Can. J. Math., Vol. XXXVIII, No. 3, 1986, pp. 751-768

A SEMIGROUP APPROACH TO LINEAR ALGEBRAIC GROUPS III. BUILDINGS

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Introduction. Let K be an algebraically closed field, $G = SL(3, K)$ the group of 3×3 matrices over K of determinant 1. Let $\mathcal{M}_3(K)$ denote the monoid of all 3×3 matrices over K. If e is an idempotent in $\mathcal{M}_3(K)$, then

$$C_G^r(e) = \{a \in G | ae = eae\},$$

$$C_G^l(e) = \{a \in G | ea = eae\}$$

are opposite parabolic subgroups of G in the usual sense [**1**], [**28**]. However the map

$$e \to (C_G^r(e), C_G^l(e))$$

does not exhaust all pairs of opposite parabolic subgroups of G. Now consider the representation $\phi:G \to SL(6, K)$ given by

$$\phi(a) = a \oplus (a^{-1})^t.$$

Let M denote the Zariski closure of $K\phi(G)$ in $\mathcal{M}_6(K)$. Let S denote the set of zero determinant elements of M. Then S is a regular semigroup. The set of idempotents of S,

$$E(S) = \{e \oplus f | e^2 = e, f^2 = f \in \mathcal{M}_3(K), \rho(e), \rho(f) \leqq 1, ef^t = f^te = 0\}.$$

Here ρ denotes rank. If $e \in E(S)$, then let

$$P(e) = \{a \in G | \phi(a)e = e\phi(a)e\},$$

$$P^-(e) = \{a \in G | e\phi(a) = e\phi(a)e\}.$$

Then the map ψ given by $\psi(e) = (P(e), P^-(e))$ is a bijection between $E(S)$ and all pairs of opposite parabolic subgroups of G. Furthermore if $e, f \in E(S)$, then $ef = f$ if and only if $P(e) \subseteq P(f)$ and $fe = f$ if and only if $P^-(e) \subseteq P^-(f)$. This example suggests that pairs of opposite parabolic subgroups of a reductive group should correspond naturally with the idempotents of a suitable regular semigroup. We will show this to be true

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Received November 20, 1984. This research was partially supported by NSF Grant MCS 8025597.

https://doi.org/10.4153/CJM-1986-039-1 Published online by Cambridge University Press

in the more general setting of a Tits system with a finite Weyl group or a Tits building with a finite Weyl complex.

1. Regular semigroups. Let S be a *regular semigroup*, i.e., $a \in aSa$ for all $a \in S$. If $a, b \in S$, then $a\mathcal{J}b$ if $SaS = SbS$, $a\mathcal{R}b$ if $aS = bS$, $a\mathcal{L}b$ if $Sa = Sb$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. The semigroups encountered in this paper turn out to have the property that $\mathcal{J} = \mathcal{D}$. If $a \in S$, then J_a, R_a, L_a, H_a will denote the $\mathcal{J}$-class, $\mathcal{R}$-class, $\mathcal{L}$-class, $\mathcal{H}$-class of a, respectively. If $a, b \in S$, then $J_a \geqq J_b$ if $SaS \supseteq SbS$, $R_a \geqq R_b$ if $aS \supseteq bS$, $L_a \geqq L_b$ if $Sa \supseteq Sb$. See [2] for details. We will denote the partially ordered set $S/\mathcal{J}$ by $\mathcal{U}(S)$. Let

$$E = E(S) = \{e \in S | e^2 = e\}.$$

If $e, f \in E$, then define $f \leqq_r e$ if $ef = f$, $f \leqq_l e$ if $fe = f$, $\leqq \, = \, \leqq_r \cap \leqq_l$, $\mathcal{R} = \leqq_r \cap (\leqq_r)^{-1}$, $\mathcal{L} = \leqq_l \cap (\leqq_l)^{-1}$. If $f \leqq_r e$, then set $e \circ f = f$, $f \circ e = fe \in E$. If $f \leqq_l e$, then set

$$f \circ e = f, \quad e \circ f = ef \in E.$$

Then the partial algebra $(E, \circ)$ satisfies certain axioms [7, Theorem 1.1] and the resulting system is called a *regular biordered set*. This is the work of Nambooripad [7] who then goes on to show that conversely every regular biordered set $(E, \circ)$ is isomorphic to the biordered set of idempotents of some regular semigroup. We denote the 'smallest' such semigroup by $\prec E \succ$. The $\prec E \succ$ is characterized by the properties of being generated by its idempotent set E and being fundamental (i.e., having no non-trivial idempotent separating congruences). See [7] for details.

A regular semigroup S is said to be an *inverse semigroup* if $ef = fe$ for all $e, f \in E(S)$. S is said to be a *locally inverse semigroup* if eSe is an inverse semigroup for all $e \in E(S)$. By [7, Theorem 7.6], S is a locally inverse semigroup if and only if the 'sandwich set' of any two idempotents in S consists of a single idempotent. The biordered set of a locally inverse semigroup is called a *local semilattice*. Local semilattices and locally inverse semigroups have also been called pseudo-semilattices and pseudo-inverse semigroups. Local semilattices were first studied by Nambooripad [8], [9], [10]. A weaker system was studied earlier by Schein [26]. Recently there has been much interest in local semilattices and locally inverse semigroups (see for example [4]-[11], [29]). We encounter local semilattices in the following special way.

Let $\Omega = (\Omega, \leqq) = (\Omega, \wedge)$ be a meet semilattice with a minimum element 0. Let $\perp$ be a symmetric relation defined on Ω such that $0 \perp 0$. We will say that $\Omega = (\Omega, \perp)$ is a *parabolic semilattice* if the following conditions hold.

(1) $\alpha\Omega = \{\beta \in \Omega | \beta \leqq \alpha \}$ is finite (and hence a lattice) for all $\alpha \in \Omega$.

(2) If $\gamma, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \Omega$, $\alpha_1 \perp \alpha_2$, $\beta_1 \perp \beta_2$, $\alpha_1 \geqq \beta_1$, $\gamma \geqq \alpha_2$, $\gamma \geqq \beta_2$, then $\alpha_2 \geqq \beta_2$.

(3) If $\alpha_1, \alpha_2, \beta_1 \in \Omega$, $\alpha_1 \perp \alpha_2$, $\alpha_1 \geqq \beta_1$, then there exists $\beta_2 \in \Omega$ (unique by (2)) such that $\alpha_2 \geqq \beta_2$, $\beta_1 \perp \beta_2$.

