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A note on magnitude bounds for the mask coefficients of the interpolatory Dubuc–Deslauriers subdivision scheme

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Abstract

We analyse the mask associated with the $2n$-point interpolatory Dubuc–Deslauriers subdivision scheme $S_{a^{[n]}}$. Sharp bounds are presented for the magnitude of the coefficients $a^{[n]}_{2i-1}$ of the mask. For scales $i \in [1, \sqrt{n}]$ it is shown that $|a^{[n]}_{2i-1}|$ is comparable to i^{-1}, and for larger power scales, exponentially decaying bounds are obtained. Using our bounds, we may precisely analyse the summability of the mask as a function of n by identifying which coefficients of the mask contribute to the essential behaviour in n, recovering and refining the recent result of Deng–Hormann–Zhang that the operator norm of $S_{a^{[n]}}$ on ℓ^∞ grows logarithmically in n.

1. *Introduction*

Subdivision schemes for curves are iterative procedures which start with an initial sequence of vectors, called control points, $v^0 = (v^0_i)_{i\in\mathbb{Z}}$, and via a linear operator S_a recursively generate new sequences of control points, v^k, for each $k \in \mathbb{N}$, through the formula

$$v^{k+1}_j = (S_a v^k)_j = \sum_{i\in\mathbb{Z}} a_{j-Mi} v^k_i.$$

Here, the fixed sequence $a = (a_i)_{i\in\mathbb{Z}}$ is called the mask and $M > 1$ is a fixed dilation parameter. In modern computer aided geometric design and modelling, subdivision is a widely used method for generating curves (and surfaces). The analysis of subdivision schemes has also become an active area of substantial mathematical interest with strong connections to geometric analysis, wavelets and Fourier analysis. Controlling the size of the linear operator S_a is a crucial aspect of this analysis as this determines the convergence of the subdivision procedure.

Binary subdivision schemes correspond to fixing the dilation parameter M as 2. In this case, the operator norm of S_a as a mapping $\ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is equal to

$$\|S_a\|_{\infty\to\infty} = \max\bigg\{\sum_{i\in\mathbb{Z}} |a_{2i}|, \sum_{i\in\mathbb{Z}} |a_{2i+1}|\bigg\}.$$

Here, for $p \in [1, \infty]$, we use $\ell^p(\mathbb{Z})$ for the Banach space of sequences $(v_j)_{j\in\mathbb{Z}}$ for which $\sum_{j\in\mathbb{Z}} |v_j|^p$ is finite, for $1 \leqslant p < \infty$, and $\sup_{j\in\mathbb{Z}} |v_j|$ is finite for $p = \infty$. Also $\|S_a\|_{p\to q}$ will be used to denote the operator norm of S_a as an operator $\ell^p(\mathbb{Z}) \to \ell^q(\mathbb{Z})$.

A particularly important example is the $2n$-point interpolatory Dubuc–Deslauriers scheme, where $n \in \mathbb{N}$ is fixed. It is shown in [3] that the mask can be constructed as the sequence of minimal support satisfying a certain polynomial filling property, and this gives rise to the explicit formula

$$a^{[n]}_{2i-1} = \begin{cases} L_{-i+1}(\frac{1}{2}) & \text{for } i = -n+1, \dots, n \\ 0 & \text{for } i \geqslant n+1 \text{ or } i \leqslant -n \end{cases}$$

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and $a^{[n]}_{2i} = \delta_{i,0}$ for the mask. Here, L_k is the fundamental Lagrange polynomial given by

$$L_k(x) = \prod_{\substack{\ell=-n+1 \\ \ell\neq k}}^{n} \frac{x-\ell}{k-\ell}$$

for $k = -n+1, \ldots, n$, and $\delta_{i,0}$ is equal to one if $i = 0$ and zero otherwise. Note that $S_{a^{[n]}}$ is a symmetric subdivision scheme in the sense that $a^{[n]}_k = a^{[n]}_{-k}$ for each integer k, and so it suffices to consider the sequence $(a^{[n]}_{2i-1})^n_{i=1}$ in what follows.

The primary purpose of this short note is to provide sharp bounds, stated in Theorem 1.1 below, for the magnitude of each coefficient $a^{[n]}_{2i-1}$ of the mask. These bounds easily permit, for example, a very precise analysis of the absolute summability of the mask; see the forthcoming Corollaries 1.2 and 1.3.

