Proceedings of the Edinburgh Mathematical Society (1990) 33, 367–379 ©

LOCAL STRUCTURE OF SOME $OUT(F_n)$-COMPLEXES

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(Received 19th July 1988, revised 25th October 1989)

In previous work of the author and M. Culler, contractible simplicial complexes were constructed on which the group of outer automorphisms of a free group of finite rank acts with finite stabilizers and finite quotient. In this paper, it is shown that these complexes are Cohen–Macauley, a property they share with buildings. In particular, the link of a vertex in these complexes is homotopy equivalent to a wedge of spheres of codimension 1.

1980 *Mathematics subject classification* (1985 Revision): 20J05.

0. Introduction

Let $Out(F_n)$ be the group of outer automorphisms of a finitely generated free group. In [1] a contractible space $X=X(n)$ was constructed on which $Out(F_n)$ acts discretely with finite stabilizers. This space may be thought of as analogous to the homogeneous space of an algebraic group, with a discrete action by an arithmetic subgroup, or to the Teichmüller space of a surface with the action of the mapping class group of the surface. Two $Out(F_n)$-invariant deformation retracts $K=K(n)$ and $L=L(n)$ of X were also described in [1]; these are locally finite simplicial complexes with finite quotient. In [1] the complex K was used to prove cohomological finiteness properties of the group $Out(F_n)$. In particular, it was shown that $Out(F_n)$ is VFL and has virtual cohomological dimension $2n$–3.

In contrast with homogeneous spaces and Teichmüller spaces, the space X is not a manifold, and standard methods in manifold theory, such as Poincaré duality, cannot be used to study the group action and quotient space. Borel and Serre encountered the same difficulty when studying S-arithmetic groups, where the role of the homogeneous space is played by a Euclidean building. Euclidean buildings are simplicial complexes, but are not triangulated manifolds; in particular, the link of a vertex is not homeomorphic to a sphere of codimension 1. However, there is a uniform local structure to buildings which makes them homotopically similar to manifolds: the link of each vertex is homotopy equivalent to a wedge of spheres of codimension 1. The purpose of this paper is to show that the simplicial complexes K and L have similar local properties.

I would like to thank the referee for helpful comments and for pointing out an error in the original version of this paper.

*Partially supported by a grant from the National Science Foundation.

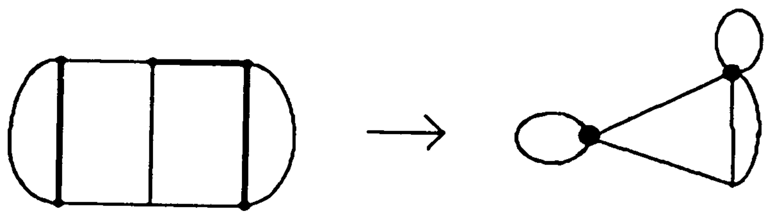


FIGURE 1. Forest collapse.

1. Background

We briefly recall from [1] the definition of the complexes K and L and some basic properties. Let R_0 be an n-leafed *rose*, i.e. a connected graph with one vertex and n edges. Vertices of L are equivalence classes of pairs (g, G), where G is a connected graph with vertices of valence at least 3, and g is a homotopy equivalence from R_0 to G. Two pairs (g, G) and (g', G') are equivalent if there is a homeomorphism $h: G \to G'$ such that $h \circ g \simeq g'$. Vertices $v_0, \ldots, v_k$ of L span a k-simplex if representatives $(g_0, G_0), \ldots, (g_k, G_k)$ can be chosen so that G_i is obtained from G_{i-1} by collapsing each component of a forest in G_{i-1} to a point, and g_i is the composition of g_{i-1} with the collapsing map. Here a *forest* in G is a subset of the edges of G which contains no cycle. This operation is called a *forest collapse* (see Fig. 1).

The complex L can be thought of as the geometric realization of the poset (partially ordered set) of its vertices, where the partial ordering is $(g, G) \geqq (g', G')$ if (g', G') can be obtained from (g, G) by a forest collapse.

An edge ε of a graph G is called a *bridge* if G minus the interior of ε is disconnected. There is a deformation retraction of L onto the subcomplex K spanned by points (g, G) such that G has no bridges.

In [1] it is shown that K and L are contractible of dimensions $2n-3$, and $Out(F_n)$ acts on K and L with finite stabilizers and finite quotient. This implies the cohomological finiteness results mentioned in the introduction.

2. The Cohen–Macauley property for L

We recall some standard facts about posets. We refer to [2] for a more complete discussion and proofs.

