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# BUCKLING ANALYSIS OF PLATES OF ARBITRARY SHAPE

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**Abstract**

A simple and efficient numerical technique for the buckling analysis of thin elastic plates of arbitrary shape is proposed. The approach is based upon the combination of the standard Finite Element Method with the constant deflection contour method. Several representative plate problems of irregular boundaries are treated and where possible, the obtained results are validated against corresponding results in the literature.

## 1. Introduction

In the analysis of buckling of thin, isotropic elastic plates, one is confronted with the solution of an intricate fourth order partial differential equation over a prescribed region with appropriate constraints on the boundary of the region. Up till now, the work on buckling of plates has been mainly restricted to plates of relatively simple geometrical shape and boundary conditions. This is confirmed by the paucity of closed-form solutins currently available in the literature.

When an analytic solution is unavailable one generally resorts to numerical techniques, two of the most popular being the Finite Difference [17] and the Finite Element [20] methods. Although it is now well-recognized that the Finite Element Method is an extremely powerful and versatile tool for solving arbitrary-shaped plate problems, it can suffer certain disadvantages of general nature. Of course, most of these, such as the large amount of programming effort peculiar to the initial implementation stages followed by the tedious task of entering vast quantities of pertinent data, have largely been resolved by the

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development of general Finite Element packages. However, one particular disadvantage inherent in the method is the fact that each element comprises a large number of degrees of freedom. Consequently when analysing plates of arbitrary shape, many such elements may be required for a realistic model, thus leading to a substantial demand in total storage requirement. While this poses no apparent problems in the case of larger machines, it may prove somewhat restrictive with the increasingly-popular smaller computers.

In an attempt to eliminate the above shortcoming, and thus enhance the viability of smaller computers, especially with regard to the efficient solution of plate problems, a number of simpler and economically attractive alternative methods have recently been proposed. One such method is concerned with the judicious use of nodal lines in favour of nodal points to generalize the definition of an element, thus leading to a substantial decrease in storage requirements [5, 12], while another focuses on the application of boundary-integral [1, 14] and boundary-element [18] techniques to the problems of linear plate theory.

More recently, a novel approach based on the combination of the conventional Finite Element Method with the deflection contour technique has been presented and exemplified with reference to the bending and vibration analysis of arbitrary-shaped plates [8, 9]. The purpose of this paper is to study the extension of the above approach to analyse buckling problems of thin plates with simply supported or clamped boundaries. It is shown that the proposed method yields accurate results with relatively few elements. A number of illustrative examples are included to demonstrate the numerical accuracy of the results.

## 2. Theory

Consider a thin elastic plate, bounded by a piecewise-smooth curve and simply connected. Let a Cartesian coordinate system be defined so that the plane $oxy$ corresponds to the undeformed middle plane of the plate, and the $z$-axis is directed positively downward.

When the plate is subjected to in-plane loads, the profile of the plate's deflected surface may be described by a family of lines of constant deflection. Hence, at any instant $\tau$, a set of contours is defined by the intersection of the parallels, $z =$ constant, with the deflected surface. After projection onto the $oxy$ plane, these contours become a set of level curves, $u(x, y) =$ constant (Figure 1).

Consider an element to contain that region $\Omega$ of the plate bounded by any two contour lines, $u = u_1$ and $u = u_2$, such that $u_1 \neq u_2$ (Figure 2). These will be taken as the two nodal lines for the element. Since the displacement function $w$ is a function of $u$, the continuity requirement across the element boundaries is

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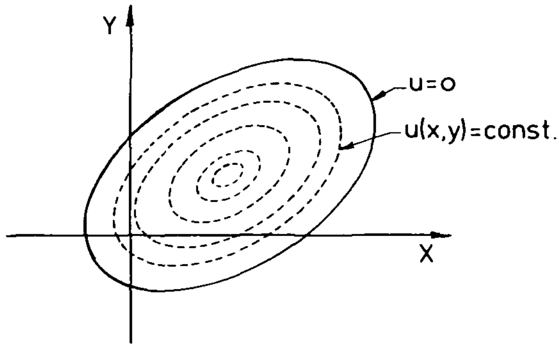


