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ANALYTIC TOEPLITZ AND COMPOSITION OPERATORS

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1. Introduction. This paper is a continuation of [**1**] where we began the study of intertwining analytic Toeplitz operators. Recall that X *intertwines* two operators A and B if $XA = BX$. Let H^2 be the Hilbert space of analytic functions in the open unit disk D for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in the L^2 norm, and H^∞ be the set of bounded functions in H^2. For $\varphi \in H^\infty$, T_φ(or $T_{\varphi(z)}$) is the *analytic Toeplitz operator* defined on H^2 by the relation $(T_\varphi f)(z) = \varphi(z)f(z)$. For $\varphi \in H^\infty$, we shall denote $\{\varphi(z): |z| < 1\}$ by Range (φ) or $\varphi(D)$. Then $\sigma_p(T_\varphi^*) \supseteq \bar{\varphi}(D)$ where $\bar{\varphi}(z) = \overline{\varphi(\bar{z})}$ and $\sigma(T_\varphi) =$ Closure$(\varphi(D))$ [**1**]. If $\varphi \in H^\infty$ maps D into D, then we define the *composition operator* C_φ on H^2 by the relation $(C_\varphi f)(z) = f(\varphi(z))$. J. Ryff has shown [**11**, Theorem 1] that C_φ is a bounded linear operator on H^2. In § 2 we investigate intertwining operators between analytic Toeplitz operators using composition operators, and in § 3 we study a special class of composition operators.

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2. Intertwining analytic Toeplitz operators.

THEOREM 1 (see [**1**]). *Let $\varphi, \psi \in H^\infty$. If $\psi(D) \not\subseteq \sigma(T_\varphi)$, then the only bounded linear operator X satisfying $XT_\varphi = T_\psi X$* is $X = 0$.

COROLLARY 1. *If $\varphi, \psi \in H^\infty$ are such that there exists $X \neq 0$, $Y \neq 0$ satisfying $XT_\varphi = T_\psi X$ and $T_\varphi Y = YT_\psi$, then $\sigma(T_\varphi) = \sigma(T_\psi)$.*

Proof. Applying Theorem 1 we see that $\psi(D) \subseteq \sigma(T_\varphi)$ and $\varphi(D) \subseteq \sigma(T_\psi)$. Since $\sigma(T_\varphi) =$ Closure$(\varphi(D))$, $\sigma(T_\varphi) = \sigma(T_\psi)$.

PROPOSITION 1. *Let $\varphi, \psi \in H^\infty$. If there exists an analytic function ω mapping D into D such that $\varphi(\omega(z)) = \psi(z)$, then there exists a nonzero X such that $XT_\varphi = T_\psi X$.*

Proof. Since C_ω is clearly nonzero and since for $f \in H^2$

$$((C_\omega T_\varphi)f)(z) = \varphi(\omega(z))f(\omega(z)) = \psi(z)f(\omega(z)) = (T_\psi C_\omega f)(z),$$

we have that

$$C_\omega T_\varphi = T_\psi C_\omega.$$

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THEOREM 2. *Let φ, $\psi \in H^\infty$, φ univalent in D. Then $\psi(D) \not\subseteq \varphi(D)$ if and only if $XT_\varphi = T_\psi X$ implies $X = 0$. In addition, $\bar{\varphi}(D) = \sigma_p(T_\varphi^*)$.*

Proof. Suppose $\psi(D) \not\subseteq \varphi(D)$.

Case 1. ψ is constant, $\psi(z) = \lambda$. Then either $\lambda \notin \sigma(T_\varphi)$ in which case $X = 0$ by Theorem 1, or $\lambda \in \sigma(T_\varphi)\backslash\varphi(D)$. Suppose X satisfies $XT_\varphi = T_\psi X = \lambda X$. Then $(T_\varphi^* - \lambda^*)X^* = 0$, so that Range $X^* \subseteq \text{Null}(R_\varphi^* - \lambda^*) =$ Range $(T_\varphi - \lambda)^\perp$. Since $\lambda \notin \varphi(D)$, the univalent function $\varphi - \lambda$ never vanishes in D. Hence $\varphi - \lambda$ contains no Blaschke products, and by Theorem 3.17 in [**4**] (see also [**9**]) $\varphi - \lambda$ contains no singular inner factor. Thus the decomposition of H^2 functions into the product of an inner and an outer function [**7**, p. 67] implies that $\varphi - \lambda$ must be outer. But if $\varphi - \lambda$ is outer, then Range $(T_\varphi - \lambda)$ is dense in H^2 [**7**, p. 101], so that Range $X^* = \{0\}$. Thus $X = 0$. This also establishes that $\sigma_p(T_\varphi^*) = \bar{\varphi}(D)$.