Let P be a poset, and let $p \in P$. The *height of* p, $\mathrm{ht}(p)$, is the length of the longest totally ordered chain of elements of P which are all less than p. The height of the poset P is the maximum of the heights of its elements.

A poset is said to be k-*spherical* if its geometric realization is k-dimensional and $(k-1)$-connected. Note that the geometric realization of a k-spherical poset is homotopy equivalent to a bouquet of k-spheres.

Definition A poset P of height h is *Cohen–Macauley* if P is h-spherical and the link of every simplex of dimension k in the geometric realization of P is $(h-k-1)$-spherical.

Define subposets $P_{>p}=\{s\in P:s<p\}$, $P_{<p}=\{s\in P:s>p\}$ and $(p,q)=\{s\in P:q<s<p\}$. The Cohen–Macauley property for P is equivalent to the following properties:

(i) P is h-spherical,
(ii) $P_{>p}$ is $(\mathrm{ht}(p)-1)$-spherical for each $p\in P$, and
(iii) $P_{<p}$ is $(h-\mathrm{ht}(p)-1)$-spherical for each $p\in P$.
(iv) (p,q) is $(\mathrm{ht}(q)-\mathrm{ht}(p)-1)$-spherical for every $p<q$ in P.

The following standard lemma is useful in determining the homotopy type of a poset. A map $f:P\to P$ is a *poset map* if $p\leqq q$ implies $f(p)\leqq f(q)$.

Lemma 2.1. (Poset Lemma) *Let $f:P\to P$ be a poset map such that $f(p)\leqq p$ for all $p\in P$ or $f(p)\geqq p$ for all $p\in P$. Then the geometric realization of P is homotopy equivalent to the geometric realization of $f(P)$.*

The rest of this section is devoted to showing that L is Cohen–Macauley. Since L is contractible and the dimension of L is equal to its height as a poset, L satisfies (i) above.

Let (g,G) be a vertex of L. Then $L_{<(g,G)}$ can be identified with the partially ordered set of non-empty forests in the graph G. The partial ordering is given by inclusion.

Proposition 2.2. *Let G be a finite connected graph, and $F(G)$ the poset of non-empty forests in G. Then the geometric realization of $F(G)$ is homotopy equivalent to a wedge of spheres of dimension $v-2$, where $v=v(G)$ is the number of vertices of G. The geometric realization of $F(G)$ is contractible if and only if G has a bridge.*

Proof. If ε is an edge of G, we denote by $G-\varepsilon$ the graph obtained by deleting the interior of the edge ε, and by G/ε the quotient graph obtained by collapsing ε to a point.

Let e denote the number of edges of G. We will proceed by induction on $e+v$. If $e+v=1$, the theorem is trivial.

If an edge ε of G is a loop, then $F(G)=F(G-\varepsilon)$; thus we may assume that G has no loops.

If G has a bridge ε, then $\phi\cup\{\varepsilon\}$ is a forest whenever ϕ is. Then $\phi\to\phi\cup\{\varepsilon\}\to\{\varepsilon\}$ are poset maps giving a contraction of the geometric realization of $F(G)$ to a point.

We now assume that G has no bridges. Fix an edge ε of G. Let $F_1=F(G)-\{\varepsilon\}$ be the set of all forests except the forest consisting of the single edge ε, and let F_0 be the set of all forests which do not contain ε. Then $\phi\to\phi-\{\varepsilon\}$ is a poset map from F_1 onto F_0 giving a homotopy equivalence of their realizations. F_0 is naturally isomorphic to $F(G-\varepsilon)$. Since ε is not a bridge, $G-\varepsilon$ is connected. By induction, F_0 is homotopy equivalent to a wedge of $(v-2)$-spheres.

Now $F(G)=F_1\cup(\mathrm{star}(\varepsilon))$, and $F_1\cap(\mathrm{star}(\varepsilon))=\mathrm{link}(\varepsilon)$, where star and link are defined as for simplicial complexes, i.e. the star is the set of forests ϕ which contain ε. The map $\phi\to\phi/\varepsilon$ is a poset isomorphism from $\mathrm{link}(\varepsilon)$ to $F(G/\varepsilon)$. G/ε is connected and has no bridges since G is connected and has no bridges. Furthermore, G/ε has one less vertex than G since ε is not a loop. Therefore the geometric realization of $F(G/\varepsilon)$ is homotopy equivalent to a wedge of $(v-3)$-spheres by induction. It follows by the Van Kampen and Mayer-Vietoris theorems that

$$F(G)=F_1\cup(\text{star}(\varepsilon))\simeq \vee S^{v-2}\vee \text{susp}(\vee S^{v-3})\simeq \vee S^{v-2}.$$

Corollary 2.3. *$L_{>(g,G)}$ is $(\text{ht}(g,G)-1)$-spherical.*

Proof. We have already remarked $L_{>(g,G)}$ can be identified with $F(G)$. We now note that the height of (g,G) is the number of edges in a maximal tree in G, which is equal to the number of vertices of G minus 1.

