J. Austral. Math. Soc. 71 (2001), 169–176

CYCLIC PERMUTABLE SUBGROUPS OF FINITE GROUPS

JOHN COSSEY and STEWART E. STONEHEWER

To Laci Kovács on his 65th birthday

(Received 22 January 2001)

Communicated by R. A. Bryce

Abstract

The authors describe the structure of the normal closure of a cyclic permutable subgroup of odd order in a finite group.

2000 *Mathematics subject classification*: primary 20D35, 20D40.

1. Introduction and main result

A subgroup A of a group G is said to be *permutable* if $AX = XA$ for all subgroups X of G. Clearly this is equivalent to the product AX being a subgroup. Many properties of permutable subgroups are known. For example, a permutable subgroup of a finite group is always subnormal [7] and of an arbitrary group is always ascendant ([10, Theorem A]).

Since permutability is preserved by homomorphisms, the structure of A modulo its core A_G is of particular interest. In finite groups G, the quotient A/A_G is always nilpotent [5]. The first non-abelian example was given by Thompson in [13]. It had class 2. Then examples of arbitrary class appeared in [2] and [11]. These were all metabelian. Further examples of arbitrary derived length were constructed in [12]. In all these cases the group was a p-group of the form

$$G = AC, \qquad (1)$$

where C is cyclic. Following on from [12], Berger and Gross in [1] constructed universal examples (depending on p and $|C|$) in which every group of the form (1) embeds [1].

https://doi.org/10.1017/S1446788700002810 Published online by Cambridge University Press

As a consequence of the subnormality of a permutable subgroup A of a finite group G and the nilpotency of A/A_G, we see from Fitting's Theorem that A^G/A_G is always nilpotent (where A^G is the normal closure of A in G). Assume for simplicity of notation that $A_G = 1$. Clearly the class of A^G will be at least that of A. But even when A is abelian, the class of A^G can be arbitrarily large. For, let p be an odd prime and n a positive integer. Consider a cyclic group A of order p^n acting faithfully on a cyclic group of order p^{n+1} and form the natural semidirect product G. Then every subgroup of G is permutable ([4, Satz 15]). Also $A_G = 1$ and A^G has order p^{2n} and class n. Of course here G is metacyclic. Therefore, writing $d(G)$ for the derived length of G, the following question is of interest:

When is $d(A^G)$ bounded in terms of $d(A)$?

The purpose of this work is to prove that this is in fact the case when A is cyclic. Moreover it is not necessary to assume that A is core-free.

Our argument quickly reduces to the case when G is a p-group. When $p = 2$, most of our preliminary results for p-groups fail to hold and this case requires a considerable amount of additional analysis. Therefore it is more convenient and appropriate to deal with the $p = 2$ situation elsewhere. Thus our objective is to prove the following result.

THEOREM 1.1. *Let A be a cyclic permutable subgroup of a finite group G. If A has odd order, then*

(1) *$[A, G]$ is abelian;*
(2) *A acts by conjugation on $[A, G]$ as a group of power automorphisms; and*
(3) *A^G is abelian-by-cyclic.*

Since $A^G = A[A, G]$, (iii) is an immediate consequence of (i). Our interest in the structure of A^G was stimulated by a conjecture of Busetto. Busetto conjectures that the normal closure of a cyclic permutable subgroup of a finite p-group (p odd) is a modular group (it has a modular subgroup lattice). It follows from Theorem 1.1 and [9, Theorem 2.3.1] that in a finite group the normal closure of a cyclic permutable subgroup of odd order is a modular group, giving a positive answer to Busetto's conjecture as a special case. We are grateful to Professor Busetto for communicating his conjecture to us.

2. Further results and proofs

We begin by showing that the proof of Theorem 1.1 reduces to the case when G is a p-group. Thus we assume, for the moment, the truth of:

THEOREM 2.1. *Let p be an odd prime and let A be a cyclic permutable subgroup of the finite p-group G. Then*

(i) *$[A, G]$ is abelian; and*
(ii) *A acts on $[A, G]$ as a group of power automorphisms.*

PROOF OF THEOREM 1.1. Let G be a finite group with a cyclic permutable subgroup A of odd order and let P be the Sylow p-subgroup of A. Then P is permutable in G, by [9, Lemma 5.2.11]. Also P is subnormal in G (by([7]) and therefore $P^G = P[P, G]$ is a p-group. All the elements in G, of order relatively prime to p, normalise P. Let G_p be any Sylow p-subgroup of G. Thus $P \leq G_p$. We claim that

(2) *$[P, G]$ is abelian and P acts on it as a group of power automorphisms.*

For, if $P \lhd G$, then (2) is trivially true. So suppose that $P_G < P$. By [6], P/P_G lies in the hypercentre of G/P_G. Therefore elements in G, of order relatively prime to p, centralise P/P_G and hence also centralise P. In this case $[P, G] = [P, G_p]$. Thus (2) follows from Theorem 2.1.