(4) If $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in \Omega$, $\alpha \geqq \alpha_i$, $\beta \geqq \beta_i$, $\alpha_i \perp \beta_i$, $i = 1, 2$, then

$$(\alpha_1 \vee \alpha_2) \perp (\beta_1 \vee \beta_2).$$

If $\Omega = (\Omega, \perp)$ is a parabolic semilattice, then we let

$$E_\Omega = \{ (\alpha, \alpha') | \alpha, \alpha' \in \Omega, \alpha \perp \alpha' \}.$$

If $e = (\alpha, \alpha')$, $f = (\beta, \beta') \in E_\Omega$, then define $f \leqq_r e$ if $\beta \leqq \alpha$, $f \leqq_l e$ if $\beta' \leqq \alpha'$. If $f \leqq_r e$, then let $ef = f$, $fe = (\beta, \beta^-)$ where $\beta^- \in \Omega$ is such that $\beta \perp \beta^-$, $\beta^- \leqq \alpha'$. If $f \leqq_l e$, then let

$$fe = f, \quad ef = (\beta_1, \beta')$$

where $\beta_1 \in \Omega$ is such that $\beta_1 \perp \beta'$ and $\beta_1 \leqq \alpha$.

THEOREM 1.1. *E_Ω is a local semilattice with an involution.*

Proof. Clearly the map $(\alpha, \alpha') \to (\alpha', \alpha)$ is an involution of E_Ω. We need to show that the axioms (B1)-(B4) of [**7**, p. 2] are satisfied and that the sandwich set $\mathscr{S}(e, f)$ consists of a single element for any $e, f \in E_\Omega$. We let

$$\leqq \; = \; \leqq_r \cap \leqq_l, \mathscr{R} = (\leqq_r) \cap (\leqq_r)^{-1}, \mathscr{L} = (\leqq_l) \cap (\leqq_l)^{-1}.$$

Let $E = E_\Omega$ and let

$$e = (\alpha, \alpha^-), f = (\beta, \beta^-), g = (\gamma, \gamma^-) \in E.$$

Suppose first that $f, g \leqq_r e$, $g \leqq_l f$. Then

$$\beta \leqq \alpha, \gamma \leqq \alpha, \gamma^- \leqq \beta^-.$$

Since $\beta \perp \beta^-$, $\gamma \perp \gamma^-$, we see that $\gamma \leqq \beta$. So $g \leqq f$. Now if $ge = (\gamma, \gamma')$, $fe = (\beta, \beta')$, then $\beta', \gamma' \leqq \alpha^-$. Since $\gamma \leqq \beta$, we see that $\gamma' \leqq \beta'$. Thus $ge \leqq fe$. So

(*) $\quad f, g \leqq_r e, g \leqq_l f$ imply $g \leqq f$, $ge \leqq fe$.

Next assume that $g \leqq_r f \leqq_r e$. Then $\gamma \leqq \beta \leqq \alpha$. If $ge = (\gamma, \gamma')$, then $\gamma' \leqq \alpha^-$. Let

$$(ge)f = (\gamma, \gamma'').$$

Then $\gamma'' \leqq \beta^-$. So by definition

$$gf = (\gamma, \gamma'') = (ge)f.$$

Thus the axioms (B1)-(B32) of [**7**, p. 2] are satisfied.

Now let $e = (\alpha, \alpha^-)$, $f = (\beta, \beta^-) \in E$ and set

$$M(e, f) = \{h \in E | h \leqq_r f, h \leqq_l e\}.$$

Then $M(e, f) \subseteq \beta\Omega \times \alpha^-\Omega$ is finite. Let

$$M(e, f) = \{h_1, \ldots, h_k\} \quad h_i = (\gamma_i, \gamma_i^-), i = 1, \ldots, k.$$

Then $\gamma_i \leqq \beta$, $\gamma_i^- \leqq \alpha^-$, $i = 1, \ldots, k$. So

$$\gamma = \gamma_1 \vee \ldots \vee \gamma_k \perp \gamma^- = \gamma_1^- \vee \ldots \vee \gamma_k^- .$$

Clearly

$$h = (\gamma, \gamma^-) \in M(e, f), \quad h_i \leqq h, i = 1, \ldots, k.$$

It follows that the sandwich set $\mathcal{S}(e, f) = \{h\}$. Now let $g \in E$ and suppose that $e, f \leqq_r g$. Then $h \leqq_r g$, $h \leqq_l e$. So by (*), $h \leqq e$, $hg \leqq eg$. Also,

$$hg\mathcal{R}h \leqq_r f\mathcal{R}fg$$

whereby

$$hg \leqq_r fg.$$

Hence $hg \in M(eg, fg)$. Let

$$\mathcal{S}(eg, fg) = \{h'\}.$$

We claim that $h' = hg$. Now

$$h\mathcal{R}hg \leqq h' \leqq_r fg\mathcal{R}f \leqq_r g.$$

So $h' \leqq_r g$. Also $h' \leqq_l eg \leqq g$. So by (*), $h' \leqq eg\mathcal{R}e$. So $h' \leqq_r e$ and $h'e \leqq e$. Now

$$h'e\mathcal{R}h' \leqq_r f.$$

So

$$h'e \leqq_r f \quad \text{and} \quad h'e \in M(e, f).$$

Hence $h'e \leqq h$. So $h'\mathcal{R}h'e \leqq h$ whereby $h' \leqq_r h$. Hence $h'\mathcal{R}h\mathcal{R}hg$. Now $hg \leqq eg$, $h' \leqq eg$. So by the dual of (*), $h' = hg$. Thus

$$\mathcal{S}(eg, fg) = \mathcal{S}(e, f)g \quad \text{whenever } e, f \leqq_r g.$$

Hence axiom (B4) of [**7**, p. 2] is also satisfied. It follows that E is a local semilattice.

2. Buildings. By a *complex* is meant a semilattice $\Omega = (\Omega, \leqq) = (\Omega, \wedge)$ with a minimum element 0 such that for all $\alpha \in \Omega$,

$$\alpha\Omega = \{\beta \in \Omega | \beta \leqq \alpha\}$$

is a finite Boolean lattice. The minimal elements of $\Omega \backslash \{0\}$ are called *vertices*. If $\alpha \in \Omega$, then the *rank* of α is defined to be the number of

https://doi.org/10.4153/CJM-1986-039-1 Published online by Cambridge University Press

vertices in $\alpha\Omega$. The maximum elements of Ω are called *chambers*. We will assume that all chambers are of the same rank d and that every element of Ω is $\leqq$ a chamber. We define the *rank* of Ω to be d. Let α, α' be chambers. We will assume that Ω is *connected* i.e., there exist chambers $\alpha = \alpha_0$, $\alpha_1, \dots, \alpha_m = \alpha'$ such that $\alpha_i \wedge \alpha_{i+1}$ has rank $d - 1$ for $i = 0, \dots, m - 1$. If m is minimal, then we set

$$\mathrm{dist}(\alpha, \alpha') = m.$$

An ideal of Ω is said to be a *subcomplex*. Ω is said to be *thick* if every element of rank $d - 1$ is less than at least three chambers. Ω is said to be *thin* if every element of rank $d - 1$ is less than exactly two chambers.

