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ON THE REPRESENTATION OF MAPPINGS OF TYCHONOV SPACES AS RESTRICTIONS OF LINEAR TRANSFORMATIONS

BY

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1. **Introduction.** Let (X, τ) be a Tychonov space and $\mathscr{P}(\tau)$ the collection of all families of pseudometrics on X generating the topology τ on X. Let $f:X\to X$ and $c>0$. Then f is said to be a topological c-homothety if there exists some B in $\mathscr{P}(\tau)$ such that $d(f(x),f(y))=cd(x,y)$ for all $d\in B$ and all x, y in X (see [4]). We say that f can be linearized in L as a c-homothety if there exists a linear topological space L, and a topological embedding $i:X\to L$ such that $i(f(x))=ci(x)$ for all x in X (see [4]). f is said to be squeezing if $\bigcap_{n=1}^{\infty} f^n[X]=\{a\}$ for some a in X. In [4] L. Janos proved the following

THEOREM 1 (JANOS). *Let X be a compact Hausdorff space and $f:X\to X$. Then the following are equivalent:*

(1) *f is a topological c-homothety for some $c\in(0, 1)$.*

(2) *f is a squeezing homeomorphism.*

(3) *f can be linearized in some linear topological space L as a c-homothety for some $c\in(0, 1)$.*

In [2], M. Edelstein and S. Swaminathan proved the following related results for normal Hausdorff spaces:

THEOREM 2 (EDELSTEIN AND SWAMINATHAN). *Let X be a normal Hausdorff space and f a homeomorphism of X onto a closed subset of X. Suppose $\bigcap_{n=1}^{\infty} f^n[X]$ is a singleton and λ a real number, $0<\lambda<1$. Then there exists a continuous one-to-one mapping of X into Q^A, where $Q=[0, 1]$ and A a suitable index set, such that hfh^{-1} is the restriction to $h[X]$ of the transformation which maps $y\in Q^A$ into λy.*

THEOREM 3 (EDELSTEIN AND SWAMINATHAN). *Let f be a homeomorphism of a normal space X onto a closed subset of X such that $\bigcap_{n=1}^{\infty} f^n[X]=\{x_1, x_2, \ldots, x_k\}$. Let λ be a real number with $0<\lambda<1$ and let p be the permutation of $(1, 2, \ldots, k)$ with the property that $p(i)=j$ if and only if $f(x_i)=x_j$. Then a continuous one-to-one*

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mapping h of X into $E^k \times Q^A$, *where* E^k *is the Euclidean k-dimensional space, exists such that* hfh^{-1} *is the restriction to* $h[X]$ *of the transformation which assigns to* $((x_1, x_2, \ldots, x_k), y)$ *the element* $((x_{p(1)}, x_{p(2)}, \ldots, x_{p(k)}), \lambda y)$.

In the present paper we shall prove related results for Tychonov spaces which generalize and strengthen those results in [2] and thus solve the problem raised in the remark of [2].

2. **Linear representations.** For a topological space X, let $C^*(X)$ denote the set of all continuous bounded real-valued functions on X. For $f \in C^*(X)$, let $Z(f)=\{x \in X: f(x)=0\}$ and $Z(X)=\{Z(f): f \in C^*(X)\}$. A subset B of X is said to be C^*-embedded in X if each f in $C^*(B)$ has an extension g in $C^*(X)$. We note that in a normal space, every closed set is C^*-embedded. For a Tychonov space X, let βX denote the Stone-Cech compactification of X. For construction of βX and notations not defined here, we shall refer to Gillman and Jerison [3]. If $f: X \to X$ is continuous then f_β will denote the unique continuous extension of f from βX into βX.