We use $A \lesssim B$ and $B \gtrsim A$ to mean that there is an absolute constant C such that $A \leqslant CB$, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

THEOREM 1.1. *Let $n \geqslant 2$. The sequence $(|a^{[n]}_{2i-1}|)^n_{i=1}$ is strictly decreasing and for each $1 \leqslant i \leqslant n-1$ satisfies*

$$|a^{[n]}_{2i-1}| \sim \frac{1}{i}\left(1-\frac{i}{n}\right)^{-1/2}\left(1+\frac{i}{n}\right)^{-(n+i)}\left(1-\frac{i}{n}\right)^{-(n-i)}. \tag{1.1}$$

Consequently,

$$|a^{[n]}_{2i-1}| \sim \frac{1}{i} \quad \text{if } 1 \leqslant i \leqslant n^{1/2}, \tag{1.2}$$

with the upper bound valid for all $1 \leqslant i \leqslant n$. For fixed $\frac{1}{2} < \tau \leqslant 1$, we have the exponentially decaying bounds

$$|a^{[n]}_{2i-1}| \lesssim \frac{1}{n^{\tau-\frac{1}{2}}} \exp(-n^{2\tau-1}) \quad \text{if } n^{\tau} \leqslant i \leqslant n. \tag{1.3}$$

We note that it is easy to extract from our proof of Theorem 1.1 some concrete values for the absolute constants that are implicit in the notation $\lesssim$ and $\sim$ in (1.1)–(1.3).

Immediately from Theorem 1.1 we may obtain the following sharp bounds on the operator norm† $\|S_{a^{[n]}}\|_{p'\to\infty}$. This provides a re-proof and extension of the impressive recent result of Deng *et al.* in [**2**], who considered the important case $p = 1$ and established (1.5) below, in order to disprove the conjecture of Conti *et al.* in [**1**] that $\sup_{n\in\mathbb{N}} \|S_{a^{[n]}}\|_{\infty\to\infty}$ is finite.

COROLLARY 1.2. *For every $p > 1$ there exists $n(p) \in \mathbb{N}$ such that*

$$\|S_{a^{[n]}}\|_{p'\to\infty} \sim \frac{1}{(p-1)^{1/p}} \quad \text{for all } n \geqslant n(p), \tag{1.4}$$

with the upper bound valid for all $n \geqslant 2$, and, corresponding to $p = 1$,

$$\|S_{a^{[n]}}\|_{\infty\to\infty} \sim \log(n) \quad \text{for all } n \geqslant 2. \tag{1.5}$$

Our approach via Theorem 1.1 is substantially more revealing and robust than the approach in [**2**], since the latter relies on some delicate combinatorial identities and a telescoping sum approach which avoids estimating the magnitude of each coefficient of the mask. Our direct approach using Theorem 1.1 allows us to simultaneously prove (1.4) and (1.5), with (1.4) highlighting that $\sup_{n\in\mathbb{N}} \|S_{a^{[n]}}\|_{p'\to\infty}$ is finite for each $p > 1$ with the exact rate of 'blow-up'

†Here, $p' \in [1,\infty]$ is the conjugate exponent to $p \in [1,\infty]$ satisfying $1/p + 1/p' = 1$, with the convention that $1' = \infty$.

as $p \to 1+$; this gives a somewhat different perspective on the falsity of the aforementioned conjecture that $\sup_{n\in\mathbb{N}} \|S_{a^{[n]}}\|_{\infty\to\infty}$ is finite.

Using our quantitative bounds in Theorem 1.1, we may refine the logarithmic growth bounds in (1.5) in the following sense[†].

Corollary 1.3. *Let $n \geqslant 2$. If $\tau \in (0,1]$ then $\sum_{i=1}^{n^\tau} |a_{2i-1}^{[n]}| \gtrsim \tau \log(n)$ and if $\tau \in [\frac{1}{2}, 1]$ then $\sum_{i=n^\tau}^{n} |a_{2i-1}^{[n]}| \lesssim 1$. Consequently, the set of real numbers $\tau \in [0,1]$ for which both*

$$\sum_{i=1}^{n^\tau} |a_{2i-1}^{[n]}| \sim C_\tau \log(n), \tag{1.6}$$

for some positive constant C_τ and all $n \geqslant 2$, and

$$\lim_{n\to\infty} \frac{1}{\log(n)} \sum_{i=n^\tau}^{n} |a_{2i-1}^{[n]}| = 0 \tag{1.7}$$

is equal to the interval $[\frac{1}{2}, 1]$.