FIGURE 1. *Iso-deflection contour lines*

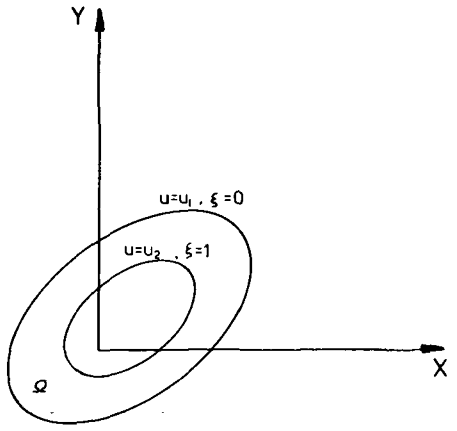


FIGURE 2. *Plate bending element*

satisfied by stipulating both $w$ and $dw/du$ as unknown degrees of freedom at each node. This implies that the total number of degrees of freedom for each element is only four.

Assuming a cubic variation for the displacement $w$ within each element, one obtains

$$w = [N]\{\delta\} \tag{1}$$

where the components of $[N]$ are given by

$$N_1 = 1 - 3\xi^2 + 2\xi^3, \qquad N_2 = l(\xi - 2\xi^2 + \xi^3),$$
$$N_3 = 3\xi^2 - 2\xi^3, \qquad N_4 = l(-\xi^2 + \xi^3),$$

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and

$$\{\delta\} = \{w_1\psi_1 w_2\psi_2\}^T,$$

$$\psi_i = \left(\frac{dw}{du}\right)_{u=u_i}, \quad i = 1,2, \qquad \xi = \frac{u-u_1}{l}, \quad l = u_2 - u_1.$$

The total potential energy $\pi_T$ of the plate element due to the in-plane forces and transverse loading may be expressed in the form [15]

$$\pi_T = \frac{1}{2}\int_\Omega\int \{\varepsilon\}^T\{\sigma\}\,d\Omega - \int_\Omega\int q(x,y)w\,d\Omega$$
$$+\frac{h}{2}\int\int\left\{N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}d\Omega, \tag{2}$$

where $N_x$, $N_y$ and $N_{xy}$ represent the in-plane forces, $q(x, y)$ is the transverse load per unit area, $h$ denotes the thickness of the plate and $\{\sigma\}$, $\{\varepsilon\}$ are respectively the generalized stress and strain vectors defined by

$$\{\sigma\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}, \qquad \{\varepsilon\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}. \tag{3}$$

In the above relations, $M_x$, $M_y$ and $M_{xy}$ represent the bending moments while the integrations are taken over the domain of the element.

If the material of the plate is assumed elastic and orthotropic, then, in accordance with Hooke's law, the generalized stresses and strains are related by the equation

$$\{\sigma\} = [D]\{\varepsilon\}, \tag{4}$$

where the elasticity matrix has the form [5]

$$[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix}, \tag{5}$$

and the orthotropic plate constants are given by

$$D_x = \frac{E_x h^3}{12(1-\nu_x\nu_y)}, \qquad D_y = \frac{E_y h^3}{12(1-\nu_x\nu_y)},$$
$$D_{xy} = Gh^3/12, \qquad D_1 = \nu_x D_x = \nu_y D_y.$$

In the above expressions, $E_x$, $E_y$, $\nu_x$, $\nu_y$ and $G$ are elastic constants dependent on the physical characteristics of the plate material.

By virtue of equation (1), the generalized strains can be expressed in the form

$$\{\varepsilon\} = [B]\{\delta\}, \tag{6}$$

where the strain matrix $[B]$ is given by

$$[B] = \frac{1}{l}\begin{bmatrix} -u_{xx} & -u_x^2/l \\ -u_{yy} & -u_y^2/l \\ 2u_{xy} & 2u_x u_y/l \end{bmatrix}\begin{Bmatrix} \dfrac{d[N]}{d\xi} \\ \dfrac{d^2[N]}{d\xi^2} \end{Bmatrix}. \tag{7}$$

While deriving the above expression, use is made of the following transformation relations;

$$\frac{\partial w}{\partial x} = \frac{\partial [N]}{\partial x}\{\delta\} = \frac{u_x}{l}\frac{d[N]}{d\xi}\{\delta\},$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 [N]}{\partial x^2}\{\delta\} = \left(\frac{u_{xx}}{l}\frac{d[N]}{d\xi} + \frac{u_x^2}{l^2}\frac{d^2[N]}{d\xi^2}\right)\{\delta\}, \quad etc.$$