A (Tits) *building* is a pair $\Delta = (\Delta, \mathscr{A})$ where Δ is a complex and $\mathscr{A}$ is a family of finite subcomplexes called *apartments* such that

(1) Δ is thick.

(2) Each apartment Σ is thin.

(3) Any two elements of Δ belong to an apartment.

(4) If $\Sigma, \Sigma' \in \mathscr{A}$ and if $\alpha, \beta \in \Sigma \cap \Sigma'$, then there exists an isomorphism $\phi:\Sigma \to \Sigma'$ such that

$$\phi(\gamma) = \gamma \quad \text{for all } \gamma \in \alpha\Delta \cap \beta\Delta.$$

We refer to [**27**, Chapter 3, Section 3], [**28**, Section 3] for details. We will follow Tits [**28**]. Let $\Sigma \in \mathscr{A}$, α a chamber in Σ. Then there exists a unique retraction $\rho_\alpha:\Sigma \to \alpha\Sigma$, i.e., (i) $\rho_\alpha(\beta) = \beta$ for all $\beta \in \alpha\Sigma$ and (ii) ρ_α restricted to $\alpha'\Sigma$ is an isomorphism for any chamber $\alpha' \in \Sigma$. If $\beta, \beta' \in \Sigma$, then β, β' are said to be of the same *type*, type (β) = type (β'), if $\rho_\alpha(\beta) = \rho_\alpha(\beta')$. This concept is independent of the choice of the chamber α. If $\alpha \in \Sigma$ is a chamber, then there exists a unique $\alpha' \in \Sigma$ called the *opposite of* α *in* Σ such that $\mathrm{dist}(\alpha, \alpha')$ is maximum. There exists a unique automorphism $\mu:\Sigma \to \Sigma$ such that for any chamber α of Σ, α and $\mu(\alpha)$ are opposite. We then define β and $\mu(\beta)$ to be *opposite* for any $\beta \in \Sigma$. Now let $\beta, \beta' \in \Delta$. Then we define β, β' to be of the same *type*, type (β) = type (β'), if they are of the same type in some (and hence every) apartment containing them. β, β' are defined to be *opposite* $(\beta \perp \beta')$ if they are opposite in some (and hence every) apartment containing them. If $\alpha, \alpha', \beta, \beta' \in \Delta$ and if $\alpha \perp \alpha'$, $\beta \perp \beta'$, then type (α) = type (β) if and only if type (α') = type (β'). Let $\alpha, \beta \in \Delta$. Then by [**28**, Proposition 3.30], type (α) = type (β) if and only if there exists $\gamma \in \Delta$ with $\alpha \perp \gamma$, $\beta \perp \gamma$. If Δ is of rank 1, then any two non-zero elements have the same type and any two non-zero unequal elements are opposite.

It is easily seen that $(\Delta, \perp)$ satisfies the axioms of a parabolic semilattice, defined in Section 1. Hence we can construct the local semilattice E_Δ by Theorem 1.1. If

$$e = (\alpha, \alpha'), f = (\beta, \beta') \in E_\Delta,$$

then we define e, f to be of the same *type* ($e \sim f$) if type (α) = type (β) or equivalently type (α') = type (β'). It follows from the above that

$$\sim \; = \mathscr{R} \circ \mathscr{L} \circ \mathscr{R} = \mathscr{L} \circ \mathscr{R} \circ \mathscr{L}.$$

In particular if $e, f_1, f_2 \in E_\Delta$, $e \geqq f_i$, $i = 1, 2$ and if $f_1 \sim f_2$, then $f_1 = f_2$. Thus by [**9**, Corollary 1.5], $\prec E \succ$ has the property that $e \prec E \succ e$ is a semilattice for all $e \in E$. Also $\mathscr{U}(\prec E \succ) \cong E/\sim$ is clearly a Boolean lattice. We have shown,

THEOREM 2.1. (i) *E_Δ is a local semilattice.*

(ii) *$e \prec E_\Delta \succ e$ is a semilattice for all $e \in E_\Delta$.*

(iii) *$\sim \; = \mathscr{R} \circ \mathscr{L} \circ \mathscr{R} = \mathscr{L} \circ \mathscr{R} \circ \mathscr{L}$ on E_Δ and $\mathscr{U}(\prec E_\Delta \succ) \cong E_\Delta/\sim$ is a finite Boolean lattice.*

Let $\alpha \in \Delta$ be a chamber, $\beta \in \Delta$. Then by [**28**, Section 3.19] there exists a unique chamber $\alpha' \in \Delta$, $\alpha' \geqq \beta$ such that dist(α, α') is minimum. α' is denoted by $\text{proj}_\beta(\alpha)$. Let $\alpha, \alpha' \in \Delta$ be chambers, $\beta, \beta' \in \Delta$ be of rank $d - 1$. Suppose $\alpha > \beta$, $\alpha' > \beta'$, $\beta \perp \beta'$. Then by [**28**, Proposition 3.29] $\alpha \perp \alpha'$ if and only if $\text{proj}_\beta \alpha' \neq \alpha$. Let $E = E_\Delta$, $E_{\max}$ be the set of maximum elements of $(E, \leqq)$.

LEMMA 2.2. *Let $e = (\alpha, \alpha^-) \in E_{\max}$, $h = (\beta, \beta^-) \in E$, e covers h. Then there exists a unique $f^* = f^*(e, h) \in E_{\max}$ such that $ef^* = f^*e = h$ in $\prec E \succ$. Moreover*

$$f^* = (\text{proj}_\beta(\alpha^-), \text{proj}_{\beta^-}(\alpha)).$$

Let $f \in E_{\max}$, $f \geqq h$. Then $ef = h$ if and only if $f \mathscr{R} f^*$, and $fe = h$ if and only if $f \mathscr{L} f^*$.

Proof. Since $\alpha \perp \alpha^-$, we see that

$$\text{proj}_{\beta^-}(\alpha) \neq \alpha^-.$$

But by [**28**, Theorem 3.28],

$$\alpha^- = \text{proj}_{\beta^-}\, \text{proj}_\beta(\alpha^-).$$

Hence

$$\text{proj}_\beta(\alpha^-) \perp \text{proj}_{\beta^-}(\alpha).$$

So

$$f^* = (\text{proj}_\beta(\alpha^-), \text{proj}_{\beta^-}(\alpha)) \in E_{\max}.$$

Let

$$f = (\gamma, \gamma^-) \in E_{\max}, f > h.$$

Then $\gamma > \beta$, $\gamma^- > \beta^-$. Suppose $f \not\!\mathscr{R} f^*$. Then $\gamma \neq \text{proj}_\beta(\alpha^-)$. So $\gamma \perp \alpha^-$. Hence

$$e = (\alpha, \alpha^{-})\mathscr{L}(\gamma, \alpha^{-})\mathscr{R}(\gamma, \gamma^{-}) = f.$$

Hence $ef\mathscr{J}e$ in $\prec E \succ$. Thus $ef \neq h$. Next suppose that $f\mathscr{R}f^*$. Then

$$\gamma = \text{proj}_\beta \alpha^{-}.$$

So γ is not opposite to α^{-}. It follows that there is no $e_1 \in E$ with $e\mathscr{L}e_1\mathscr{R}f$. So $ef \not\!\mathscr{J} e$. Now

$$efh = hef = h.$$

Since J_e covers J_f, $ef\mathscr{J}h$. It follows that $ef = h$. This proves the lemma.