Since A is subnormal in G, $A^G = A[A, G]$ is nilpotent of odd order. Therefore if $A = P_1 \times \cdots \times P_s$ is the decomposition of A into its primary components, we have

$$[A, G] = [P_1, G] \times \cdots \times [P_s, G].$$

Thus $[A, G]$ is abelian, by (2). Finally, again let $P = \langle x \rangle$ be any one of the P_i corresponding to the prime p. Then

(3) *P acts as a group of power automorphisms on $[A, G]$.*

For, write any element of $[A, G]$ in the form uv, with u a p-element and v a p'-element. So $u \in [P, G]$ and $u^x = u^n$, by (2). Also $v^x = v$. Let $|u| = p^a$, $|v| = m$. Then $(p, m) = 1$ and there are integers k and l such that

$$n - 1 = kp^a + lm.$$

Put $r = n - kp^a$. Thus $(uv)^x = u^n v$ and $(uv)^r = u^r v^r = u^n v$. Therefore P normalises every cyclic subgroup of $[A, G]$ and so acts as a group of power automorphisms. Hence (3) is true and the theorem follows. □

Now we may restrict our attention to finite p-groups, where p is an odd prime. Thus let $A = \langle a \rangle$ be a cyclic permutable subgroup of the finite p-group G. It turns out that every element of $[A, G]$ has the form $[a, g]$, for some g in G. The situation when A has order p is well understood.

https://doi.org/10.1017/S1446788700002810 Published online by Cambridge University Press

LEMMA 2.2. *Let $A = \langle a \rangle$ be a permutable subgroup of prime order p in the finite p-group G. Then $[A, G] \leq Z(G)$, the centre of G, and each element of $[A, G]$ has the form $[a, g]$, for some g in G.*

PROOF. If $A \lhd G$, then $A \leq Z(G)$ and the result is trivial. Therefore suppose that A is not normal in G. By [9, Theorem 5.2.9 (c)], $A \leq Z_2(G)$, the second centre of G. Thus $[A, G] \leq Z(G)$. Therefore, for any integer i and element g in G,

$$[a^i, g] = [a, g^i]$$

and it follows that each element of $[A, G]$ has the required form. □

We recall the elementary properties of finite p-groups that are the product of two cyclic subgroups.

LEMMA 2.3. *Let p be an odd prime and let $G = AX$ be a finite p-group, where $A = \langle a \rangle$ and $X = \langle x \rangle$ are cyclic subgroups. Then*

(i) *G is metacyclic;*
(ii) *every subgroup of G is permutable;*
(iii) *$G' = \langle [a, x] \rangle$;*
(iv) *for each integer i, $\langle [a^i, x] \rangle = \langle [a, x^i] \rangle = \langle [a, x]^i \rangle$; and*
(v) *every element of G' has the form $[a, g]$, for some g in X.*

PROOF. (i) and (ii) are, respectively, Hauptsatz I and Satz 15 in [4].

(iii) Let $N = \langle [a, x] \rangle$. Then $N \leq G'$ and $N \lhd G$, by (i). Since G/N is abelian, we must have $N = G'$.

(iv) We may assume, without loss of generality, that $i \geq 1$. Let p^t be the highest power of p dividing i. So $\langle a^{p^t} \rangle = \langle a^i \rangle$ and, by (ii) and (iii),

$$\langle [a^i, x] \rangle = \langle [a^{p^t}, x] \rangle.$$

Also $\langle [a, x]^i \rangle = \langle [a, x]^{p^t} \rangle$. Thus, from the symmetry of A and X, we may assume that $i = p^t$. Then, by induction on t, it suffices to establish the case $t = 1$. Modulo N^p, N is central in G and so $\langle [a^p, x] \rangle \leq \langle [a, x]^p \rangle$. Conversely, factoring G by $\langle [a^p, x] \rangle$, a^p becomes central. By (i), there is a normal cyclic subgroup K of G with G/K cyclic. Then we may assume that $N < K$. But a induces an automorphism of order at most p in K and therefore a must centralise N. Thus modulo $\langle [a^p, x] \rangle$, $[a, x]^p \equiv 1$, and so $\langle [a, x]^p \rangle \leq \langle [a^p, x] \rangle$. Using symmetry, (iv) follows.