Upon substitution of equations (1), (4) and (6) into the total potential energy expression, one obtains

$$\pi_T = \tfrac{1}{2}\{\delta\}^T[K]\{\delta\} - \{\delta\}^T\{F\} + \tfrac{1}{2}\{\delta\}^T[K_s]\{\delta\}, \tag{8}$$

where the stiffness matrix $[K]$ and the stability coefficient matrix $[K_s]$ are respectively defined by

$$[K] = \int_\Omega\int [B]^T[D][B]\, d\Omega, \tag{9}$$

$$[K_s] = h\int_\Omega\int [G]^T[\bar\sigma][G]\, d\Omega, \tag{10}$$

while the consistent load vector is defined by

$$\{F\} = \int_\Omega\int [N]^T q(x, y)\, d\Omega. \tag{11}$$

In the above expressions, the matrix $[G]$ has the form

$$[G] = \frac{1}{l}\begin{Bmatrix} u_x \\ u_y \end{Bmatrix}\frac{d[N]}{d\xi}, \tag{12}$$

while $[\bar\sigma]$ contains the in-plane loads in the manner

$$[\bar\sigma] = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix}. \tag{13}$$

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Minimizing the total potential energy, $\pi_T$, with respect to the nodal degrees of freedom, $\{\delta\}$, yields the equilibrium equations of the plate subjected to both lateral and in-plane forces as

$$([K] + [K_s])\{\delta\} = \{F\}. \tag{14}$$

Consequently, the general process of assembly [20] finally leads to

$$([\bar{K}] + [\bar{K}_s])\{\bar{\delta}\} = \{\bar{F}\}, \tag{15}$$

where $[\bar{K}]$ is the structural stiffness matrix, $[\bar{K}_s]$ is the overall stability coefficient matrix, $\{\bar{F}\}$ is a column vector of externally applied nodal forces and moments while $\{\bar{\delta}\}$ is a vector comprising all the nodal degrees of freedom of the discretized plate.

Of particular practical interest is the class of problems obtained by assuming zero lateral load and uniform compressive in-plane forces in the form $N_x = N_y = -\lambda$, $N_{xy} = 0$. The system of equations, given by (15), now reduces to

$$([\bar{K}] - \lambda[\bar{S}])\{\bar{\delta}\} = \{0\} \tag{16}$$

where the matrix $[\bar{S}]$ is defined by

$$[\bar{K}_s] = -\lambda[\bar{S}]. \tag{17}$$

Thus the plate buckling problem reduces to an eigenvalue problem, the solution of which can be achieved quite efficiently with the aid of any standard eigenvalue routine.

Equation (15) can be readily extended to enable the analysis of vibrating plates under the influence of in-plane loads. In this case, the use of D'Alembert's Principle yields

$$([\bar{K}] + [\bar{K}_s])\{\bar{\delta}\} = \omega^2[\bar{M}]\{\bar{\delta}\}, \tag{18}$$

where $\omega$ represents the frequency of free vibration and $[\bar{M}]$ is the overall consistent mass matrix derived in reference [4].

## 3. Determination of the contour function $u(x, y)$

It is clear that isodeflection contour lines form a family of non-intersecting closed curves starting with the boundary of the plate as one of the lines, especially when the boundary is either clamped or simply suppported or a combination of both. In principle, it is always possible to determine the exact equation of such lines. In some cases, the equation of these lines can be selected by symmetry considerations or by intuition [11]. However, for the cases in which the preassigned equation of the contour function $u(x, y)$ is not an exact one, we get an approximate solution to the problem and obviously this approximation will be as close to the exact solution as the assigned equation to the contour lines has been selected close to the exact one.

Guidelines for arriving at a suitable contour function $u(x, y)$ have already been amply set out in reference [10]. However, as pointed out therein, only the case for which $N_x = N_y =$ constant, $N_{xy} = 0$, received attention. In this regard, the procedure essentially relies on the solution of a second order partial differential equation for $u$ over the region of interest, depending on which buckling mode is being analysed. Obviously for practical reasons, only the fundamental buckling mode needs to be investigated, which, in fact, is the subject of the present study.