If $e, f \in E_{\max}$, then define $e\delta f$ if in $\prec E \succ$, $ef = fe$ is covered by e. Let δ^* denote the transitive closure of δ. Let

$$[e] = \{h \in E | h \leqq f, f\delta^* e \text{ for some } f \in E_{\max}\}.$$

Now let

$$e = (\alpha, \alpha^{-}) \in E_{\max}.$$

Let Σ be the unique apartment of Δ containing α, α^{-}. If $\beta \in \Sigma$, then let β^{-} denote the unique opposite of β in Σ. Let

$$\hat{\Sigma} = \{ (\beta, \beta^{-}) | \beta \in \Sigma\}.$$

By Lemma 2.2, $[e] = \hat{\Sigma}$. So

$$([e], \leqq) \cong \Sigma.$$

Let $\lambda: E \to E/\mathscr{R}$ denote the natural map. Let

$$\mathscr{A}' = \{\lambda([e]) | e \in E_{\max}\}.$$

We then clearly have,

THEOREM 2.3. *$(E/\mathscr{R}, \mathscr{A}')$ is a building isomorphic to* $(\Delta, \mathscr{A})$.

Let Aut*E denote the group of all automorphisms ϕ of E such that $e \sim e\phi$ for all $e \in E$. Let Aut*$\prec E \succ$ denote the group of all automorphisms of the semigroup $\prec E \succ$ such that $a\mathscr{J}a\phi$ for all $a \in \prec E \succ$. Let Aut*Δ denote the group of all automorphisms ϕ of Δ such that type (α) = type $(\alpha\phi)$ for all $\alpha \in \Delta$.

THEOREM 2.4. $\text{Aut}^*E_\Delta \cong \text{Aut}^* \prec E_\Delta \succ \cong \text{Aut}^*\Delta$.

Proof. That $\text{Aut}^*E \cong \text{Aut}^* \prec E \succ$ follows from [7]. So we need to show that

$$\text{Aut}^*E \cong \text{Aut}^*\Delta.$$

First let $\phi \in \text{Aut}^*\Delta$. Then $\bar{\phi} \in \text{Aut}^*E$ where

$$(\alpha, \alpha')\bar{\phi} = (\alpha\phi, \alpha'\phi).$$

Conversely let $\psi \in \text{Aut}^*E$. Then for all $e, e' \in E$, $e\mathscr{R}e'$ if and only if $e\psi\mathscr{R}e'\psi$ and $e\mathscr{L}e'$ if and only if $e\psi\mathscr{L}e'\psi$. It follows that there exist ϕ_1, $\phi_2 \in \text{Aut}^*\Delta$ such that

$$(\alpha, \beta)\psi = (\alpha\phi_1, \beta\phi_2) \quad \text{for all } (\alpha, \beta) \in E.$$

We claim that $\phi_1 = \phi_2$. For suppose $\alpha\phi_1 \neq \alpha\phi_2$ for some $\alpha \in \Delta$. Then $\alpha\phi_1\phi_2^{-1} \neq \alpha$. Now $\alpha, \alpha\phi_1\phi_2^{-1} \in \Sigma$ for some apartment Σ. So $\alpha \perp \beta$ for a unique $\beta \in \Sigma$. Then $\alpha\phi_1\phi_2^{-1}$ is not opposite to β. So $\alpha\phi_1$ is not opposite to $\beta\phi_2$. Thus $(\alpha, \beta) \in E$, $(\alpha\phi_1, \beta\phi_2) \notin E$. But

$$(\alpha\phi_1, \beta\phi_2) = (\alpha, \beta)\psi \in E,$$

a contradiction. Hence $\phi_1 = \phi_2$. It follows that

$$\text{Aut}^*E \cong \text{Aut}^*\Delta.$$

This proves the theorem.

Many important classes of groups, including reductive algebraic groups and finite simple groups of Lie type, admit what has come to be called a Tits system (see [**27**, Section 3.3], [**3**, Section 29]). A *Tits system* is a quadruple (G, B, N, S) where G is a group, B, N are subgroups of G generating G, $T = B \cap N \lhd N$, S a generating set of order 2 elements of $W = N/T$ such that

(1) $\rho B\rho \neq B$ for any $\rho \in S$

(2) $\sigma B\rho \subseteq B\sigma B \cup B\sigma\rho B$ for all $\sigma \in W$, $\rho \in S$.

We will assume that the Weyl group W is finite. If $I \subseteq S$, then let

$$W_I = \langle I \rangle,\ P_I = BW_IB.$$

The P_I's are exactly the subgroups of G containing B. For any $x \in G$, I, $I' \subseteq S$, we have by [**28**, Section 3.2.3] that $x^{-1}P_Ix \subseteq P_{I'}$, if and only if $x \in P_{I'}$, $I \subseteq I'$. Thus for $x, y \in G$,

$$x^{-1}P_Ix \subseteq y^{-1}P_{I'}y$$

if and only if $P_Ix \subseteq P_{I'}y$. Let

$$\Sigma = \{\sigma^{-1}P_I\sigma | \sigma \in W, I \subseteq S\}.$$

Let

$$\mathscr{A} = \{x^{-1}\,\Sigma\, x | x \in G\}.$$

Let

$$\Delta = \{x^{-1}P_Ix | x \in G, I \subseteq S\}.$$

If $P_1, P_2 \in \Delta$, then define $P_1 \geqq P_2$ if $P_1 \subseteq P_2$. Then by [**28**, Theorem 3.2.6], $\Delta = \Delta_G = (\Delta, \mathscr{A})$ is a building which we call the *building of* G. The elements of Δ are called *parabolic subgroups* of G. Two parabolic subgroups are of the same *type* if and only if they are conjugate. The conjugates of B are called *Borel subgroups*. We will call $E_G = E_\Delta$ the *local semilattice of* G.