Case 2. ψ is not constant. Now $N = \psi(D) \cap \mathbf{C}\backslash\varphi(D)$ is nonempty by hypothesis. Since φ is a univalent analytic function, $\varphi(D)$ is an open simply connected set, hence $\mathbf{C}\backslash\varphi(D)$ contains no isolated points. Since ψ is nonconstant, $\psi(D)$ is an open set. Thus N is the nonempty intersection of an open set and a closed set containing no isolated points, and hence N must be uncountable. The proof of Theorem 1 then implies $X = 0$.

Suppose $\psi(D) \subseteq \varphi(D)$. Since φ is univalent, $F(z) = \varphi^{-1}(\psi(z))$ is an analytic function mapping D into D such that $\varphi(F(z)) = \psi(z)$. Hence Proposition 1 implies there exists an $X \neq 0$ such that $XT_\varphi = T_\psi X$.

PROPOSITION 2. *Let $\varphi, \psi \in H^\infty$ map D into D. If $\overline{C_\varphi H^2}$ reduces T_φ and if there exists $K > 0$ such that*

$$(*) \qquad \|C_\psi g\| \leqq K\|C_\varphi g\| \text{ for all } g \in H^2,$$

then there exists a bounded $X \neq 0$ such that $XT_\varphi = T_\psi X$. (We remark that (*) is equivalent to the existence of $Y \in \mathscr{B}(H)$ satisfying $YC_\varphi = C_\psi$ and to $C_\psi^*H^2 \subseteq C_\varphi^*H^2$ [**2**].)

Proof. Write $H^2 = \overline{C_\varphi H^2} \oplus (C_\varphi H^2)^\perp$ and define X on $C_\varphi H^2 \oplus (C_\varphi H^2)^\perp$ by

$$X(C_\varphi g) = C_\psi g \text{ for } g \in H^2$$
$$Xf = 0 \qquad \text{for } f \perp C_\varphi H^2.$$

Then X is well defined and (*) implies that X is bounded, so we can continuously extend it to all of H^2. Also

$$(XT_\varphi)f = XT_\varphi(C_\varphi g \oplus h) = X(\varphi C_\varphi g \oplus \varphi h)$$
$$= X(\varphi C_\varphi g) = \psi C_\psi g$$

and

$$(T_\psi X)f = T_\psi X(C_\varphi g \oplus h) = T_\psi X C_\varphi g$$
$$= T_\psi C_\psi g = \psi C_\psi g.$$

Hence $XT_\varphi = T_\psi X$ on $C_\varphi H^2 \oplus (C_\varphi H^2)^\perp$ and thus on H^2.

Remarks. 1. There is no loss of generality in assuming φ, ψ map D into D, since $\tilde{\varphi} = \varphi/2M$ and $\tilde{\psi} = \psi/2M$, where $M = \max\{||\varphi||_\infty, ||\psi||_\infty\}$, map D into D, and $XT_{\tilde{\varphi}} = T_{\tilde{\psi}}X$ if and only if $XT_\varphi = T_\psi X$.

2. $\overline{C_\varphi H^2}$ is always invariant for T_φ, since $T_\varphi C_\varphi = C_\varphi T_z$. However, $\overline{C_\varphi H^2}$ need not always reduce T_φ (example: if $\varphi(z) = \frac{1}{2}z^2 + \frac{1}{2}z^3$ then $e_1(z) = z \in$ Null $(C_\varphi^*) = (C_\varphi H^2)^\perp$ but $C_\varphi^* T_\varphi e_1 = \frac{1}{2}e_1 \neq 0$).