Heuristically, it has been found that a suitable approximate form of the function describing the contours to the above class of problems can be constructed on the basis of the equation of the plate boundary. Thus, if the boundary of the plate is generated by the equation $g(x, y) = 0$ and the function $g$ is different from zero everywhere within the region of the plate, then a first approximation to the expression for $u$ has the form

$$u(x, y) = g(x, y). \tag{19}$$

It should be mentioned here that if one speaks of the rule rather than the exception, then the considerations adduced above for selecting a contour function appear sufficiently convincing.

## 4. Illustrative examples

To test the accuracy of the proposed method, the buckling of plates of various boundary shapes will now be considered and, where possible, the ensuing results will be compared to previously published results.

### (a) Equilateral triangular plate

As a first example of the method, consider the deformation of a simply supported equilateral triangular plate under the influence of compressive in-plane forces given by $N_x = N_y = -\lambda$, $N_{xy} = 0$, where $\lambda$ is a constant. The problem here is to find the critical value of the compressive forces, $\lambda_{cr}$, at which the plane stress system becomes unstable and the plate buckles. In general, there exists as many values of $\lambda_{cr}$ as there are degrees of freedom for the plate structure, but it is the lowest value of $\lambda_{cr}$ which is of vital importance in practice.

It will be assumed that the lines of constant deflection for the first buckling mode of a thin elastic plate subject to hydrostatic edge loading coincide with the

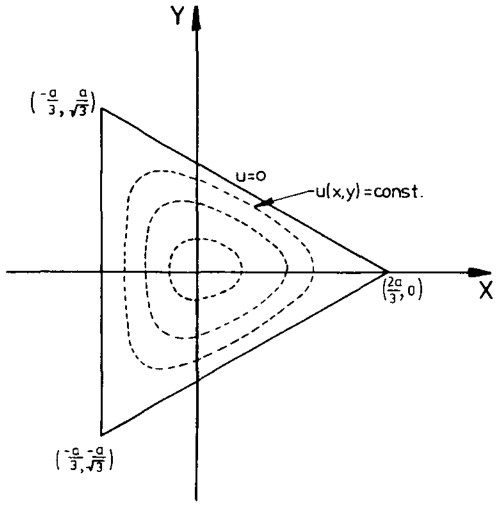


FIGURE 3. *Equilateral triangular plate idealized by four elements*

lines of constant deflection for the same plate under uniform transverse loading and with identical boundary conditions. Thus, the expression for the contour function, in this case, has the form [3]

$$u(x, y) = \left(x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27}\right)\left(\frac{4}{9}a^2 - x^2 - y^2\right), \quad (20)$$

where $a$ denotes the perpendicular height of the triangle as depicted in Figure 3.

If the material of the plate is assumed isotropic and only four elements are considered in the discretization process, then, after imposing the appropriate boundary conditions the solution to equation (16) gives the value of the critical buckling load $\lambda_{cr}$ as

$$\frac{a^2\lambda_{cr}}{\pi^2 D} = 4.024, \quad (21)$$

where $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the plate while $E$, $\nu$ represent Young's modulus and Poisson's ratio, respectively. The above value of $a^2\lambda_{cr}/\pi^2 D$ compares very well with the corresponding value of 4.00 obtained by Woinowsky-Krieger [19].

Corresponding results for the same plate composed of orthotropic material are presented in Table 1. The material constants assumed in the analysis coincide with those adopted in an earlier paper [4].

TABLE 1: Critical buckling load for an orthotropic equilateral triangular plate simply supported on all edges ($\nu = 0.3$).

| Material | $D_x/D_y$ | $H/D_y$ | $H/\sqrt{D_x D_y}$ | $a^2\lambda_{cr}/\pi^2 D$ |
|---|---|---|---|---|
| Veneer | 14.143 | 3.286 | 0.874 | 26.133 |
| Glass-Epoxy | 2.963 | 1.156 | 0.672 | 7.139 |
| 5-Plywood | 2.690 | 0.620 | 0.378 | 6.187 |
| Grooved Steel | 1.265 | 0.958 | 0.852 | 4.381 |

### (b) Limacon-shaped plate

As a second example, consider the problem of determining the critical buckling load, $\lambda_{cr}$, of a hydrostatically compressed thin elastic plate, the boundary of which is rigidly clamped and described by the general equation

$$r(\theta) = a(1 + \varepsilon \cos \theta), \qquad 0 \leqslant \varepsilon \leqslant 1, \tag{22}$$

where $r, \theta$ are polar coordinates and $a, \varepsilon$ are parameters characteristic of the limacon (Figure 4). In fact, a range of shapes from the circle ($\varepsilon = 0$) to the cardioid ($\varepsilon = 1$) may be generated by the above equation.