We now consider $L_{<(g,G)}$. If G is a rose, $L_{>(g,G)}$ is the entire link of (g,G) in L.

In [1], the contractability of L was shown by reducing the problem to a local problem in the link of a rose. We recall the formalism used to understand the link of a rose. Fix a rose $\rho=(r,R)$. Let $E(R)=\{\varepsilon_1,\bar{\varepsilon}_1,\ldots,\varepsilon_n,\bar{\varepsilon}_n\}$ denote the set of oriented edges of R.

An *ideal edge* of R is a partition of $E(R)$ into two subsets S and $\bar{S}=E(R)-S$ which each contain at least two elements.

Two partitions $\{S,\bar{S}\}$ and $\{T,\bar{T}\}$ of a set are *compatible* if one of the following inclusions holds:

$$S\subseteq T,\quad S\subseteq \bar{T},\quad \bar{S}\subseteq T,\quad \bar{S}\subseteq \bar{T}.$$

A set of k distinct pairwise compatible ideal edges partitions $E(R)$ into $k+1$ subsets. If two partitions are not compatible, they are said to *cross.*

The motivation for this terminology comes from considering Venn diagrams for ideal edges. Represent the elements of $E(R)$ by $2n$ points in the plane. Then an ideal edge $\{S,\bar{S}\}$ can be represented by a simple closed curve in the plane which separates the edges in S from those in $\bar{S}$. If two ideal edges are compatible, simple closed curves representing them can be drawn to be disjoint. If they cross, the simple closed curves must cross (Fig. 2).

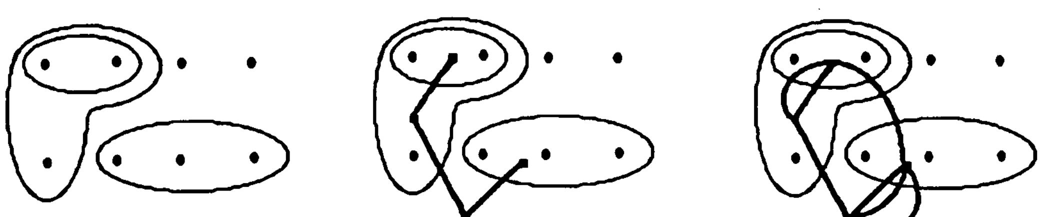
The set of sets of distinct, pairwise compatible ideal edges of R is partially ordered by inclusion. The geometric realization of this poset can be identified with the link of ρ as follows. A set of k distinct, pairwise compatible ideal edges can be represented by a set $\mathscr{S}$ of simple closed curves in the plane (Fig. 3a). The corresponding graph in link(ρ) has a vertex for each component of the plane minus $\mathscr{S}$ and a maximal tree which is dual to these components (Fig. 3b). The rest of the graph is obtained by adding an edge for each pair $\{\varepsilon,\bar{\varepsilon}\}$ in $E(R)$ (Fig. 3c).

We extend the concept of ideal edges to arbitrary graphs G. Let x be a vertex of G, and let $E(G,x)$ denote the set of oriented edges of G which terminate at x. Define an ideal edge of G *at the vertex* x to be a partition of $E(G,x)$ into two sets, each with at least two elements. An *ideal edge* of G is an ideal edge at some vertex of G.

We introduce the convention that ideal edges at distinct vertices of G are compatible. With this convention, a collection $\mathscr{S}$ of distinct, pairwise compatible ideal edges of G corresponds to a graph in $L_{>(g,G)}$ as follows. For each vertex x of G, construct the tree dual to a set of curves representing the ideal edges of $\mathscr{S}$ which lie at x (if there are no

(a) Compatible ideal edges (b) Crossing ideal edges

FIGURE 2



(a) (b) (c)

FIGURE 3

ideal edges at x in $\mathscr{S}$, this dual tree is a point). Each vertex of each dual tree thus corresponds to a set of oriented edges in G. Now connect the original edges of G to the appropriate vertices of the dual trees to obtain the graph in $L_{>(g,G)}$.