LEMMA 4. *Let f be a homeomorphism of a Tychonov space X into itself. Then* $f[X]$ *is* C^**-embedded in X if and only if* f_β *is a homeomorphism.*

Proof. Assume that $f[X]$ is C^*-embedded in X. Since βX is a compact Hausdorff space, to show that f_β is a homeomorphism, we need only to show that f_β is one-to-one. Let $x, y \in \beta X$ and $x \neq y$; there exists a continuous function $h: \beta X \to [0, 1]$ such that $h(x) \neq h(y)$. By assumption, the function $hf^{-1}: f[X] \to [0, 1]$ has a continuous extension $H: \beta X \to [0, 1]$. We shall show that $Hf_\beta(x) \neq Hf_\beta(y)$. Let $(x_\alpha)_{\alpha \in \Gamma}$ and $(y_\alpha)_{\alpha \in \Gamma}$ be nets in X which converge to x and y respectively. Then $Hf_\beta(x)=\lim_\alpha Hf_\beta(x_\alpha)=\lim_\alpha Hf(x_\alpha)=\lim_\alpha hf^{-1}f(x_2)=\lim_\alpha h(x_2)=h(x)$. Similarly, we can show that $Hf_\beta(y)=h(y)$. Since $h(x)\neq h(y)$, $Hf_\beta(x)\neq Hf_\beta(y)$. Therefore $f_\beta(x)\neq f_\beta(y)$. This shows that f_β is one-to-one.

Conversely suppose that $f_\beta: \beta X \to \beta X$ is a homeomorphism. Let $g \in C^*(f[X])$. Then the function $gf \in C^*(X)$ has an extension G in $C^*(\beta X)$. Consider the function $Gf_\beta^{-1}: f_\beta[\beta X] \to R$. For each x in X, $Gf_\beta^{-1}(f(x))=G(x)=g(f(x))$. Hence $Gf_\beta^{-1}|_{f[X]}=g$. Since $f_\beta[\beta X]$, being compact, is C^*-embedded in βX ([3](c) p. 43), Gf_β^{-1} has an extension H in $C^*(\beta X)$ and we have $H|_X \in C^*(X)$ and $H|_X$ is an extension of g. Therefore $f[X]$ is C^*-embedded in X.

LEMMA 5. *Let B be a nonempty compact subset of a Tychonov space X and* $f: X \to X$ *be continuous such that* $\bigcap_{n=1}^\infty \overline{f^n[X]}=B$. *Then* $f^n[X]\to B$ *i.e., for each neighborhood U of B, there is an n such that* $f^k[X]\subseteq U$ *for* $k \geq n$, *if and only if* $\bigcap_{n=1}^\infty f_\beta^n[\beta X]=B$.

Proof. Assume that $f^n[X]\to B$ and let $\mathscr{F}=\{Z \in Z(X) \mid Z \supseteq f^n[X]$ for some $n\}$. Then $\mathscr{F}$ is a z-filter and we shall show that if A^p contains $\mathscr{F}$ where A^p denotes the

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z-ultrafilter on X with limit p in βX, then A^p is fixed and $p \in B$. Let $A^p \supseteq \mathscr{F}$. For each $Z \in A^p$, we have $Z \cap B \neq \varnothing$. Otherwise, since B is compact, there exists zero-set neighborhood Z' of B such that $Z' \cap Z = \varnothing$. (By (a) 3.11 and theorem 1.15 of [3]). Hence $Z' \notin A^p$ but $Z' \in \mathscr{F}$ since $f^n[X] \to B$. This contradicts that $A^p \supseteq \mathscr{F}$. Thus $\{Z \cap B \mid Z \in A^p\}$ is a collection of closed subsets of the compact set B with finite intersection property, hence $\varnothing \neq \bigcap \{Z \cap B \mid Z \in A^p\} \subseteq \bigcap A^p$. Therefore A^p is fixed and $\bigcap A^p = \{p\}$. Hence $p \in B$. Now let $p \in \bigcap_{n=1}^{\infty} f_\beta^n[\beta X] = \bigcap_{n=1}^{\infty} cl_{\beta X} f^n[X]$, then $p \in cl_{\beta X} Z$ for every Z in $\mathscr{F}$. By (c) p. 87 of [3] $\mathscr{F} \subseteq A^p$ and hence $p \in B$. Thus $\bigcap_{n=1}^{\infty} f_\beta^n[\beta X] \subseteq B$. Since $\bigcap_{n=1}^{\infty} f_\beta^n[\beta X] \supseteq \overline{\bigcap_{n=1}^{\infty} f^n[X]} = B$, we have $\bigcap_{n=1}^{\infty} f_\beta^n[\beta X] = B$.