(v) This is clear if $|G'| = p$, for then $G' \leq Z(G)$. Thus suppose that $|G'| = p^n$, $n \geq 2$, and proceed by induction on $|G'|$. We see that $[a, x^{p^{n-1}}]$ has order p and lies in $Z(G)$. By induction, each element of $[A, G]$ has the form

$$[a, g_1][a, x^{p^{n-1}}]^i,$$

for some g_1 belonging to X and some integer i. But this product of commutators equals $[a, x^{ip^{n-1}} g_1]$ and (v) follows. $\square$

Another special case can easily be proved.

LEMMA 2.4. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of the finite p-group G, where p is an odd prime. Suppose that $A \cap [A, G] = 1$. Then every element of $[A, G]$ has the form $[a, g]$, for some g in G.*

PROOF. We proceed by induction on $|G|$. If $A \lhd G$, then $[A, G] = 1$. Therefore we may suppose that $A_1 = A_G < A$.

Let A_2 be the unique subgroup of A with $|A_2 : A_1| = p$. Then A_2 is permutable in G and, by Lemma 2.2,

$$[A_2, G]A_1/A_1 \leq Z(G/A_1).$$

Hence $[A_2, G, G] \leq A_1 \cap [A, G] = 1$, and so $[A_2, G] \leq Z(G)$. Since $[A_2, G] \neq 1$, it follows from Lemma 2.3 that there is an element g in G such that

$[a, g]$ *is a non-trivial central element of* G.

Let $N = \langle [a, g] \rangle$. By induction, every element of $[A, G]$ has the form $[a, g_1]$ modulo N, for some g_1 in G. Therefore every element of $[A, G]$ has the form

$$[a, g_1][a, g]^i = [a, g^i g_1],$$

as required. $\square$

REMARK. The above result can fail when $p = 2$. For example, let A be the cyclic subgroup of order 8 in the dihedral group of order 16. Then all the elements $[a, g]$ are either trivial or of order 4.

Theorem 2.1 will follow from our next result.

THEOREM 2.5. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of the finite p-group G, where p is an odd prime. Then each element of $[A, G]$ has the form $[a, g]$, for some g in G.*

PROOF. Suppose that the Theorem is false and let G be a minimal counter-example. We also choose A of smallest possible order. By Lemma 2.2, $|A| \geq p^2$. Therefore, since A^p is permutable in G, each element of $[A^p, G]$ has the form $[a^p, g]$. Now we may assume that

$$(4) \qquad [A^p, G] = 1.$$

https://doi.org/10.1017/S1446788700002810 Published online by Cambridge University Press

For, if $[A^p, G] \neq 1$, then let N be a minimal normal subgroup of G contained in $[A^p, G]$. Thus $N \leq Z(G)$ and, by Lemma 2.3, each element of N has the form $[a, g_1]$, for some g_1 in G. Also each element of $[A, G]$ has the form $[a, g_2]$, modulo N, for some g_2 in G. Therefore each element of $[A, G]$ can be written as

$$[a, g_2][a, g_1] = [a, g_1 g_2]$$

as required. Thus we may assume that (4) holds.

By Lemma 2.2, $[A, G, G] \leq A^p$. Therefore $[A, G] \leq Z_2(G) \cap G'$, and so $[A, G]$ is abelian (see [8, 5.1.11 (iii)]). Then, by Lemma 2.3, $[A, G]$ is elementary abelian.

Let $|A| = p^n$ and write A_i for the subgroup of index p^i in A, $1 \leq i \leq n$. By Lemma 2.4, we must have $A \cap [A, G] = A_{n-1}$ of order p. Thus

$$[A, G, G] \leq A^p \cap [A, G] = A_{n-1}.$$

If $[A, G] \leq Z(G)$, then the Theorem will be true. So we may assume that

(5) $$[A, G, G] = A_{n-1} = \langle b \rangle,$$

say. It follows that there are elements g, h in G such that $[a, g, h] = b$. Thus $[a, gh] = [a, h][a, g]b$ and hence $b = [a, g]^{-1}[a, h]^{-1}[a, gh]$. Therefore, by Lemma 2.3, there are elements u, v, w in G such that

(6) $$b = [a, u][a, v][a, w].$$

From (5) we have

(7) $$[a, u]^v = b^i[a, u],$$

some $0 \leq i \leq p - 1$; and of course we have

(8) $$[a, uv] = [a, v][a, u]^v.$$

Since $[A, G]$ is abelian, (6), (7) and (8) give $[a, uv] = [a, v]b^i[a, u] = b^{i+1}[a, w]^{-1}$. Hence $b^{i+1} = [a, uv][a, w]$. But $b \in A^p \leq Z(G)$ and $[a, uv]$ has order at most p and lies in the centre of $\langle a, uv \rangle$, by Lemma 2.3. It follows that uv centralises $[a, w]$ and so $b^{i+1} = [a, wuv]$. If $i \neq p - 1$, then b has the form $[a, g]$, and since the theorem holds for $G/\langle b \rangle$, it will then also hold for G. Therefore $i = p - 1$.