3. Algebraic monoids. Let K be an algebraically closed field and $\mathcal{M}_n(K)$ the monoid of all $n \times n$ matrices over K. A (Zariski) closed and irreducible submonoid of $\mathcal{M}_n(K)$ will be called *connected*. Let M be a connected monoid with zero and group of units G. Then by [**16**] and [**21**], G is a reductive group if and only if M is a regular semigroup. The theory of connected regular monoids with zero is being developed by the author [**12**]-[**20**] and Renner [**21**]-[**24**].

Let M be a connected regular monoid with zero and group of units G. Let $E = E(M)$ denote the biordered set of idempotents of M. For the purposes of this section, we need only consider the weaker system $(E, \leqq_r, \leqq_l)$ where $f \leqq_r e$ if $ef = f$, $f \leqq_l e$ if $fe = f$. As usual,

$$\mathcal{R} = \leqq_r \cap (\leqq_r)^{-1}, \mathcal{L} = \leqq_l \cap (\leqq_l)^{-1} \text{ and } \leqq = \leqq_r \cap \leqq_l.$$

We wish to show that E_G (and hence the building Δ_G of G) is completely determined by E. The length of (any) maximal chain in $(E, \leqq)$ is called the *rank* of E. If $e, f \in E$, we define $e \sim f$ if there exist $e', f' \in E$ such that

$$e\mathcal{R}e'\mathcal{L}f'\mathcal{R}f.$$

By [**13**, Lemma 1.12], $e \sim f$ if and only if $e\mathcal{J}f$ in M. If $e, f \in E$, then $J_e \geqq J_f$ if and only if $e \geqq f'$ for some $f' \in E$ with $f \sim f'$. Thus the finite lattice $\mathcal{U} = \mathcal{U}(M) \cong E/\sim$ is completely determined by E.

LEMMA 3.1. *Let $e, h, f \in E$ such that $e > h > f$, e covers h covers f. Then there exists a unique $h^* = h^*(e, f) \in E$ such that $e > h^* > f$ and $hh^* = h^*h = f$. Let $h_1 \in E$, $e > h_1 > f$. Then $h_1 \neq h^*$ if and only if there exists $h_2 \in E$, $e > h_2 > f$ such that either $h\mathcal{R}h_2\mathcal{L}h_1$ or $h\mathcal{L}h_2\mathcal{R}h_1$. In particular h^* is determined by E.*

Proof. By general considerations [**14**, Theorems 3, 11], we reduce to the case when $e = 1, f = 0$. If G is a torus, then $|E| = 4$, and the lemma is trivial. Otherwise dim $M = 4$ and the width of h is 2. We are then done by [**18**, Theorem 13].

A useful concept in the theory of linear algebraic monoids is that of cross-section lattices [**15**], [**17**], [**19**]. A subset Λ of E is a *cross-section lattice* if (i) for all $e \in E$ there exists a unique $e' \in \Lambda$ such that $e \sim e'$ and (ii) for all $e, f \in \Lambda$, $e \geqq f$ if and only if $J_e \geqq J_f$. If $\Gamma \subseteq \Lambda$, then we let

$$C^r_G(\Gamma) = \{a \in G | ae = eae \text{ for all } e \in \Gamma\},$$

$$C^l_G(\Gamma) = \{a \in G | ea = eae \text{ for all } e \in \Gamma\},$$

$$C_G(\Gamma) = \{a \in G | ae = ea \text{ for all } e \in \Gamma\}.$$

Fix a cross-section lattice Λ of E. Let

$$B = C_G^r(\Lambda),\ B^- = C_G^l(\Lambda),\ T = C_G(\Lambda).$$

By [**17**, Theorem 10], [**19**, Theorem 1.2], B, B^- are opposite Borel subgroups of G with respect to the maximal torus T. Any parabolic subgroup P of G containing B is of the form $P = C_G^r(\Gamma)$ for some chain $\Gamma \subseteq \Lambda$. Moreover $P^- = C_G^l(\Gamma)$ is the parabolic subgroup G opposite to P with respect to T. See [**18**, Theorem 4], [**19**, Theorem 2.7]. Let $[\Lambda]$ denote the smallest subset of E containing Λ such that for all e, h, $f \in [\Lambda]$ with e covering h covering f in $(E, \leqq)$, we have $h^*(e, f) \in [\Lambda]$ where $h^*(e, f)$ is as in Lemma 3.1. Since $E(\bar{T})$ is a finite relatively complemented lattice, we see that $[\Lambda] = E(\bar{T})$. The point here is that $[\Lambda]$ is determined by E. Clearly

$$\mathscr{E} = \mathscr{E}_\Lambda = ([\Lambda], \leqq, \sim)$$

is the $\mathscr{E}$-structure of M studied by the author [**15**, Section 3]. In particular by [**15**, Theorem 3.9], the Weyl group $W = N_G(T)/T$ is recovered from $\mathscr{E}$ as the group of all permutations σ of $[\Lambda]$ such that (i) $e \leqq f$ if and only if $e^\sigma \leqq f^\sigma$ for any $e, f \in [\Lambda]$, and (ii) $e \sim e^\sigma$ for all $e \in [\Lambda]$. Let

$$S_\Lambda = \{\sigma \in W | \sigma \neq 1, \sigma^2 = 1,$$
$$\sigma \text{ fixes a chain of length rank } E - 1 \text{ in } \Lambda\}.$$

By [**19**, Corollary 2.8], S_Λ is just the set of simple reflections with respect to the Borel subgroup B and maximal torus T. Let $I \subseteq S_\Lambda$, W_I the subgroup of W generated by I. Then by [**3**, Theorem 29.3], $P_I = BW_IB$ is the (unique) parabolic subgroup of G containing B and having W_I as its Weyl group. We let

$$\Lambda_I = \{e \in \Lambda | e^\sigma = e \text{ for all } \sigma \in I\},$$

$$\mathscr{U}_I = \{J_e | e \in \Lambda_I\}.$$

By [**19**, Theorem 2.7], $P_I = C_G^r(\Gamma)$ for some chain $\Gamma \subseteq \Lambda$. By [**15**, Theorem 2.3], $\Gamma \subseteq \Lambda_I$. Clearly

$$B \subseteq C_G^r(\Lambda_I) \subseteq C_G^r(\Gamma) = P_I$$

and W_I is contained in the Weyl group of $C_G^r(\Lambda_I)$. It follows that

$$P_I = C_G^r(\Lambda_I).$$

Similarly

$$P_I^- = B^- W_I B^- = C_G^l(\Lambda_I).$$

Let

$$\mathscr{U}^* = \{\mathscr{U}_I | I \subseteq S_\Lambda\}.$$

If I, $I' \subseteq S_\Lambda$, then

$$\mathscr{U}_I \cap \mathscr{U}_{I'} = \mathscr{U}_{I \cup I'}.$$

Also $I \subseteq I'$ if and only if $\mathscr{U}_{I'} \subseteq \mathscr{U}_I$. Note also that $\mathscr{U}^*$ is a family of sublattices of $\mathscr{U}$. Since any two cross-section lattices of E are conjugate by [**17**, Theorems 10, 12], we see that $\mathscr{U}^*$ is independent of the particular choice of the cross-section lattice Λ.