Corollary 1.3 says that, for arbitrarily small $\tau > 0$, one only needs to use the first n^τ terms of the mask in order to obtain the essential size of the full sum (that is logarithmic growth in n) and that the sum of the terms $|a_{2i-1}^{[n]}|$ for $\sqrt{n} \leqslant i \leqslant n$ is uniformly bounded in $n \in \mathbb{N}$ (which can be viewed as verifying a weak form of the original conjecture by Conti *et al.*).

Before giving the proofs of the above results in the subsequent section, we remark that except for the bound (1.5) it is not possible to obtain any of the above results from the approach in [**2**] since their analysis only yields estimates on the full sum $\sigma_n = \sum_{i=1}^{n} |a_{2i-1}^{[n]}|$ as a function of n and avoids estimating each coefficient $|a_{2i-1}^{[n]}|$.

2. *Proofs of Theorem* 1.1 *and Corollaries* 1.2 *and* 1.3

It is convenient for us to write $[r] := \{1, 2, \ldots, r_0\}$, for $r \geqslant 1$ and where r_0 is the largest integer less than or equal to r.

The expression for $a_{2i-1}^{[n]}$ that we use is

$$a_{2i-1}^{[n]} = L_{-i+1}\left(\frac{1}{2}\right) = \frac{2(-1)^{i+1}}{\pi(2i-1)} \frac{\Gamma(n+1/2)^2}{(n+i-1)!(n-i)!} \tag{2.1}$$

for $i \in [n]$, and Γ is the usual Gamma function given by $\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\, dt$ for $x > 0$. To see this, we use that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to write

$$\prod_{\substack{\ell=-n+1 \\ \ell\neq -i+1}}^{n} \left(\frac{1}{2} - \ell\right) = \frac{(-1)^n \Gamma(n+1/2)^2}{\pi(i-1/2)}.$$

Remark. Applying the duplication formula $\Gamma(x)\Gamma(x+\frac{1}{2}) = 2^{1-2x}\sqrt{\pi}\Gamma(2x)$ for the Gamma function to (2.1) yields the expression

$$a_{2i-1}^{[n]} = \frac{(-1)^{i+1}}{2^{4n-1}} \frac{n+i}{2i-1} \binom{2n}{n} \binom{2n}{n+i}$$

[†] When we write $\sum_{i=r}^{s}$ and r and s are not integers, we mean $\sum_{i=r_0}^{s_0}$ where r_0 and s_0 are the largest integers less than or equal to r and s respectively

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derived in [3] and used in [2]. However, we will make use of Stirling's approximation

$$\Gamma(x+1) \sim \sqrt{x}\left(\frac{x}{e}\right)^x \tag{2.2}$$

for the Gamma function and $x \geqslant 1$, and thus we find (2.1) more convenient.

Along with Stirling's approximation, the following elementary bounds underpin Theorem 1.1.

LEMMA 2.1. *Let $n \geqslant 2$. For each $i \in [n^{1/2}]$,*

$$\left(1-\frac{i}{n}\right)^i\left(1+\frac{i}{n}\right)^{-i} \gtrsim 1, \tag{2.3}$$

and for each $i \in [n-1]$,

$$\left(1+\frac{i}{n}\right)^{n+i}\left(1-\frac{i}{n}\right)^{n-i} \geqslant \exp\left(\frac{i^2}{n}\right). \tag{2.4}$$

Proof of Lemma 2.1. Since $\log(1+t) \leqslant t$ for all $t \geqslant 0$, for $i \in [n^{1/2}]$ we have

$$\log\left(\frac{1+i/n}{1-i/n}\right)^i = i\log\left(1+\frac{2i}{n-i}\right) \leqslant \frac{2i^2}{n-i} \leqslant \frac{2}{1-n^{-1/2}} \lesssim 1$$

for all $n \geqslant 2$, and (2.3) follows. Taking logarithms and dividing by n, (2.4) is equivalent to $\Psi(t) \geqslant t^2$, where $t = i/n$, and $\Psi(t) = (1+t)\log(1+t) + (1-t)\log(1-t)$. Note that $\Psi(0) = \Psi'(0) = 0$, $\Psi''(0) = 2$ and

$$\Psi'''(t) = \frac{4t}{(1-t^2)^2} \geqslant 0$$

for $t \in (0,1)$. Thus, $\Psi(t) \geqslant t^2$ follows from Taylor's theorem.