The converse is clear.

THEOREM 6. *Let $f: X \to X$ be continuous from a Tychonov space X into itself and $x_0 \in X$. Then the following conditions* (i) *and* (ii) *are equivalent;*

(i) (a) *f is a homeomorphism,* (b) *$f[X]$ is C^*-embedded in X and* (c) $f^n[X] \to \{x_0\}$

(ii) *$f_\beta: \beta X \to \beta X$ is a homeomorphism and* $\bigcap_{n=1}^{\infty} f_\beta^n[\beta X] = \{x_0\}$.

Proof. By lemmas 4, 5 and note that $f^n[X] \to \{x_0\}$ implies $\bigcap_{n=1}^{\infty} f^n[X] = \{x_0\}$.

Combining Theorem 6, Lemma 5 with results of Janos (theorem 1) and of Edelstein and Swaminathan (theorems 2 and 3), we have the following theorems:

THEOREM 7. *Let $f: X \to X$ be a homeomorphism of a Tychonov space into itself such that $f[X]$ is C^*-embedded and $f^n[X] \to \{x_0\}$ for some $x_0 \in X$. Then*

(a) *f is a topological c-homothety for some $c \in (0, 1)$*

(b) *f can be linearized in some linear topological space L as a c-homothety for some $c \in (0, 1)$.*

THEOREM 8. *Let $f: X \to X$ be a homeomorphism of a Tychonov space X into itself such that $f[X]$ is C^*-embedded in X and $f^n[X] \to \{x_0\}$ for some $x_0 \in X$. Then for each $\lambda \in (0, 1)$, there exists a homeomorphism h of X into Q^A, where $Q = [0, 1]$ and A is a suitable index set, such that hfh^{-1} is the restriction to $h[X]$ of the transformation which maps $y \in Q^A$ into λy.*

THEOREM 9. *Let f be a homeomorphism of a Tychonov space X into itself such that $\bigcap_{n=1}^{\infty} f^n(X) = \{x_1, x_2, \ldots, x_K\}$, $f[X]$ is C^*-embedded in X and $f^n[X] \to \{x_1, x_2, \ldots, x_K\}$. Let $\lambda \in (0, 1)$ and let p be a permutation of $(1, 2, \ldots, k)$ with the property that $p(i) = j$ if and only if $f(x_i) = f(x_j)$. Then there exists a homeomorphism h of X into $E^k \times Q^A$ where E^k is the Euclidean k-dimensional space, such that hfh^{-1} is the restriction to $h[X]$ of the transformation which assigns to $((x_1, x_2, \ldots, x_K), y)$ the element $((x_{p(1)}, x_{p(2)}, \ldots, x_{p(K)}), \lambda y)$.*

The above theorems generalize and strengthen those theorems in [2] and answer the question raised in [2].

Next, we shall represent selfmap on Tychonov space X in product of l_2. In

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case X is compact metrizable, M. Edelstein [1] proved that each squeezing selfmap can be linearly represented in l_2 in the following way:

THEOREM 10 (M. EDELSTEIN). *Let f be a continuous mapping of a compact metrizable space X into itself with $\bigcap_{n=1}^{\infty} f^n[X]$ a singleton and $P:l_2\to l_2$ the linear transformation defined by $P(y)=(y_2, y_4, \ldots, y_{2n}, \ldots)$ for $y=(y_1, y_2, \ldots, y_n, \ldots)\in l_2$. Given λ, $0<\lambda<1$, there is a homeomorphism h of X into l_2 such that hfh^{-1} is the restriction of λP to $h[X]$.*

We shall use the above theorem and the method in [4] to prove the following theorems.