If $\mathscr{V} \in \mathscr{U}^*$ and if Λ is any cross-section lattice of E, then we let

$$\Lambda_{\mathscr{V}} = \{e \in \Lambda | J_e \in \mathscr{V}\}.$$

Let

$$\hat{E} = \{\Lambda_{\mathscr{V}} | \Lambda \text{ is a cross-section lattice of } E, \mathscr{V} \in \mathscr{U}^*\}.$$

If $A, A' \in \hat{E}$, then define $A \leqq_r A'$ if for all $e \in A$ there exists (necessarily unique) $e' \in A'$ such that $e\mathscr{R}e'$. Similarly we define $A \leqq_l A'$ if for all $e \in A$, there exists $e' \in A'$ such that $e\mathscr{L}e'$. As usual we let

$$\leqq \ = \ \leqq_r \cap \leqq_l, \mathscr{R} = \leqq_r \cap (\leqq_r)^{-1}, \mathscr{L} = \leqq_l \cap (\leqq_l)^{-1}.$$

If $A \in \hat{E}$, then we define

$$\mathit{type}\,(A) = \{J_e | e \in A\} \in \mathscr{U}^*.$$

Let $A, A' \in \hat{E}$, $A \leqq_r A'$. Then type $(A) \subseteq$ type A'. We define

$$A'A = A, \quad AA' = \{f \in A' | J_f \in \text{type}\,(A)\}.$$

Similarly if $A \leqq_l A'$, we define

$$AA' = A, \quad A'A = \{f \in A' | J_f \in \text{type}\,(A)\}.$$

Note that $A \leqq A'$ if and only if $A \subseteq A'$. Define

$$\theta: \hat{E} \to E_G \text{ as } \theta(A) = (C^r_G(A), C^l_G(A)).$$

THEOREM 3.2. *$\hat{E}$ is a local semilattice and $\theta: \hat{E} \cong E_G$ is a type preserving isomorphism.*

Proof. It is clear from the preceding discussion that θ is a surjection. Let $A, A' \in \hat{E}$. Suppose $A \leqq_r A'$. Then $A \subseteq \Lambda$, $A' \subseteq \Lambda'$ for some cross-section lattices Λ, Λ'. By [**17**], $x\Lambda x^{-1} = \Lambda'$ for some $x \in G$. Since type $(A) \subseteq$ type (A'), we see that $xAx^{-1} \subseteq A'$. Thus

$$xex^{-1}\mathscr{R}e \quad \text{for all } e \in A.$$

Hence

$$xe = exe \quad \text{for all } e \in A \text{ and } x \in C^r_G(A).$$

Thus

$$C^r_G(A') \subseteq C^r_G(xAx^{-1}) = xC^r_G(A)x^{-1} = C^r_G(A).$$

Thus $\theta(A) \leqq_r \theta(A')$. Clearly $\theta(A')\theta(A) = \theta(A)$. Since $A\mathscr{R}AA' \subseteq A'$, we have

$$\theta(AA') = (C^r_G(A), C^l_G(AA')), C^l_G(A') \subseteq C^l_G(AA').$$

It follows that

$$\theta(A)\theta(A') = \theta(AA').$$

Next assume that $A, A' \in \hat{E}$, $\theta(A) \leqq_r \theta(A')$. Let $\mathscr{V}$ = type (A), $\mathscr{V}'$ = type A'. Now $A \subseteq \Lambda$, $A' \subseteq \Lambda'$ for some cross-section lattices Λ, Λ'. So $A = \Lambda_{\mathscr{V}}$; $A' = \Lambda'_{\mathscr{V}'}$. Now $x\Lambda x^{-1} = \Lambda'$ for some $x \in G$. So

$$xC^r_G(\Lambda_{\mathscr{V}'})x^{-1} = C^r_G(\Lambda'_{\mathscr{V}'}) = C^r_G(A') \subseteq C^r_G(A) = C^r_G(\Lambda_{\mathscr{V}}).$$

By [**28**, Section 3.2.3], $x \in C^r_G(\Lambda_{\mathscr{V}})$ and $\mathscr{V} \subseteq \mathscr{V}'$. So for all $e \in \Lambda_{\mathscr{V}}$;

$$e\mathscr{R}xex^{-1} \in \Lambda'_{\mathscr{V}} \subseteq \Lambda'_{\mathscr{V}'}.$$

Hence

$$A = \Lambda_{\mathscr{V}} \leqq_r \Lambda'_{\mathscr{V}'} = A'.$$

The dual statements concerning $\leqq_l$ are similarly proved. In particular for $A, A' \in \hat{E}$, $\theta(A) = \theta(A')$ implies $A \leqq A' \leqq A$. Thus θ is an isomorphism. Let $A, A \in \hat{E}$,

$$\text{type } (A) = \text{type } (A') = \mathscr{V}.$$

Let $A \subseteq \Lambda$, $A' \subseteq \Lambda'$ where Λ, Λ' are cross-section lattices. Then $A = \Lambda_{\mathscr{V}}$; $A' = \Lambda'_{\mathscr{V}}$. Now $x^{-1}\Lambda x = \Lambda'$ for some $x \in G$. So

$$x^{-1}Ax = A' \quad \text{and} \quad x^{-1}C^r_G(A)x = C^r_G(A').$$

So $\theta(A)$, $\theta(A')$ are of the same type. Assume conversely that $A, A' \in \hat{E}$ such that $\theta(A)$, $\theta(A')$ have the same type. So

$$x^{-1}C^r_G(A)x = C^r_G(A') \quad \text{for some } x \in G.$$

Thus

$$\theta(x^{-1}Ax)\mathscr{R}\theta(A').$$

Hence by the above, $x^{-1}Ax\mathscr{R}A'$. So

$$\text{type } (A) = \text{type } (x^{-1}Ax) = \text{type } (A').$$

This proves the theorem.

If $A, A' \in \hat{E}$, then define $A \sim A'$ if type (A) = type (A'). Then we have by Theorems 2.1, 3.2,

COROLLARY 3.3. *In* $\hat{E}$, $\sim \,= \mathscr{R} \circ \mathscr{L} \circ \mathscr{R} = \mathscr{L} \circ \mathscr{R} \circ \mathscr{L}$.