Now (7) becomes

(9) $$[a, u]^v = b^{-1}[a, u].$$

Again $[a, v]$ has order $\leq p$ and is centralised by v. Thus from (8) we obtain

$$[a, uv, v^{-1}] = [a, uv]^{-1}[a, v][a, u] = b^j,$$

https://doi.org/10.1017/S1446788700002810 Published online by Cambridge University Press

$0 \le j \le p-1$, by (5). Therefore, by (6),

$$b = b^j[a, uv][a, w] = b^j[a, uvw],$$

since w commutes with b and $[a, w]$ and hence with $[a, uv]$. Since b cannot have the form $[a, g]$, we must have

$$j = 1 \quad \text{and} \quad [a, uvw] = 1.$$

Finally, since the factors of (6) can be permuted arbitrarily, analogous to (9) we get $[a, u]^w = b^{-1}[a, u]$. Hence

$$[a, u]^{vw} = (b^{-1}[a, u])^w = b^{-2}[a, u].$$

Therefore $1 = [a, uvw] = [a, vw]b^{-2}[a, u]$ and so

$$b^2 = [a, vw][a, u] = [a, uvw] = 1.$$

This contradiction proves the theorem. □

PROOF OF THEOREM 2.1. We have $A = \langle a \rangle$, a cyclic permutable subgroup of the finite p-group G, where p is an odd prime. For each element g in G, it follows from Lemma 2.3 that $\langle [a, g] \rangle$ is normalised by A. Therefore, by Theorem 2.5, A normalises every subgroup of $[A, G]$. Thus $A^G = A[A, G]$ also normalises every subgroup of $[A, G]$ and hence every subgroup of $[A, G]$ is normal. Then $[A, G]$ is a Hamiltonian group of odd order, that is, $[A, G]$ is abelian. Clearly A acts on $[A, G]$ as a group of universal power automorphisms, since $[A, G]$ is abelian, by [3]. □

References

[1] T. E. Berger and F. Gross, 'A universal example of a core-free permutable subgroup', *Rocky Mountain J. Math.* **12** (1982), 345–365.

[2] R. H. Bradway, F. Gross and W. R. Scott, 'The nilpotence class of core-free permutable subgroups', *Rocky Mountain J. Math.* **1** (1971), 375–382.

[3] C. D. H. Cooper, 'Power automorphism of a group', *Math. Z.* **107** (1968), 335–356.

[4] B.Huppert, 'Über das Produkt von paarweise vertauschbaren zyklischen Gruppen', *Math. Z.* **58** (1953), 243–264.

[5] N. Itô and J. Szép, 'Über die Quasinormalteiler von endlichen Gruppen', *Acta Sci. Math. (Szeged)* **23** (1962), 168–170.

[6] R. Maier and P. Schmid, 'The embedding of permutable subgroups in finite groups', *Math. Z.* **131** (1973), 269–272.

[7] O. Ore, 'On the application of structure theory to groups', *Bull. Amer. Math. Soc.* **44** (1938), 801–806.

[8] D. J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Math. 80, 2nd edition (Springer, New York, 1996).
[9] R. Schmidt, *Subgroup lattices of groups* (Walter de Gruyter, Berlin, 1994).
[10] S. E. Stonehewer, 'Permutable subgroups of infinite groups', *Math. Z.* **125** (1972), 1–16.
[11] ———, 'Permutable subgroups of some finite p-groups', *J. Austral. Math. Soc.* **16** (1973), 90–97.
[12] ———, 'Permutable subgroups of some finite permutation groups', *Proc. London Math. Soc. (3)* **28** (1974), 222–236.
[13] J. G. Thompson, 'An example of core-free permutable subgroups of p-groups', *Math. Z.* **96** (1967), 226–227.

Mathematics Department
School of Mathematical Sciences
Canberra ACT 0200
Australia

Mathematics Institute
University of Warwick
Coventry CV4 7AL
UK

https://doi.org/10.1017/S1446788700002810 Published online by Cambridge University Press