Notation.** In this section, we fix the notation which will be kept throughout the rest of this note.

Let A be a commutative artinian local ring with maximal ideal m and assume that A contains the residue field $k = A/m$. Since $\dim_k m/m^2 \leq 1$ implies A selfinjective, we assume that $\dim_k m/m^2 = 2$.

Let $x \in m$ with $x \notin m^2$ and put $R = k[x]$ and $\mu = \dim_k A/(x)$. It should be noted that both R and $A/(x)$ are serial local rings. Note also that $k \otimes_R - \simeq A/(x) \otimes_A -$ and $\mathrm{Hom}_R(k, -) \simeq \mathrm{Hom}_A\big(A/(x), -\big)$ on A modules. We denote by $L(i)$ the uniserial $A/(x)$-module of length i for $0 \leq i \leq \mu$.

Let D denote $\mathrm{Hom}_k(-, k)$ and put $b_i = \dim_k \mathrm{Tor}_i^A(k, DA)$ for $i \geq 0$. Let

$$\cdots \longrightarrow A^{b_1} \longrightarrow A^{b_0} \longrightarrow DA \longrightarrow 0$$

be a minimal free resolution of DA and put $\Omega_i = \mathrm{Coker}(A^{b_{i+1}} \longrightarrow A^{b_i})$ for $i \geq 1$.

LEMMA 3. *We have a decomposition of the form:*

$$\mathrm{Hom}_R(k, A) \simeq \bigoplus_{n=1}^{b_0} L(i_n)$$

with $i_1 \geq \cdots \geq i_{b_0} \geq 1$ and $i_1 + \cdots + i_{b_0} = \mu$.

PROOF. Since $\dim_k \mathrm{Hom}_{A/(x)}\big(k, \mathrm{Hom}_R(k, A)\big) = \dim_k \mathrm{Hom}_A(k, A) = b_0$ and $\dim_k \mathrm{Hom}_R(k, A) = \dim_k k \otimes_R A = \mu$, the assertion follows.

COROLLARY 4. *We have isomorphisms:*
(1) $\mathrm{Tor}_1^A\big(A/(x), DA\big) \simeq L(\mu - i_1)$, *and*
(2) $\mathrm{Tor}_2^A\big(A/(x), DA\big) \simeq \oplus_{n=2}^{b_0} L(i_n)^{2n-1}$.

PROOF. Let $(\ )^*$ denote $\mathrm{Hom}_A(-, A)$. We have an exact sequence of the form (see [4], Proposition 6.3):

$$0 \longrightarrow \mathrm{Ext}_A^1\big(A/(x), A\big) \longrightarrow A/(x) \stackrel{\varepsilon}{\longrightarrow} A/(x)^{**} \longrightarrow \mathrm{Ext}_A^2\big(A/(x), A\big) \longrightarrow 0$$

where ε denotes the usual evaluation map. Note that $A/(x)^* \simeq \mathrm{Hom}_R(k,A)$. It follows by Lemma 3 that $\mathrm{Im}\,\varepsilon \simeq L(i_1)$. Thus $\mathrm{Ker}\,\varepsilon \simeq L(\mu - i_1)$ and hence

$$\begin{aligned}\mathrm{Tor}_1^A\big(A/(x), DA\big) &\simeq D\,\mathrm{Ext}_A^1\big(A/(x), A\big)\\ &\simeq \mathrm{Ext}_A^1\big(A/(x), A\big)\\ &\simeq L(\mu - i_1).\end{aligned}$$

Next, since by Lemmas 2 and 3

$$\begin{aligned}A/(x)^{**} &\simeq \mathrm{Hom}_A\big(\mathrm{Hom}_R(k,A), A\big)\\ &\simeq \mathrm{Hom}_A\big(A/(x)\otimes_A \mathrm{Hom}_R(k,A), A\big)\\ &\simeq \mathrm{End}_A\big(\mathrm{Hom}_R(k,A)\big)\\ &\simeq \mathrm{End}_{A/(x)}\Big(\bigoplus_{n=1}^{b_0} L(i_n)\Big)\\ &\simeq \bigoplus_{n=1}^{b_0} L(i_n)^{2n-1},\end{aligned}$$

we get an exact sequence of the form:

$$0 \longrightarrow L(i_1) \longrightarrow \bigoplus_{n=1}^{b_0} L(i_n)^{2n-1} \longrightarrow \mathrm{Ext}_A^2\big(A/(x), A\big) \longrightarrow 0,$$

which splits because $i_1 \geq i_n$ for all $1 \leq n \leq b_0$. Thus

$$\begin{aligned}\mathrm{Tor}_2^A\big(A/(x), DA\big) &\simeq D\,\mathrm{Ext}_A^2\big(A/(x), A\big)\\ &\simeq \mathrm{Ext}_A^2\big(A/(x), A\big)\\ &\simeq \bigoplus_{n=2}^{b_0} L(i_n)^{2n-1}.\end{aligned}$$

4. **Proof of theorem.** Note that $H^i(A, A\otimes_k A) \simeq \mathrm{Ext}_A^i(DA, A)$ for all $i \geq 1$ (see for instance [7], p. 114). Thus the theorem stated in the introduction is a consequence of the following Lemmas 6 and 7.