Since the boundary of the plate is clamped and analysis is confined to the non-nodal or fundamental buckling mode, then the contour function $u$ must satisfy [10]

$$\nabla^2 u = \text{constant} = -2 \quad \text{(say)}, \tag{23}$$

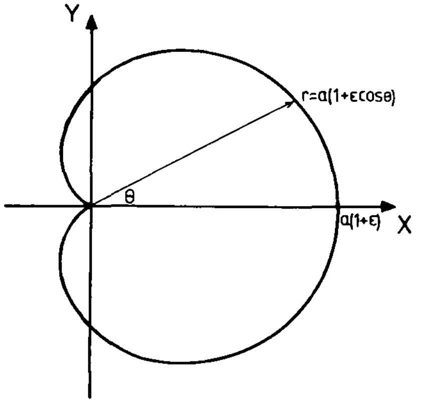


FIGURE 4. *Limacon-shaped plate*

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where the Laplacian operator $\nabla^2(\;)$ is defined by

$$\nabla^2(\;) \equiv \frac{\partial^2(\;)}{\partial x^2} + \frac{\partial^2(\;)}{\partial y^2}.$$

The solution to equation (23), subject to the condition that $u = 0$ on the boundary, may be expressed in the following form [9];

$$u(x, y) = a^2\left(\frac{\varepsilon^2}{4} - \frac{1}{4} + \frac{\varepsilon}{2}\alpha^{1/2}\cos\frac{\beta}{2} + \frac{\varepsilon^2}{4}\alpha\cos\beta\right) - \frac{1}{2}(x^2 + y^2), \quad (24)$$

where

$$\alpha = \left\{\left(\frac{2x}{a\varepsilon} - 1 + \frac{1}{\varepsilon^2}\right)^2 + \frac{4y^2}{a^2\varepsilon^2}\right\}^{1/2}, \qquad \cos\beta = \left(\frac{2x}{a\varepsilon} - 1 + \frac{1}{\varepsilon^2}\right)/\alpha.$$

The numerical values of critical buckling loads, obtained with the aid of equation (24) are displayed in Table 2 for various values of the shape parameter $\varepsilon$. Since in the authors' knowledge, there are no exact theoretical or experimental results for buckling of thin limacon-shaped plates with which comparisons of our results can be made, one may as a basis for comparisons and also as confirmation of the proposed method consider the limiting case ($\varepsilon = 0$) when the plates become circular. It is interesting to note from the results of Table 2 that the value of the critical parameter coincides exactly with the corresponding value of 14.68 available in the literature [16].

TABLE 2: Critical buckling load $\lambda_{cr}$ for a clamped limacon-shaped plate with shape parameter $\varepsilon$.

| $\varepsilon$ | $\lambda_{cr}a^2/D$ | $\varepsilon$ | $\lambda_{cr}a^2/D$ |
|---|---|---|---|
| 0.0 | 14.682 | 0.6 | 12.487 |
| 0.1 | 14.582 | 0.7 | 12.015 |
| 0.2 | 14.308 | 0.8 | 11.586 |
| 0.3 | 13.914 | 0.9 | 11.232 |
| 0.4 | 13.820 | 1.0 | 10.956 |
| 0.5 | 13.327 | | |

### (c) Skew plate

As a final example, consider the buckling analysis of a clamped skew plate. Since the plate boundary is assumed to be rigidly clamped, the contour function $u(x, y)$ can be obtained by solving the equation (23) over the region of the plate with the condition that it vanishes on the edges. A solution to this problem in non-dimensional skew coordinates has recently been given by Coleby and Mazumdar in reference [6] as

$$u = \sum_{m=1,3,5\ldots}^{\infty} \frac{A_m}{\alpha_m \sinh s_m}\{\phi_{1m} + \phi_{2m}\tan c_m \tanh s_m\}$$
$$+ \sum_{m=1,3,5,\ldots}^{\infty} \frac{B_m}{\alpha_m \sinh s_m}\{\psi_{1m} - \psi_{2m}\tan c_m \tanh s_m\}$$
$$- \sum_{m=1,3,5,\ldots}^{\infty} \frac{k_m \phi_{2m}}{\cos c_m \cosh s_m} + (1-\xi^2)\sin^2\gamma, \tag{25}$$