THEOREM 11. *Let f be a continuous mapping on a compact Hausdorff space (X, τ) into itself such that $f^n[X]\to\{x_0\}$ for some x_0 in X and let $P:l_2\to l_2$ be the linear transformation defined by $P(y)=(y_2, y_4, \ldots, y_{2n}, \ldots)$ for $y=(y_1, y_2, \ldots, y_n, \ldots)\in l_2$. Given λ, $0<\lambda<1$, there is a homeomorphism h of X into $\prod_{\alpha\in A} l_2$ where A is a suitable index set such that hfh^{-1} is the restriction of $\lambda\prod_{\alpha\in A} P$ to $h[X]$, where $(\lambda\prod_{\alpha\in A} P)(x_\alpha)=\lambda\prod_{\alpha\in A} P(x_\alpha)$, for any $(x_\alpha)\in\prod_{\alpha\in A} l_2$.*

Proof. Let $\mathscr{D}=\{d_\alpha \mid \alpha\in A\}$ be a family of pseudometrics on X generating the topology τ of X such that f is nonexpansive with respect to $\mathscr{D}$ i.e. $d_\alpha(f(x), f(y))\leq d_\alpha(x, y)$ for all $\alpha\in A$ and all x, y in X. Such $\mathscr{D}$ exists according to Lemma 2.1 of [4]. For each $\alpha\in A$, let $X_\alpha=\{[x]_\alpha \mid x\in X\}$ be the family of all equivalent classes $[x]_\alpha$ where $[x]_\alpha=\{y\in X \mid d_\alpha(x, y)=0\}$. Then (X_α, ρ_α) is a compact metric space where $\rho_\alpha([x]_\alpha, [y]_\alpha)=d_\alpha(x, y)$. Since $f:X\to X$ is nonexpansive with respect to $\mathscr{D}$, the function $f_\alpha:X_\alpha\to X_\alpha$ defined by $f_\alpha([x]_\alpha)=[f(x)]_\alpha$ is well-defined and continuous and it can be easily shown that $\bigcap_{n=1}^{\infty} f_\alpha^n[X_\alpha]=\{[x_0]_\alpha\}$. Thus by Theorem 10, there exists a homeomorphism $h_\alpha:X_\alpha\to l_2$ such that $h_\alpha f_\alpha h_\alpha^{-1}$ is the restriction of λP to $h_\alpha[X_\alpha]$. Define $h:X\to\prod_{\alpha\in A} l_2$ by $h(x)=(h_\alpha([x]_\alpha))_{\alpha\in A}$. Then h is a homeomorphism and furthermore if $y=(y_\alpha)_{\alpha\in A}\in h[X]$ then $y=h(x)=(h_\alpha[X]_\alpha)_{\alpha\in A}$ for some x in X and

$$
\begin{aligned}
hfh^{-1}(y) &= hfh^{-1}(h(x)) = h(f(x)) = (h_\alpha[f(x)]_\alpha))_{\alpha\in A} \\
&= (h_\alpha f_\alpha([x]_\alpha))_{\alpha\in A} \\
&= (\lambda P h_\alpha([x]_\alpha))_{\alpha\in A} \\
&= (\lambda P(y_\alpha))_{\alpha\in A} \\
&= \Big(\prod_{\alpha\in A} \lambda P\Big)(y) \\
&= \Big(\lambda \prod_{\alpha\in A} P\Big)(y).
\end{aligned}
$$

Hence hfh^{-1} restricted to $h[X]$ is $\lambda\prod_{\alpha\in A} P$.

THEOREM 12. *Let $f:X\to X$ be a continuous function from a Tychonov space X into itself such that $f^n[X]\to\{x_0\}$ for some x_0 in X and let $P:l_2\to l_2$ be defined as in*

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Theorem 11. *Then given* λ, $0<\lambda<1$, *there exists a homeomorphism* $h:X\to\prod_{\alpha\in A} l_2$ *such that* hfh^{-1} *is the restriction of* $\lambda\prod_{\alpha\in A} P$ *to* $h[X]$.

Proof. Let $f_\beta:\beta X\to\beta X$ be the unique continuous extension of f to βX. Then $f^n_\beta[\beta X]\to\{x_0\}$ from Lemma 5. By Theorem 11, there exists a homeomorphism $g:\beta X\to\prod_{\alpha\in A} l_2$ such that $gf_\beta g^{-1}$ is the restriction of $\lambda\prod_{\alpha\in A} P$ to $g[\beta X]$. Let $h=g\mid_X$ then h is a homeomorphism from X into $\prod_{\alpha\in A} l_2$ such that hfh^{-1} is the restriction of $\lambda\prod_{\alpha\in A} P$ to $h[X]$.

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