Proof of Theorem 1.1. The claimed monotonicity of $(\,|a_{2i-1}^{[n]}|\,)_{i=1}^n$ is straightforward since

$$\frac{|a_{2i+1}^{[n]}|}{|a_{2i-1}^{[n]}|} = \frac{(2i-1)(n-i)}{(2i+1)(n+i)} < 1$$

for all $n \in \mathbb{N}$ and $i \in [n]$. Next, we establish the sharp estimates in (1.1). Using Stirling's approximation (2.2) on $\Gamma(n+\frac{1}{2})$, $(n+i-1)!$ and $(n-i)!$ it follows that

$$\frac{\Gamma(n+1/2)^2}{(n+i-1)!(n-i)!} \sim \left(\frac{n+i-1}{n-i}\right)^{1/2}\frac{(n-1/2)^{2n}}{(n+i-1)^{n+i}(n-i)^{n-i}}.$$

Since $i \in [n-1]$ it follows that

$$\left(\frac{n+i-1}{n-i}\right)^{1/2} \sim \left(\frac{n}{n-i}\right)^{1/2} = \left(1-\frac{i}{n}\right)^{-1/2}$$

and since $(1-1/x)^x \sim 1$ for $x \geqslant 2$, it follows that $(n-\frac{1}{2})^{2n} \sim n^{2n}$ and $(n+i-1)^{n+i} \sim (n+i)^{n+i}$. Consequently,

$$\begin{aligned}\frac{\Gamma(n+1/2)^2}{(n+i-1)!(n-i)!} &\sim \left(1-\frac{i}{n}\right)^{-1/2}\frac{n^{2n}}{(n+i)^{n+i}(n-i)^{n-i}}\\ &= \left(1-\frac{i}{n}\right)^{-1/2}\left(1+\frac{i}{n}\right)^{-(n+i)}\left(1-\frac{i}{n}\right)^{-(n-i)}\end{aligned}$$

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and then (2.1) implies

$$|a_{2i-1}^{[n]}| \sim \frac{1}{i}\left(1-\frac{i}{n}\right)^{-1/2}\left(1+\frac{i}{n}\right)^{-(n+i)}\left(1-\frac{i}{n}\right)^{-(n-i)}.$$

This finishes our proof of (1.1).

We shall now use (1.1) to establish the bounds in (1.2) and (1.3), and begin by showing the upper bound in (1.2) for each $i \in [n]$. For this, observe that $\Phi(t) = (t+1/2n)\log(t)$ is a convex function for $t \geqslant 1/2n$ which implies

$$\Phi\left(1+\frac{i}{n}\right)-\Phi\left(1+\frac{1}{n}\right) \geqslant \Phi\left(1-\frac{1}{n}\right)-\Phi\left(1-\frac{i}{n}\right).$$

Therefore

$$\begin{aligned}\left(1-\frac{i}{n}\right)^{-1/2}\left(1+\frac{i}{n}\right)^{-(n+i)}\left(1-\frac{i}{n}\right)^{-(n-i)} &\leqslant \frac{(1+i/n)^{1/2}}{(1+1/n)^{n+3/2}(1-1/n)^{n-1/2}} \\ &\lesssim 1,\end{aligned}$$

where the second upper bound follows since $i \in [n]$ and a further application of the estimate $(1-1/x)^x \sim 1$ for $x \geqslant 2$. From (1.1) we now obtain $|a_{2i-1}^{[n]}| \lesssim 1/i$ for each $i \in [n]$. Turning to the lower bound in (1.2), which is claimed to be true in restricted range $i \in [n^{1/2}]$, we clearly have

$$\left(1-\frac{i}{n}\right)^{-1/2}\left(1+\frac{i}{n}\right)^{-n}\left(1-\frac{i}{n}\right)^{-n} = \left(1-\frac{i}{n}\right)^{-1/2}\left(1-\frac{i^2}{n^2}\right)^{-n} \geqslant 1$$

and the claimed lower bound $|a_{2i-1}^{[n]}| \gtrsim 1/i$ now follows from (1.1) and (2.3).

Finally we must show (1.3), and for this we first fix $\tau \in (\frac{1}{2}, 1)$. For $n^\tau \leqslant i \leqslant n-1$ we use (1.1) and (2.4) to obtain

$$|a_{2i-1}^{[n]}| \lesssim \frac{1}{i}\left(1-\frac{i}{n}\right)^{-1/2}\exp\left(-\frac{i^2}{n}\right) \leqslant \frac{1}{n^{\tau-1/2}}\exp(-n^{2\tau-1})$$

as required. The same bound is also true in the missing cases with $i = n$ or $\tau = 1$, and this can be shown by directly computing $|a_{2n-1}^{[n]}| \sim n^{-1/2}4^{-n}$ or by using that $|a_{2i-1}^{[n]}|$ is strictly decreasing in i.