3. Nevertheless there are examples where $\overline{C_\varphi H^2}$ reduces T_φ. If $C_\varphi H^2$ is dense, then $\overline{C_\varphi H^2}$ trivially reduces T_φ. If φ is an inner function, then $\overline{C_\varphi H^2}$ reduces T_φ since, in this case, $T_\varphi^* C_\varphi = C_\varphi(T_z^* + \bar{\varphi}(0)E)$ where $(Ef)(z) = f(0)$. Also, if ω is an inner function and $C_\psi H^2$ is dense in H^2, then $\overline{C_\varphi H^2}$ reduces T_φ for $\varphi(z) = \psi(\omega(z))$.

COROLLARY 2. *Let φ, $\psi \in H^\infty$, φ an inner function. Then $\bar{\psi}(D) \not\subseteq \sigma_p(T_\varphi^*)$ if and only if $XT_\varphi = T_\psi X$ implies $X = 0$.*

Proof. If φ is constant the statement is clear, so we assume φ is nonconstant. Hence $\sigma_p(T_\varphi^*) = D$[**5**, p. 230].

Suppose $\bar{\psi}(D) \subseteq D$. By Remark 3, $\overline{C_\varphi H^2}$ reduces T_φ, and by Theorem 1 in [**10**], C_φ is bounded below, hence Proposition 2 implies there exists $X \neq 0$ such that $XT_\varphi = T_\psi X$. An alternative proof is to observe that there exists $Y \neq 0$ such that $YT_\varphi = T_z Y$, since T_φ and T_z are both isometries. Hence $X = C_\psi Y \neq 0$ satisfies $XT_\varphi = T_\psi X$.

Suppose $\bar{\psi}(D) \not\subseteq D$. The result then follows from Corollary 1 in [**1**] with (i) replaced by

(i)′ $$\text{Interior (Closure}(\sigma_p(T_\varphi^*))) = \sigma_p(T_\varphi^*).$$

In [**1**] we conjectured that $\bar{\psi}(D) \not\subseteq \sigma_p(T_\varphi^*)$ is necessary and sufficient for $XT_\varphi = T_\psi X$ to imply $X = 0$. Theorem 2 and Corollary 2 establish this conjecture if φ is univalent or inner. In case φ is a polynomial, $\bar{\varphi}(D) = \sigma_p(T_\varphi^*)$ (see [**3**]). Since it can be shown that Interior (Closure$(\varphi(D))) = \varphi(D)$, Corollary 1 in [**1**] implies the sufficiency of our conjecture in case φ is a polynomial.

3. Composition operators. In this section we study the special class of composition operators C_φ of the form $\varphi(z) = \alpha + \beta z$, that is, $|\alpha| < 1$, $|\alpha| + |\beta| \leqq 1$. E. Nordgren [**10**] has studied C_φ when φ is an inner function, while H. Schwartz [**12**] has obtained numerous results concerning composition operators.

THEOREM 3. (i) *If $|\beta| = 1$, then $C_{\alpha+\beta z}$ is a unitary operator whose spectrum is the closure of the set $\{1, \beta, \beta^2, \ldots\}$.*

(ii) *If $|\alpha| + |\beta| < 1$, then $C_{\alpha+\beta z}$ is a compact operator whose spectrum is the closure of $\{1, \beta, \beta^2, \ldots\}$.*

(iii) *If $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, β not positive, then $C_{\alpha+\beta z}$ is a noncompact operator, whose square is compact, and whose spectrum is the closure of $\{1, \beta, \beta^2, \ldots\}$.*

(iv) *If* $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, β *positive, then* $C_{\alpha+\beta z}$ *is a cosubnormal operator whose spectrum is the closed disk of radius* $\beta^{-\frac{1}{2}}$ *centered at the origin.*

Proof. Before beginning the proof, notice that under the natural identification between H^2 and $l_+{}^2$ (i.e., $\sum_{n=0}^{\infty} a_n z^n \rightarrow \{a_n\}_0^{\infty}$), $C_{\alpha+\beta z}$ has a matrix representation on $l_+{}^2$ as

$$C_{\alpha+\beta z}\cdot \sim \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \ldots \\ & \beta & 2\alpha\beta & 3\alpha^2\beta \ldots \\ & & \beta^2 & 3\alpha\beta^2 \ldots \\ & 0 & & \beta^3 \ddots \end{bmatrix}$$

that is, $C_{\alpha+\beta z} \sim (a_{ij})$ where $a_{ij} = 0$ if $j < i$ and $a_{ij} = \binom{j}{i}\alpha^{j-i}\beta^i$ if $j \geqq i$.