LEMMA 5. *We have inequalities:*
(1) $b_1 \leq b_0 + 1$*; and*
(2) $b_0^2 - 1 \leq b_2$.

PROOF. By Corollary 4(1) we have an exact sequence of the form:

$$0 \longrightarrow L(\mu - i_1) \longrightarrow A/(x)\otimes_A \Omega_1 \longrightarrow A/(x)^{b_0}.$$

Applying $\mathrm{Hom}_{A/(x)}(k, -)$, we get

$$\begin{aligned}b_1 &= \dim_k k\otimes_A \Omega_1\\ &= \dim_k k\otimes_{A/(x)} A/(x)\otimes_A \Omega_1\\ &= \dim_k \mathrm{Hom}_{A/(x)}\big(k, A/(x)\otimes_A \Omega_1\big)\\ &\leq b_0 + 1.\end{aligned}$$

Next, by Corollary 4(2) we have an exact sequence of the form:

$$0 \longrightarrow \bigoplus_{n=2}^{b_0} L(i_n)^{2n-1} \longrightarrow A/(x) \otimes_A \Omega_2.$$

Applying $\mathrm{Hom}_{A/(x)}(k, -)$, we get

$$\begin{aligned} b_0^2 - 1 &= \sum_{n=2}^{b_0} (2n-1) \\ &\leq \dim_k \mathrm{Hom}_{A/(x)}\big(k, A/(x) \otimes_A \Omega_2\big) \\ &= \dim_k k \otimes_{A/(x)} A/(x) \otimes_A \Omega_2 \\ &= \dim_k k \otimes_A \Omega_2 \\ &= b_2. \end{aligned}$$

LEMMA 6. *Assume that* $\mathrm{Ext}_A^i(DA, A) = 0$ *for* $i = 1$ *and* 2. *Then* $b_0 \leq 2$.

PROOF. By Lemma 5 it suffices to show that $b_2 \leq b_1 + 1$. We have an exact sequence of the form:

$$0 \longrightarrow \mathrm{Hom}_R(k, A) \longrightarrow A \longrightarrow A \longrightarrow A/(x) \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(DA, -)$, we get an exact sequence of the form:

$$\mathrm{Hom}_A\big(DA, A/(x)\big) \longrightarrow \mathrm{Ext}_A^2\big(DA, \mathrm{Hom}_R(k, A)\big) \longrightarrow 0.$$

Apply $k \otimes_{A/(x)} -$. Since

$$\begin{aligned} \mathrm{Ext}_A^2\big(DA, \mathrm{Hom}_R(k, A)\big) &\simeq \mathrm{Ext}_A^2\big(\mathrm{Hom}_R(k, A), A\big) \\ &\simeq D\,\mathrm{Tor}_2^A\big(\mathrm{Hom}_R(k, A), DA\big) \\ &\simeq \mathrm{Tor}_2^A\big(\mathrm{Hom}_R(k, A), DA\big) \\ &\simeq \mathrm{Tor}_4^A\big(A/(x), DA\big) \\ &\simeq \mathrm{Tor}_2^A\big(A/(x), \Omega_2\big) \end{aligned}$$

and $\mathrm{Hom}_A\big(DA, A/(x)\big) \simeq \mathrm{Hom}_A\big(A/(x), A\big) \simeq \mathrm{Hom}_R(k, A)$, we get

$$\begin{aligned} b_0 &= \dim_k \mathrm{Hom}_A(k, A) \\ &= \dim_k \mathrm{Hom}_{A/(x)}\big(k, \mathrm{Hom}_R(k, A)\big) \\ &= \dim_k k \otimes_{A/(x)} \mathrm{Hom}_R(k, A) \\ &\geq \dim_k k \otimes_{A/(x)} \mathrm{Tor}_2^A\big(A/(x), \Omega_2\big) \\ &= \dim_k \mathrm{Hom}_{A/(x)}\Big(k, \mathrm{Tor}_2^A\big(A/(x), \Omega_2\big)\Big). \end{aligned}$$

Also, applying $\mathrm{Hom}_A(k, -)$ to the exact sequence $0 \longrightarrow \Omega_2 \longrightarrow A^{b_1}$, we get

$$\dim_k \mathrm{Hom}_A(k, \Omega_2) \leq b_0 b_1.$$

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Note that we have an exact sequence of the form:

$$0 \longrightarrow \operatorname{Tor}_2^A\big(A/(x), \Omega_2\big) \longrightarrow \operatorname{Hom}_R(k, A) \otimes_A \Omega_2 \longrightarrow \Omega_2.$$