With these definitions, $L_{>(g,G)}$ is the geometric realization of the poset of sets of ideal edges of G for any vertex (g, G) of L. This poset is the barycentric subdivision of a simplicial complex $\mathscr{I}(G)$. It often simplfies notation to work with $\mathscr{I}(G)$ instead of the geometric realization of $L_{>(g,G)}$. A k-simplex in $\mathscr{I}(G)$ is simply a set of $k+1$ distinct, pairwise compatible ideal edges. $\mathscr{I}(G)$ can be decomposed as the simplicial join $\mathscr{I}_1 * \ldots * \mathscr{I}_{v(G)}$, where $\mathscr{I}_i$ is the subcomplex spanned by the ideal edges at the ith vertex of G.

To show that $\mathscr{I}(G)$ (and therefore $L_{>(g,G)}$) is $(h-\mathrm{ht}(g,G)-1)$-spherical we will need the following combinatorial theorem.

Let E be a finite set, and let P be the set of partitions of E into two subsets, each with at least two elements. Let $\Sigma(E)$ be the simplicial complex whose vertices are elements of P, and whose k-simplices consist of sets of $k+1$ distinct, pairwise compatible vertices.

Theorem 2.4. *If E has m elements, then $\Sigma(E)$ is homotopy equivalent to a wedge of $(m-4)$-spheres.*

To prove this theorem, we make the following observations.

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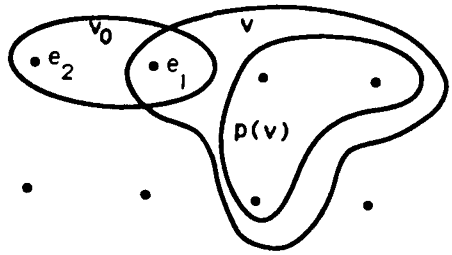

FIGURE 4

Notation. Let $B_1, \ldots, B_k$ be subcomplexes of a simplicial complex A. We denote by $\langle B_1, \ldots, B_k \rangle$ the full subcomplex of A spanned by $B_1 \cup \cdots \cup B_k$. If v is a single vertex, the subcomplex $\langle B, v \rangle$ is said to be obtained from B by *adjoining* the vertex v.

Lemma 2.5. (Coning Lemma) *Let A be a simplicial complex with the property that a set $\{v_0, \ldots, v_k\}$ spans a k-simplex if and only if any pair $\{v_i, v_j\}$ spans an edge. Let $B \subset A$ be a full subcomplex, and let v be a vertex of $A - B$. Then the subcomplex $\langle B, v \rangle$ obtained from B by adjoining v is equal to*

$$B \bigcup_J C(J)$$

where $J = \langle$vertices $w \in B$ such that $\{w, v\}$ spans an edge$\rangle$, and $C(J)$ is the cone on the subcomplex J with cone point v.

Corollary 2.6 *Let A, B and v be as in Lemma 2.5. If B is contractible, then $\langle B, v \rangle \simeq \text{susp}(J)$. If $B \simeq \vee S^k$ and $J \simeq \vee S^{k-1}$, then $\langle B, v \rangle \simeq \vee S^k$.*

Note that the Coning Lemma applies to our complex $\Sigma(E)$. We are now ready to prove that $\Sigma(E)$ is spherical.

Proof of theorem. Denote the elements of E by $e_1, \ldots, e_m$. The proof proceeds by induction on m. If $m = 4$, the only allowable partitions are $\{S, \bar{S}\}$ with $S = \{e_1, e_2\}$, $\{e_1, e_3\}$ or $\{e_1, e_4\}$. Since no two of these are compatible, $\Sigma(E)$ consists of three distinct points, i.e. $\Sigma(E) \simeq \vee S^0$.

Now assume $m > 4$. Let $v_0 = \{S_0, \bar{S}_0\}$ with $S_0 = \{e_1, e_2\}$. Let $\Sigma_0 \subset \Sigma(E)$ be the subcomplex spanned by partitions compatible with v_0. Then Σ_0 is a cone on the vertex v_0, and is hence contractible.

Any vertex $v = \{S, \bar{S}\}$ in $\Sigma(E) - \Sigma_0$ crosses v_0, i.e. the partition $\{S, \bar{S}\}$ separates e_1 from e_2. Define the *inside* of v to be the subset of the partition which contains e_1, and the *size* of v to be the number of elements in the inside of v. If S is the inside of v, let $T = S - \{e_1\}$. If size $(v) > 2$, then $p(v) = \{T, \bar{T}\}$ is in P and is compatible with both v and v_0 (Fig. 4). In fact, *any* vertex which is compatible with both v and v_0 is compatible with $p(v)$.

