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NILPOTENT SUBSETS OF NEAR-RINGS WITH MINIMAL CONDITION

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Throughout this paper all near-rings will be zero-symmetric and left distributive. A near-ring with minimal condition on right N-subgroups will be called an $\mathcal{M}$ near-ring. It is well known (see (**1**), 3.40, p. 90)) that a nil right N-subgroup of an $\mathcal{M}$ near-ring is nilpotent. However, in a deeper study of $\mathcal{M}$ near-rings a stronger result than this is sometimes required (**2**, p. 77).

Let N be an $\mathcal{M}$ near-ring and M_i, $i \in I$, a collection of nil (hence nilpotent) right N-subgroups of N. Let $H = \bigcup_{i \in I} M_i$. Then H may no longer be a right N-subgroup, although, $HN \subseteqq H$. A natural question to ask is whether H is nilpotent. This question can be answered in the affirmative (**2**, p. 77).

If N is a near-ring, then a subset H of N will be called a *right N-subset*, if $HN \subseteqq H$. Left and two-sided N-subsets are defined in an analogous way. We prove the following theorem:

Theorem. *If N is an $\mathcal{M}$ near-ring and H a nil right N-subset of N, then H is nilpotent.*

Proof. Set $L_1 = HN$, and $L_{i+1} = L_i^2$, for $i = 1, 2, \ldots$. We shall show the subsets L_i, $i = 1, 2, \ldots$, have the following properties:

(a) each L_i is a right N-subset of N;
(b) $L_{i+1} \subseteqq L_i$, for $i = 1, 2, \ldots$;
(c) each L_i is a union of nil right N-subgroups of N; and
(d) if $NL_j = \{0\}$, for some positive integer j then H is nilpotent.

(a) We note that L_1 is a right N-subset of N. Suppose it has been shown that L_k (k is a positive integer) is a right N-subset of N. Clearly

$$L_{k+1}N = L_k^2 N \subseteqq L_k^2 = L_{k+1}$$

and L_{k+1} is a right N-subset of N. Thus (a) follows.

(b) For $i = 1, 2, \ldots$, $L_i^2 \subseteqq L_i N$, and, by (a) $L_i N \subseteqq L_i$. Thus $L_{i+1} \subseteqq L_i$, for $i = 1, 2, \ldots$, and (b) follows.

(c) Since by (b) we have each L_i is contained in L_1 and L_1 is contained in H it follows that each L_i, $i=1,2,\ldots$, is nil. It remains to prove that each L_i is a union of N-subgroups. Clearly

$$L_1 = HN = \bigcup_{\alpha \in H} \alpha N$$

and (c) holds for $i=1$. Suppose for some positive integer k, L_k is a union $\bigcup_{j \in J} M_j$ of N-subgroups. It follows that

$$L_{k+1} = L_k^2 = \bigcup_{\lambda \in L_k} \bigcup_{j \in J} \lambda M_j$$

and (c) holds.

(d) Now if $NL_j = \{0\}$, then $L_j^2 = L_{j+1} = \{0\}$. Also $H^2 \subseteqq HN = L_1$, $(H^2)^2 \subseteqq L_2$, etc. Thus if $n = 2^{j+2}$, $H^n \subseteqq L_{j+1} = \{0\}$, and $H^n = \{0\}$. Therefore (d) holds.

We now proceed with the proof of the theorem. By (d) we may assume that $NL_i \neq \{0\}$, for $i=1,2,\ldots$. Let M_i, $i=1,2,\ldots$, be right N-subgroups of N minimal for the property that $M_iL_i \neq \{0\}$. Let $T(L_i)$ denote the ideal of N generated by $L_i = 1, 2, \ldots$. By (b), $L_{i+1} \subseteqq L_i$, and $T(L_i) \geqq T(L_{i+1})$. If for all i, $M_iL_{i+1} = \{0\}$, then $L_{i+1} \leqq (0 : M_i)$ and $T(L_{i+1}) \leqq (0 : M_i)$. But since $M_iL_i \neq \{0\}$, $M_iT(L_i) \neq \{0\}$, and $T(L_i) \not\leqq (0 : M_i)$. Thus $T(L_i) > T(L_{i+1})$, for $i = 1, 2, \ldots$. This contradicts the minimal condition. Hence we may assume that $M_jL_{j+1} \neq \{0\}$, for some positive integer j. Now L_j is a union $\bigcup_{k \in K} P_k$, of nil right N-subgroups P_k, $k \in K$, of N by (c). Thus

$$M_jL_j = \bigcup_{\alpha \in M_j} \bigcup_{k \in K} \alpha P_k.$$

Since $M_jL_{j+1} \neq \{0\}$ we have $M_jL_j^2 \neq \{0\}$ and thus $\alpha P_kL_j \neq \{0\}$ for some α in M_j and k in K. But $\alpha P_k \leqq M_j$ and, by the minimality of M_j, $\alpha P_k = M_j$. Thus $\alpha\beta = \alpha$ where β is in P_k. Hence $\alpha\beta^m = \alpha$, for $m = 1, 2, \ldots$. Since β is in P_k which is nil $\alpha = 0$ and $M_j = \{0\}$. This contradicts the fact that $M_jL_j \neq \{0\}$. The proof of the theorem is complete.

Corollary. *A nil left N-subset of an $\mathcal{M}$ near-ring N is nilpotent.*

Proof. Let P be a nil left N-subset of N. It is easily seen that PN is a nil right N-subset of N. By the above theorem $(PN)^k = \{0\}$, for some positive integer k. Since $P \subseteqq N$, $P^{2k} = \{0\}$, and the corollary holds.

Corollary. *Any union of nilpotent right N-subgroups of an $\mathcal{M}$ near-ring N is nilpotent.*

Proof. A union of nilpotent right N-subgroups of N forms a nil right N-subset and the above theorem is applicable.

If N is an $\mathcal{M}$ near-ring, then the union H of all nilpotent right N-subgroups of N is a two-sided N-subset of N. This two-sided N-subset is, by the second corollary, nilpotent. The subset H of N has a further property, the proof of which is omitted as it presents certain difficulties. This property is that the right ideal $R(H)$ of N generated by H is simply the radical $(J_2(N))$.

REFERENCES

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