Applying $\operatorname{Hom}_A(k, -)$, we get

$$\dim_k \operatorname{Hom}_{A/(x)}\big(k, \operatorname{Hom}_R(k, A) \otimes_A \Omega_2\big) \leq b_0 b_1 + b_0.$$

It only remains to see that $\dim_k \operatorname{Hom}_{A/(x)}\big(k, \operatorname{Hom}_R(k, A) \otimes_A \Omega_2\big) = b_0 b_2$. Since $\dim_k k \otimes_{A/(x)} A/(x) \otimes_A \Omega_2 = \dim_k k \otimes_A \Omega_2 = b_2$, we have a decomposition of the form:

$$A/(x) \otimes_A \Omega_2 \simeq \bigoplus_{m=1}^{b_2} L(j_m).$$

Thus by Lemma 3

$$\begin{aligned}\operatorname{Hom}_R(k, A) \otimes_A \Omega_2 &\simeq \operatorname{Hom}_R(k, A) \otimes_{A/(x)} A/(x) \otimes_A \Omega_2 \\ &\simeq D \operatorname{Hom}_{A/(x)}\big(\operatorname{Hom}_R(k, A), A/(x) \otimes_A \Omega_2\big) \\ &\simeq \operatorname{Hom}_{A/(x)}\big(\operatorname{Hom}_R(k, A), A/(x) \otimes_A \Omega_2\big) \\ &\simeq \bigoplus_{n=1}^{b_0} \bigoplus_{m=1}^{b_2} \operatorname{Hom}_{A/(x)}\big(L(i_n), L(j_m)\big)\end{aligned}$$

and hence by Lemma 2

$$\dim_k \operatorname{Hom}_{A/(x)}\big(k, \operatorname{Hom}_R(k, A) \otimes_A \Omega_2\big) = b_0 b_2.$$

Therefore $b_0 b_2 \leq b_0 b_1 + b_0$, so that $b_2 \leq b_1 + 1$, as required.

LEMMA 7. *Assume that* $b_0 = 2$. *Then* $\operatorname{Ext}_A^1(DA, A) \neq 0$.

PROOF. Apply $A/(x) \otimes_A$- to the exact sequence

$$0 \longrightarrow \Omega_1 \longrightarrow A^2 \xrightarrow{\pi} DA \longrightarrow 0.$$

Note that $A/(x) \otimes_A DA \simeq k \otimes_R DA \simeq D \operatorname{Hom}_R(k, A) \simeq \operatorname{Hom}_R(k, A)$. Thus by Lemma 3

$$\operatorname{Ker}\big(A/(x) \otimes_A \pi\big) \simeq \bigoplus_{n=1}^{2} L(\mu - i_n) = \bigoplus_{n=1}^{2} L(i_n)$$

and hence we get an exact sequence of the form:

$$A/(x) \otimes_A \Omega_1 \longrightarrow \bigoplus_{n=1}^{2} L(i_n) \longrightarrow 0.$$

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Applying $\operatorname{Hom}_{A/(x)}\big(-, \oplus_{n=1}^{2} L(i_n)\big)$, we get by Lemmas 2 and 3

$$\begin{aligned}\mu + 2i_2 &= i_1 + 3i_2\\ &= \dim_k \operatorname{End}_{A/(x)}\Big(\bigoplus_{n=1}^{2} L(i_n)\Big)\\ &\leq \dim_k \operatorname{Hom}_{A/(x)}\Big(A/(x)\otimes_A \Omega_1, \bigoplus_{n=1}^{2} L(i_n)\Big)\\ &= \dim_k \operatorname{Hom}_{A/(x)}\big(A/(x)\otimes_A \Omega_1, \operatorname{Hom}_R(k,A)\big)\\ &= \dim_k \operatorname{Hom}_A\big(k\otimes_R A/(x)\otimes_A \Omega_1, A\big)\\ &= \dim_k \operatorname{Hom}_A\big(A/(x)\otimes_A \Omega_1, A\big)\\ &= \dim_k A/(x)\otimes_A \Omega_1 \otimes_A DA.\end{aligned}$$

Now, suppose to the contrary that $\operatorname{Ext}_A^1(DA, A) = 0$. Then we have an exact sequence of the form (see [3], Remark 2.2):

$$0 \longrightarrow \Omega_1 \otimes_A DA \longrightarrow A.$$

Applying $\operatorname{Hom}_R(k, -)$, we get

$$\begin{aligned}\mu + 2i_2 &\leq \dim_k A/(x)\otimes_A \Omega_1 \otimes_A DA\\ &= \dim_k k\otimes_R \Omega_1 \otimes_A DA\\ &= \dim_k \operatorname{Hom}_R(k, \Omega_1 \otimes_A DA)\\ &\leq \dim_k \operatorname{Hom}_R(k, A)\\ &= \dim_k k\otimes_R A\\ &= \mu,\end{aligned}$$

a contradiction.

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