Proof of Corollary 1.2. Firstly, we have $\|S_{a^{[n]}}\|_{p'\to\infty} \sim (\sum_{i=1}^n |a_{2i-1}^{[n]}|^p)^{1/p}$. Also, trivially $\sum_{i=1}^N (1/i) \sim \log(N)$ for $N \geqslant 2$ and for $p > 1$ we have

$$\left(\sum_{i=1}^N \frac{1}{i^p}\right)^{1/p} \sim \frac{1}{(p-1)^{1/p}},$$

where the upper bound is valid for all $N \geqslant 2$ and the lower bound for $N \geqslant 2^{1/(p-1)}$. Hence, with $N = \sqrt{n}$, the lower bounds in (1.4) and (1.5) follow immediately from (1.2). Moreover, we have $|a_{2i-1}^{[n]}| \lesssim 1/i$ for all $i \in [n]$ so, with $N = n$, we get upper bounds in (1.4) and (1.5).

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Proof of Corollary 1.3. For the first claim, we let $\tau^* = \min\{\tau, \frac{1}{2}\}$ and use (1.2) to get

$$\sum_{i=1}^{n^\tau} |a_{2i-1}^{[n]}| \geqslant \sum_{i=1}^{n^{\tau^*}} |a_{2i-1}^{[n]}| \gtrsim \sum_{i=1}^{n^{\tau^*}} \frac{1}{i} \sim \tau^* \log(n) \sim \tau \log(n).$$

The second claim clearly follows if we show that

$$\sum_{i=n^{1/2}}^{n} |a_{2i-1}^{[n]}| \lesssim 1. \tag{2.5}$$

For this, we fix any $\sigma \in (\frac{1}{2}, 1)$, say $\sigma = \frac{2}{3}$, and split the sum into $n^{1/2} \leqslant i \leqslant n^\sigma$ and $n^\sigma \leqslant i \leqslant n$. Using (1.1) and (2.4) we have the upper bound

$$|a_{2i-1}^{[n]}| \lesssim \frac{1}{i} \exp\left(-\frac{i^2}{n}\right)$$

for $n^{1/2} \leqslant i \leqslant n^\sigma$. Also

$$\sum_{i=n^{1/2}}^{n^\sigma} \frac{1}{i} \exp\left(-\frac{i^2}{n}\right) \sim \int_{n^{1/2}}^{n^\sigma} \exp\left(-\frac{t^2}{n}\right) \frac{dt}{t} = \int_{1}^{n^{\sigma-1/2}} \exp(-r^2) \frac{dr}{r},$$

using that $t^{-1}\exp(-t^2/n)$ is decreasing in t, and the change of variables $r = t/\sqrt{n}$ in the integral. The function $r \mapsto r^{-1}\exp(-r^2)$ is Lebesgue integrable on $(1, \infty)$ so that

$$\int_{1}^{n^{\sigma-1/2}} \exp(-r^2) \frac{dr}{r} \lesssim 1$$

uniformly in $n \geqslant 2$, and therefore

$$\sum_{i=n^{1/2}}^{n^\sigma} |a_{2i-1}^{[n]}| \lesssim 1.$$

To deal with the contribution from $n^\sigma \leqslant i \leqslant n$ we use (1.3) to obtain

$$\sum_{i=n^\sigma}^{n} |a_{2i-1}^{[n]}| \lesssim \sum_{i=n^\sigma}^{n} n^{1/2-\sigma} \exp(-n^{2\sigma-1}) \leqslant n^{3/2-\sigma} \exp(-n^{2\sigma-1}) \lesssim 1$$

uniformly in $n \geqslant 2$, and hence we have shown (2.5).

It follows from the above argument that (1.6) and (1.7) hold with $\tau = \frac{1}{2}$. Thus, the remaining claim in Corollary 1.3 follows by noting that no smaller value of τ has this property because (1.2) implies

$$\frac{1}{\log(n)} \sum_{i=n^\tau}^{n} |a_{2i-1}^{[n]}| \gtrsim \frac{1}{\log(n)} \sum_{i=n^\tau}^{n^{1/2}} \frac{1}{i} \geqslant \frac{1}{\log(n)} \int_{n^\tau}^{n^{1/2}} \frac{dt}{t} = \frac{1}{2} - \tau.$$

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