Proof of 3(i). Since $|\beta| = 1$, α equals 0. Hence $C_{\alpha+\beta z}$ corresponds to a diagonal matrix all of whose entries have modulus 1. Thus $C_{\alpha+\beta z}$ is unitary with spectrum = Closure(Diagonal) = Closure$(1, \beta, \beta^2, \ldots)$.

Proof of 3(ii). Since $|\alpha| + |\beta| = r < 1$, we have $|\alpha + \beta z| \leqq r < 1$ for $|z| \leqq 1$. Hence Theorem 5.2 in [**12**] implies that $C_{\alpha+\beta z}$ is compact with spectrum = Closure$\{1, \beta, \beta^2, \ldots\}$, and that if $\beta \neq 0$ then each β^n is a simple eigenvalue. An alternative proof is to first notice that $\sigma_p(C_{\alpha+\beta z}) \supseteq \{1, \beta, \beta^2, \ldots\}$. In fact, if $f_n(z) = (z - \alpha/(1-\beta))^n$ then $C_{\alpha+\beta z} f_n = \beta^n f_n$. Next notice that the matrix (a_{ij}) of $C_{\alpha+\beta z}$ satisfies $\sum_{i,j=0}^{\infty} |a_{ij}| = 1/(1-r) < \infty$, so that $C_{\alpha+\beta z}$ is compact. From this it is not hard to conclude that spectrum = Closure $\{1, \beta, \beta^2, \ldots\}$ and that each eigenvalue is simple if $\beta \neq 0$.

Proof of 3(iii). Since $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, and β is not positive, we have $|1 + \beta| < 1 + |\beta|$ and hence $|\alpha(1+\beta)| + |\beta^2| < 1$. Because $C_{\alpha+\beta z}{}^2 = C_{\alpha(1+\beta)+\beta^2 z}$, 3(ii) and the spectral mapping theorem [**5**, p. 38] imply that $C_{\alpha+\beta z}{}^2$ is compact and that

$$(\sigma(C_{\alpha+\beta z}))^2 = \sigma(C_{\alpha+\beta z}{}^2) = \sigma(C_{\alpha(1+\beta)+\beta^2 z}) = \text{Closure}\{1, \beta^2, \beta^4, \ldots\}.$$

Hence

$$\sigma(C_{\alpha+\beta z}) \subseteq \text{Closure}\{\pm 1, \pm \beta, \pm \beta^2, \ldots\}.$$

As usual, $\sigma_p(C_{\alpha+\beta z}) \supseteq \{1, \beta, \beta^2, \ldots\}$. Recall that β^{2n} is a simple eigenvalue for $C_{\alpha+\beta z}{}^2$ with eigenvector $f_n(z) \equiv (z - \alpha(1+\beta)/(1-\beta^2))^n = (z - \alpha/(1-\beta))^n$, which is also the eigenvector for $C_{\alpha+\beta z}$ corresponding to the eigenvalue β^n. Hence

$$\mathcal{N} \equiv \text{Null}(C_{\alpha+\beta z}{}^2 - \beta^{2n}) = \text{Null}(C_{\alpha+\beta z} - \beta^n),$$

and

$$\text{Null}(C_{\alpha+\beta z} + \beta^n) = \{0\},$$

since $\text{Null}(C_{\alpha+\beta z} + \beta^n) \subseteq \text{Null}(C_{\alpha+\beta z}{}^2 - \beta^{2n})$ and $\beta \neq 0$. We need to show that $-\beta^n \notin \sigma(C_{\alpha+\beta z})$ for $n = 0, 1, 2, \ldots$. If $-\beta^n \in \sigma(C_{\alpha+\beta z})$, then $-\beta^n \in \partial\sigma(C_{\alpha+\beta z}) \subset \sigma_a(C_{\alpha+\beta z})$ [**5**, p. 39]. Hence there exist y_m, $||y_m|| = 1$ such that