Consider the natural map $\xi:\hat{E} \to \hat{E}/\mathscr{R}$. If Λ is a cross-section lattice of E, then let

$$\Sigma_\Lambda = \{\Lambda_{\mathscr{V}} | \mathscr{V} \in \mathscr{U}^*\}.$$

Let

$$\mathscr{A}' = \{\xi(\Sigma_\Lambda) \mid \Lambda \text{ is cross-section lattice of } E\}.$$

Then we have,

COROLLARY 3.4. *$(\hat{E}/\mathscr{R}, \mathscr{A}')$ is a building isomorphic to Δ_G.*

Aut*E is the group of all permutations ϕ of E such that (i) $f \leqq_r e$ if and only if $f\phi \leqq_r e\phi$ for all $e, f \in E$, (ii) $f \leqq_l e$ if and only if $f\phi \leqq_l e\phi$ for all $e, f \in E$, and (iii) $e \sim e\phi$ for all $e \in E$. Aut*$\hat{E}$ is the group of all permutations ϕ of $\hat{E}$ such that (i) $A \leqq_r A'$ if and only if $A\phi \leqq_r A'\phi$ for all $A, A' \in \hat{E}$, (ii) $A \leqq_l A'$ if and only if $A\phi \leqq_l A'\phi$ for all $A, A' \in \hat{E}$, and (iii) type (A) = type $(A\phi)$ for all $A \in \hat{E}$.

THEOREM 3.5. Aut*$E \cong$ Aut*$\hat{E} \cong$ Aut*$E_G \cong$ Aut*Δ_G.

Proof. That Aut*$\hat{E} \cong$ Aut*$E_G \cong$ Aut*Δ_G follows from Theorems 2.4, 3.2. So we need to show that Aut*$E \cong$ Aut*$\hat{E}$. Let $\psi \in$ Aut*E. Define $\hat{\psi}:\hat{E} \to \hat{E}$ as

$$A\hat{\psi} = \{e\psi \mid e \in A\} \in \hat{E}.$$

It is routinely verified that $\hat{\psi} \in$ Aut*$\hat{E}$ and that the map $\psi \to \hat{\psi}$ is a homomorphism. Now let $\phi \in$ Aut*$\hat{E}$, $e \in E$. Let $\mathscr{U}(e)$ denote the smallest element of $\mathscr{U}^*$ containing J_e. Let Λ be a cross-section lattice of E with $e \in \Lambda$. Let

$$I = \{\sigma \in S_\Lambda \mid e^\sigma = e\}.$$

Clearly

$$e \in \Lambda_I = \Lambda_{\mathscr{U}(e)}.$$

If $B = C_G^r(\Lambda)$ then

$$C_G^r(e) = BW_IB = C_G^r(\Lambda_I) = C_G^r(\Lambda_{\mathscr{U}(e)}).$$

Similarly

$$C_G^l(e) = C_G^l(\Lambda_{\mathscr{U}(e)}).$$

If Λ' is another cross-section lattice with $e \in \Lambda'$, then

$$\begin{aligned}(C_G^r(\Lambda'_{\mathscr{U}(e)}), C_G^l(\Lambda'_{\mathscr{U}(e)})) &= (C_G^r(e), C_G^l(e)) \\ &= (C_G^r(\Lambda_{\mathscr{U}(e)}), C_G^l(\Lambda_{\mathscr{U}(e)})).\end{aligned}$$

By Theorem 3.2,

$$\Lambda_{\mathscr{U}(e)} = \Lambda'_{\mathscr{U}(e)}.$$

Hence $\Lambda_{\mathscr{U}(e)}$ is independent of the choice of the cross-section lattice Λ containing e. Let

$$\Gamma_e = \Lambda_{\mathscr{U}(e)} \in \hat{E}.$$

Now

$$\text{type } \Gamma_e = \text{type } \Gamma_e\phi.$$

Hence there exists a unique $e\bar{\phi} \in \Gamma_e\phi$ with $e \sim e\bar{\phi}$. Clearly

$$(e\bar{\phi})\overline{\phi^{-1}} = e.$$

Hence $\bar{\phi}:E \to E$ is a bijection. Let $e, f \in E$. Suppose $e\mathscr{R}f$. Then by [**12**, Theorems 1, 9], $C_G^r(e) = C_G^r(f)$. Hence

$$C_G^r(\Gamma_e) = C_G^r(\Gamma_f).$$

So by Theorem 3.2, $\Gamma_e\mathscr{R}\Gamma_f$. Hence

$$\Gamma_e\phi\mathscr{R}\Gamma_f\phi.$$

So $e\bar{\phi}\mathscr{R}f\bar{\phi}$. Next assume that $e \geqq f$. Then by [**15**, Theorem 6.2], there exists a cross-section lattice Λ of E such that $e, f \in \Lambda$. Since $\Gamma_e \leqq \Lambda$, we have

$$e\bar{\phi} \in \Gamma_e\phi \leqq \Lambda\phi.$$

So $e\bar{\phi} \in \Lambda\phi$. Similarly $f\bar{\phi} \in \Lambda\phi$. Since $e \sim e\bar{\phi}$, $f \sim f\bar{\phi}$, we see that $e\bar{\phi} \geqq f\bar{\phi}$. Next suppose that $f \leqq_r e$. Then for some $f' \in E$, $f\mathscr{R}f' \leqq e$. So

$$f\bar{\phi}\mathscr{R}f'\bar{\phi} \leqq e\bar{\phi}.$$

Hence $f\bar{\phi} \leqq_r e\bar{\phi}$. Conversely if $f\bar{\phi} \leqq_r e\bar{\phi}$, then

$$f = (f\bar{\phi})\overline{\phi^{-1}} \leqq_r (e\bar{\phi})\overline{\phi^{-1}} = e.$$

Similarly $f \leqq_l e$ if and only if

$$f\bar{\phi} \leqq_l e\bar{\phi}.$$

Hence $\bar{\phi} \in \mathrm{Aut}^*E$. Let Λ be a cross-section lattice of E. Then clearly

$$\Lambda\phi = \{e\bar{\phi} | e \in \Lambda\}.$$

Let $\mathscr{V} \in \mathscr{U}^*$. Then $\Lambda_{\mathscr{V}}\phi \leqq \Lambda\phi$, type $(\Lambda_{\mathscr{V}}\phi) = \mathscr{V}$. So

$$\Lambda_{\mathscr{V}}\phi = \{e\bar{\phi} | e \in \Lambda_{\mathscr{V}}\}.$$

Thus for all $A \in \hat{E}$,

$$A\phi = \{e\bar{\phi} | e \in A\}.$$

Hence the maps $\phi \to \bar{\phi}$, $\psi \to \hat{\psi}$ are inverses of each other. This proves the theorem.