where

$$\phi_{1m} = \sin(\xi\alpha_m + \eta c_m)\sinh(\eta s_m), \quad \phi_{2m} = \cos(\xi\alpha_m + \eta c_m)\cosh(\eta s_m),$$
$$\psi_{1m} = \sin(\xi c_m + \eta\alpha_m)\sinh(\xi s_m), \quad \psi_{2m} = \cos(\xi c_m + \eta\alpha_m)\cosh(\xi s_m),$$
$$k_m = \frac{(-1)^{(m-1)/2}(32\sin^2\gamma)}{m^3\pi^3}, \quad c_m = \alpha_m\cos\gamma, \tag{26}$$
$$s_m = \alpha_m \sin\gamma, \quad \alpha_m = m\pi/2.$$

In the above formulation, the coordinate transformations are carried out *via* the relations

$$\xi = 2x'/a, \qquad \eta = 2y'/b, \tag{27}$$

where $a$, $b$ are the dimensions of the plate and

$$x' = x - y\cot\gamma, \qquad y' = y\operatorname{cosec}\gamma, \tag{28}$$

while $\gamma$ denotes the skew angle as depicted in Figure 5.

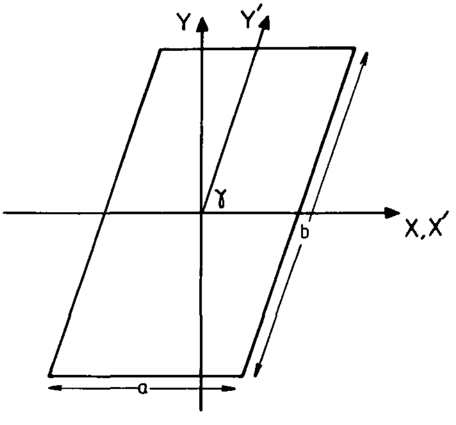


FIGURE 5. *Skew-shaped plate*

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The coefficients $A_m$ and $B_m$ are obtained by solving a system of coupled linear equations. For numerical purposes in this study, these coefficients were computed for several values of the skew angle $\gamma$, namely, 45°, 60°, 75° and 90°. The case $\gamma = 90°$, which corresponds to a rectangular plate, resulted in the conclusion that both $A_m$ and $B_m$ are identically zero for all values of $m$. This indicates that the contour function, for this case, reduces to the well-known expression for the torsion of a beam having a rectangular shaped boundary [13]. The values of $A_m$ and $B_m$ for $\gamma = 45°$ and 75° are presented in Table 3. The corresponding values of these coefficients for $\gamma = 60°$ are not included in the table as they have been previously reported in reference [6].

TABLE 3: Value of coefficients $A_m/\alpha_m$ and $B_m/\alpha_m$ for $\gamma = 45°$ and 75°.

| $m$ | $\gamma = 45°$ $A_m/\alpha_m$ | $\gamma = 45°$ $B_m/\alpha_m$ | $\gamma = 75°$ $A_m/\alpha_m$ | $\gamma = 75°$ $B_m/\alpha_m$ |
|---|---|---|---|---|
| 1 | $-.5557 \times 10^{0}$ | $.1179 \times 10^{0}$ | $-.4468 \times 10^{0}$ | $.9654 \times 10^{-1}$ |
| 3 | $-.2643 \times 10^{-1}$ | $-.4815 \times 10^{-2}$ | $.3781 \times 10^{-1}$ | $.1156 \times 10^{-2}$ |
| 5 | $-.2492 \times 10^{-2}$ | $-.2486 \times 10^{-2}$ | $-.4969 \times 10^{-2}$ | $.1090 \times 10^{-2}$ |
| 7 | $.1755 \times 10^{-2}$ | $.1602 \times 10^{-3}$ | $-.1160 \times 10^{-2}$ | $-.1392 \times 10^{-2}$ |
| 9 | $.1934 \times 10^{-2}$ | $.1145 \times 10^{-2}$ | $.1683 \times 10^{-2}$ | $.8113 \times 10^{-3}$ |
| 11 | $.9954 \times 10^{-3}$ | $.9255 \times 10^{-3}$ | $.8838 \times 10^{-3}$ | $-.1535 \times 10^{-3}$ |

Once the coefficients $A_m$ and $B_m$ are known for a particular plate geometry, the contour function $u$ can then be utilized, in conjunction with the Finite Element method, to determine the critical buckling load, $\lambda_{cr}$, at which the plate buckles. Table 4 summrizes the critical buckling load so obtained as a function of aspect ratio $a/b$ as well as skew angle $\gamma$. For the sake of comparison, the corresponding results given in reference [2] are also included in the table.