$$||(C_{\alpha+\beta z} + \beta^n)y_m|| \rightarrow 0 \text{ as } m \rightarrow \infty$$

so

$$||(C_{\alpha+\beta z}{}^2 - \beta^{2n})y_m|| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Let $y_m = y'_m \oplus y''_m \in \mathcal{N} \oplus \mathcal{N}^\perp$. Since $C_{\alpha+\beta z}{}^2$ is compact and $\beta \neq 0$, $C_{\alpha+\beta z}{}^2 - \beta^{2n}$ is bounded below on $\mathcal{N}^\perp$ [**5**, p. 91]. Hence $y_m'' \to 0$. Because $1 = ||y_m||^2 + ||y_m||^2$, there is a subsequence $\{y'_{mk}\}$ that converges weakly to g_n where $g_n \in \mathcal{N}$, $||g_n|| = 1$. Hence

$$(C_{\alpha+\beta z} + \beta^n) y_{mk} \to (C_{\alpha+\beta z} + \beta^n) g_n = 0,$$

which contradicts $\mathrm{Null}(C_{\alpha+\beta z} + \beta^n) = \{0\}$. Thus $\sigma(C_{\alpha+\beta z}) = \mathrm{Closure}\{1, \beta, \beta^2, \ldots\}$.

In order to see that $C_{\alpha+\beta z}$ is not compact, we employ the argument on page 23 of [**12**]. By hypothesis $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, so that $\alpha = \rho e^{i\theta}$ and $\beta = (1-\rho)e^{i\eta}$ where $0 < \rho < 1$. If we define $f_n(z) = 1/\sqrt{n}\ (e^{i\theta} - z + z/n)^{-1}$ then $f_n \in H^2$, $\frac{1}{2} \leqq ||f_n||^2 \leqq 1$, and $f_n \to 0$ uniformly on compact subsets of D. Also $||C_{\alpha+\beta z} f_n||^2 \geqq ||f_n||^2 \geqq \frac{1}{2}$. Theorem 2.5 in [**12**] then implies that $C_{\alpha+\beta z}$ is not compact.

Proof of 3(iv). We first consider the case when α is positive. Then $\alpha + \beta = 1$. Define C_0^* to be that operator on H^2 whose matrix representation under the natural identification between H^2 and l_+^2 is

$$C_0^* \sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \cdots \\ & \frac{1}{2} & \frac{1}{3} \cdots \\ \text{\Large 0} & & \frac{1}{3} \cdots \\ & & \ddots \end{bmatrix}$$

That is, $C_0^* \sim (b_{ij})$ where $b_{ij} = 0$ if $j < i$ and $b_{ij} = 1/j$ if $j \geqq i$. Then C_0^* is a bounded linear operator on H^2 and a simple calculation shows that $C_{\alpha+\beta z}$ commutes with C_0^*. The operator C_0 on l_+^2 is called the Cesaro operator [**6**, p. 96]. A theorem of Shields and Wallen [**13**] then implies that there is a bounded analytic function F on $\{z: |1 - z| < 1\}$ such that $C_{\alpha+\beta z} = F(C_0^*)$, $\sigma(C_{\alpha+\beta z}) = \mathrm{Closure}\{F(z): |1 - z| < 1\}$ and $||C_{\alpha+\beta z}|| = \sup\{|\lambda|: \lambda \in \sigma(C_{\alpha+\beta z})\}$. Since we obviously must have $F(1/n) = \beta^{n-1}$ for $n = 1, 2, \ldots$; $F(z) = \beta^{(1/z)-1}$ is the required function. Hence

$$\begin{aligned} \sigma(C_{\alpha+\beta z}) &= \mathrm{Closure}\{F(z): |1 - z| < 1\} \\ &= \mathrm{Closure}\{\beta^{(1/z)-1}: |1 - z| < 1\} \\ &= \{\lambda: |\lambda| \leqq \beta^{-\frac{1}{2}}\}. \end{aligned}$$

and

$$||C_{\alpha+\beta z}|| = \sup\{|\lambda|: \lambda \in \sigma(C_{\alpha+\beta z})\} = \beta^{-\frac{1}{2}}.$$