Let R denote the radical of G. Then $G = RG_1 \ldots G_m$, where $G_1, \ldots, G_m$ are simple algebraic groups, $(G_i, G_j) = 1$ for $i \neq j$ (see [**3**, Theorem 27.5]). Let C_i denote the center of G_i, $G_i' = G_i/C_i$. Then

$$\Delta_{G_i} \cong \Delta_{G_i'}.$$

If the rank of $G_i \geqq 2$, then by a theorem of Tits [**28**, Corollaries 5.9, 5.10], $\mathrm{Aut}^*\Delta_{G'_i}$ is an extension of G'_i by Aut K. Here Aut K denotes the automorphism group of K. Now

$$\mathrm{Aut}^*\Delta_G \cong \mathrm{Aut}^*\Delta_{G_1} \times \ldots \times \mathrm{Aut}^*\Delta_{G_m}.$$

Let C denote the center of G. We then clearly have,

THEOREM 3.6. *Suppose no reflection in W is in the center of W (i.e., each G_i has rank $\geqq 2$). Then* Aut^*E *is an extension of G/C by the m-fold direct product* Aut $K \times \ldots \times$ Aut K.

Besides being a biordered set, E is a closed subset of M. Let $\mathrm{Aut}^{**}(E)$ denote the subgroup of Aut^*E consisting of those ϕ which are also automorphisms of the affine variety E.

CONJECTURE 3.7. $\mathrm{Aut}^{**}(E) \cong G/C$.

THEOREM 3.8. *Let $S = M\backslash G$. Then $E(S) \cong E_G$ if and only if $\mathcal{U}(S)$ is a Boolean lattice. In such a case, S is a locally inverse semigroup.*

Proof. Suppose first that $\mathcal{U}(S)$ is a Boolean lattice. By [**14**, Theorem 14], for any $e \in E(S)$, H_e is a torus. In particular $eSe = \bar{H}_e$ is commutative. Hence S is a locally inverse semigroup. It is also clear that for any $e, f_1, f_2 \in E(S)$, $e \geqq f_1$, $e \geqq f_2$, $f_1\mathcal{J}f_2$ imply $f_1 = f_2$. Define $\phi:E(S) \rightarrow E_G$ as

$$\phi(e) = (C^r_G(e), C^l_G(e)).$$

Let $e, f \in E(S)$. Suppose $f \leqq_r e$. Then by [**12**, Theorem 1],

$$f \in C^r_G(e).$$

Let B be any Borel subgroup of $C^r_G(e)$ and let T be a maximal torus of B. By [**19**, Theorem 1.2], $B = C^r_G(\Lambda)$ for some cross-section lattice $\Lambda \subseteq E(\bar{T})$. There exists $a \in C^r_G(e)$ such that

$$e' = aea^{-1} \in E(\bar{T}).$$

Then $e\mathcal{R}e'$. Hence by [**12**, Theorems 1, 9],

$$C^r_G(e) = C^r_G(e').$$

So $B \subseteq C^r_G(e')$. By [**19**, Theorem 1.2], $e' \in \Lambda$. Let $f' \in J_f \cap \Lambda$. Then since $J_e \geqq J_f$, we have $e' \geqq f'$. Since $f \leqq_r e'$, we have

$$f\mathcal{R}fe' \leqq e'.$$

Hence $e' \geqq f'$, $e' \geqq fe'$, $f'\mathcal{J}fe'$ in S. Thus $f' = fe$. Hence $f\mathcal{R}f'$. So

$$B \subseteq C^r_G(f') = C^r_G(f).$$

Since B is an arbitrary Borel subgroup of $C^r_G(e)$, we see that

$$C^r_G(e) \subseteq C^r_G(f).$$

So $\phi(f) \leqq_r \phi(e)$. Clearly

$$\phi(e)\phi(f) = \phi(f).$$

Now $f\mathscr{R}fe \leqq e$. So

$$\phi(fe) = (C^r_G(fe), C^l_G(fe)), C^r_G(fe) = C^r_G(f), C^l_G(fe) \supseteq C^l_G(e).$$

Thus $\phi(f)\phi(e) = \phi(fe)$.

Assume now that $e, f \in E(S)$. $\phi(f) \leqq_r \phi(e)$. Then

$$C^r_G(e) \subseteq C^r_G(f).$$

Let T be a maximal torus of $C^r_G(e)$ with $e \in E(\bar{T})$. Let J denote the maximum $\mathscr{J}$-class of S,

$$A = \{h \in J \cap E(\bar{T}) \,|h \geqq e\} = \{h_1, \ldots, h_k\}.$$

Since $E(\bar{T})$ is a relatively complemented lattice, $e = h_1 \ldots h_k$. There exists $a \in C^r_G(f)$ such that

$$f' = afa^{-1} \in E(\bar{T}).$$

Then $f\mathscr{R}f'$. Let $h \in A$. Then $h \geqq e$. So by [**15**, Theorem 6.2], there exists a cross-section lattice $\Lambda \subseteq E(\bar{T})$ such that $e, h \in \Lambda$. So

$$B = C^r_G(\Lambda) \subseteq C^r_G(e) \subseteq C^r_G(f) = C^r_G(f').$$

So by [**19**, Theorem 1.2], $f' \in \Lambda$. Since $J \geqq J_f$, we see that $h \geqq f'$. So

$$e = h_1 \ldots h_k \geqq f'.$$

Hence $f \leqq_r e$. Similarly $\phi(f) \leqq_l \phi(e)$ if and only if $f \leqq_l e$. In particular ϕ is injective. Now let $(P, P^-) \in E_G$. Then by [**19**, Theorem 2.7], there exists a chain Γ in $E(S)$ such that

$$P = C^r_G(\Gamma), P^- = C^l_G(\Gamma).$$

Let e denote the maximum element of Γ. Then by the above,

$$(P, P^-) = (C^r_G(e), C^l_G(e)) = \phi(e).$$

Hence $E(S) \cong E_G$. Conversely if $E(S) \cong E_G$, then by Theorem 2.1, $\mathscr{U}(S) \cong E(S)/\sim$ is a Boolean lattice. This proves the theorem.

Remark. Renner [**25**] has been studying algebraic monoids M for which $\mathscr{U}(M)\setminus\{0\}$ is a Boolean lattice. Thus the monoids encountered in Theorem 3.8 are dual to these.

Let G be a reductive group, $\phi:G \to GL(n, K)$ a representation. Let

$$M(\phi) = \overline{K\phi(G)} \subseteq \mathscr{M}_n(K)$$

denote the Zariski closure of $K\phi(G)$ in $\mathscr{M}_n(K)$. We call $M(\phi)$ the *monoid of* ϕ. We call $E(\phi) = E(M(\phi))$ the *biordered set of* ϕ. $E(\phi)$ is a geometrical object which, in light of the results of this paper, may be viewed as a generalized building. Thus the same group G gives rise to an infinite number of biordered sets $E(\phi)$. We conjecture that for irreducible representations ϕ, the biordered sets $E(\phi)$ are finite in number.

For a finite simple group G of Lie type we get one finite biordered set (a local semilattice) directly from its Tits system. Getting other natural finite biordered sets related to the representations of G and finding geometrical interpretations for them, remains an important open problem.

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