TABLE 4: Critical buckling load $\lambda_{cr}b^2/D$ for a clamped skew plate of varying skew angle $\gamma$.

| $\gamma$ | $a/b$ | Present | Reference [2] |
|---|---|---|---|
| 90° | 1.0 | 53.16 | 53.2 |
| | 1.5 | 41.31 | – |
| | 2.0 | 38.99 | 39.1 |
| 75° | 1.0 | 56.82 | 56.6 |
| | 1.5 | 44.12 | – |
| | 2.0 | 41.89 | 41.8 |
| 60° | 1.0 | 70.63 | 68.9 |
| | 1.5 | 54.60 | – |
| | 2.0 | 52.48 | 51.8 |
| 45° | 1.0 | 103.71 | 99.6 |
| | 1.5 | 82.57 | – |
| | 2.0 | 78.31 | 77.5 |

Consider now the problem of determining the fundamental frequency of free vibration of the same skew plate subjected to a compressive normal in-plane load. Using equation (18), the fundamental frequency parameter $\Omega = (\omega a^2/\pi^2)\sqrt{\rho h/D}$, for various values of the non-dimensional parameter $\Gamma = \lambda a^2/\pi^2 D$ and for the particular case in which $a/b = 1$, is displayed in Table 5. The numerical values given in the table are derived by the use of skew angles of 90° and 75°. For the case, $\gamma = 90°$, corresponding results given by Dickinson [7] are also included in the table for comparison, while for $\gamma = 75°$, a thorough search of the literature failed to provide results other than for the case $\Gamma = 0$.

Suppose now that the contour function $u$, as given in equation (25), was not available in this case. Then, in accordance with the rationale presented in Section 3 for selecting this function, it is plausible to write

$$u_0 = (\xi^2 - 1)(\eta^2 - 1) \tag{29}$$

where $\xi$, $\eta$ are defined in equation (27), and $u_0$ which in fact is the equation of the boundary, is assumed as an approximation to $u$. Using this expression for the equation of the contour lines enables the fundamental frequency parameter $\Omega_0 = (\omega_0 a^2/\pi^2)\sqrt{\rho h/D}$ to be computed. All results are displayed in Table 5.

TABLE 5: Fundamental frequency of vibration for a clamped skew plate under the influence of in-plane forces, $N_x = N_y = -\lambda$, $N_{xy} = 0$.

| Case $\gamma = 90°$ | | | |
|---|---|---|---|
| $\Gamma$ | Present Method $\Omega_0$ | Present Method $\Omega$ | Reference [7] |
| 2 | 2.90 | 2.92 | 2.89 |
| 0 | 3.65 | 3.66 | 3.64 |
| -10 | 6.07 | 6.08 | 6.07 |
| -20 | 7.72 | 7.72 | 7.71 |
| -50 | 11.21 | 11.21 | 11.26 |
| -200 | 21.08 | 21.07 | 21.37 |

| Case $\gamma = 75°$ | | | |
|---|---|---|---|
| $\Gamma$ | Present Method $\Omega_0$ | Present Method $\Omega$ | Reference [8] |
| 2 | 3.17 | 3.16 | |
| 0 | 3.90 | 3.89 | 3.87 |
| -10 | 6.34 | 6.33 | |
| -20 | 8.01 | 8.01 | |
| -50 | 11.59 | 11.58 | |
| -200 | 21.73 | 21.72 | |

The results presented in the above table clearly demonstrate the accuracy achievable by the present method.

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## 5. Conclusion

A simple and accurate method for the buckling analysis of plates of arbitrary shape subject to in-plane forces has been proposed. It has been shown that the problem of determining the lowest buckling load for a particular plate can be studied in a very simple manner by a Finite Element-Contour Method. The proposed method has the advantage of using only relatively small matrices, thereby requiring a reduction in core storage over conventional Finite Element-Methods and minimum time for execution of problems. However, this is not to claim that the proposed method is always superior to the conventional Finite Element Method because the latter method has a much wider applicability than to the class of problems that are being discussed in the present study.

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