A theorem of Kriete and Trutt [**8**] states that C_0 is a subnormal operator with a cyclic vector and hence every operator commuting with C_0 is subnormal [**14**]. Thus $C_{\alpha+\beta z}$ is cosubnormal. We remark that $C_{\alpha+\beta z}$ is the adjoint of the Euler summability matrix of order $\alpha/(1 - \alpha)$ [**6**, p. 178]. Thus the spectrum of the Euler matrix of order $\alpha/(1 - \alpha)$ on l^2 is $\{z: |z| \leqq (1 - \alpha)^{-\frac{1}{2}}\}$.

We next consider the case when α is not positive. Then $\alpha = |\alpha|e^{i\theta}$ and $|\alpha| + \beta = 1$. However, it is easily checked using the unitary operator $C_{e^{i\theta}z}$ that $C_{\alpha+\beta z}$ is unitarily equivalent to $C_{|\alpha|+\beta z}$. Hence $C_{\alpha+\beta z}$ is again a cosubnormal operator whose spectrum is the closed disk of radius $\beta^{-\frac{1}{2}}$ centered at the origin.

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An alternative proof for 3(iv) would be to first try and prove that $||C_{\alpha+\beta z}|| = \beta^{-(\frac{1}{2})}$ and then notice that $(1 - z)^{(1/\lambda)-1}$ is an eigenvector for $C_{\alpha+\beta z}$ corresponding to the eigenvalue $\beta^{(1/\lambda)-1}$ where $|1 - \lambda| < 1$.

Notice that if $\beta \neq 1$ then $\alpha/(1 - \beta)$ is the only fixed point of $\varphi(z) = \alpha + \beta z$. We remark that the real distinction between 3(iv) and 3(i–iii) is that in 3(iv) the fixed point of φ is on the unit circle, while in 3(i–iii) the fixed point of φ is in D.

Theorem 3(iii) can be generalized in the following manner. If $\varphi \in H^\infty$ maps D into D, define $\varphi_n \in H^\infty$ inductively by $\varphi_1(z) = \varphi(z)$, $\varphi_n(z) = \varphi_{n-1}(\varphi(z))$.

PROPOSITION 3. *Suppose that $\varphi \in H^\infty$ maps D into D and that for some integer n there is an r, $0 < r < 1$, such that $|\varphi_n(z)| \leqq r < 1$ for all $|z| < 1$. Then C_φ^n is compact. Furthermore, if φ has a fixed point z_0 in D and $\beta = \varphi'(z_0)$, then* $\sigma(C_\varphi) = \text{Closure}\{1, \beta, \beta^2, \ldots\}$.

Proof. By Theorem 5.2 in [**12**], $C_\varphi^n = C_{\varphi_n}$ is compact. The last statement follows as in Theorem 3(iii).

Remarks. 4. H. Schwartz in [**12**] proves that if $\varphi \in H^\infty$ maps D into D and has a fixed point z_0 in D and if $\varphi'(z_0) \neq 0$ then $\{\varphi'(z_0)^n\}_{n=0}^\infty$ are eigenvalues for C_φ and these are the only eigenvalues. In Theorem 3(iv) and Theorems 5 and 6 in [**10**] the eigenvalues are related to the fixed points of φ on the unit circle. Is there some general connection between fixed points of φ on the unit circle and eigenvalues for C_φ?

5. Using Schur's test [**5**, p. 22] one can show that $||C_{\alpha+\beta z}|| \leqq (1 - |\alpha|)^{-\frac{1}{2}}$. Is this an equality?

6. Theorem 3(iii) yields perhaps the worst possible example of a noncompact operator T whose square is compact, since T and T^2 possess common simple eigenvectors that span H^2.

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