Let Σ_1 be the subcomplex of $\Sigma(E)$ spanned by Σ_0 and all vertices in $\Sigma(E)-\Sigma_0$ of size greater than 2. We now show that Σ_1 is contractible, by adjoining the vertices in $\Sigma_1-\Sigma_0$ to Σ_0 in order of decreasing size, and showing that the complex remains contractible after each such adjunction.

Let v be a vertex of size $r>2$. Suppose we have adjoined all vertices of size bigger than r, as well as possibly some vertices of size r, to Σ_0 to obtain a contractible subcomplex B of Σ_1. Let J_v be the subcomplex of B spanned by vertices w which are compatible with v. Note that $p(v)$ is in J_v, since $p(v)$ is in Σ_0. In fact J_v is a cone on $p(v)$: if w is in J_v, w is compatible with v and either compatible with v_0 (and so compatible with $p(v)$) or of size at least r (and hence compatible with $p(v)$). Since J_v is contractible, $\langle B, v\rangle$ is contractible by the Coning Lemma. We adjoin all vertices in $\Sigma_1-\Sigma_0$ in this way, to see that Σ_1 is contractible.

Now let v be a vertex in $\Sigma(E)-\Sigma_1$, i.e. v is a vertex of size 2 which crosses v_0. Let J_v be the subcomplex of Σ_1 spanned by vertices w which are distinct from v but compatible with v. Then J_v in fact consists of all vertices in $\Sigma(E)$ distinct from v but compatible with v. If S is the inside of $w=\{S,\bar{S}\}$, the map $S\to S-\{e_1\}$ induces a poset isomorphism from J_v to $\Sigma(E-\{e_1\})$. By induction $\Sigma(E-\{e_1\})\simeq \vee S^{m-5}$. By the Coning Lemma, $\langle\Sigma_1, v\rangle\simeq \vee S^{m-4}$. Since no two vertices in $\Sigma(E)-\Sigma_0$ are compatible, we see that $\Sigma(E)\simeq \vee S^{m-4}$.

Corollary 2.7. *$L_{<(g,G)}$ is $(h-\mathrm{ht}(g,G)-1)$-spherical.*

Proof. By the remarks preceding the statement of Theorem 2.4, we have

$$L_{<(g,G)}\simeq \mathscr{I}(G)\simeq \mathscr{I}_1 * \cdots * \mathscr{I}_{v(G)}$$

where $v(G)$ is the number of vertices of G. If $E(G,x_i)$ has k_i elements, then $\mathscr{I}_i\simeq \vee S^{k_i-4}$ by the theorem. Since $\Sigma k_i=2(\#E(G))=2(v(G)+n-1)$, we have

$$L_{<(g,G)}\simeq \vee S^{2n-v(G)-3}.$$

To complete the proof we note that $h=\mathrm{height}(L)=2n-3$ and $\mathrm{ht}(g,G)=v(G)-1$.

To complete the proof that L is Cohen–Macauley, we need:

Proposition 2.8. *$((g',G'),(g,G))$ is $(\mathrm{ht}(g,G)-\mathrm{ht}(g',G')-1$-spherical for $(g,G)>(g',G')$.*

Proof. (g',G') is obtained from (g,G) by collapsing a forest in G, and $((g',G'),(g,G))$ can be identified with the poset of proper subsets of the set of edges in this forest. The geometric realization of this poset is the barycentric subdivision of the boundary of a simplex of dimension $\mathrm{ht}(g,G)-\mathrm{ht}(g',G')$ and is hence spherical of dimension $\mathrm{ht}(g,G)-\mathrm{ht}(g',G')-1$.

3. The Cohen–Macauley property for K

The methods used to study the link of a vertex in L can be extended to apply to the subcomplex K spanned by vertices (g,G) such that G has no bridges. K has building-like properties missing in L. For example, any simplex of codimension 1 in K is a face of at least two maximal-dimensional simplices, whereas L has codimension 1 simplices contained in only one maximal simplex. In addition, the complex K has a more uniform local structure than L: we will show in this section that the link of every vertex in K is homotopy equivalent to a non-trivial wedge of spheres of codimension 1. In contrast, some links of L are contractible.

Since K is a deformation retract of L, K is contractible. To show that K is Cohen–Macauley, therefore, we need only to check the local conditions on $K_{>v}$, on $K_{>v}$ and on (v, v') for vertices $v=(g, G)$ and $v'=(g', G')$ in K.

The proof that (v, v') is spherical of the correct dimension is identical to the proof for L.

The complex $K_{<v}$ is isomorphic to the poset of non-empty forests in G, as for L. Since $v=(g, G)$ is in K, G has no bridges; thus by Proposition 2.2, $K_{<v}$ is homotopy equivalent to a non-trivial wedge of $(v(G)-2)$-spheres.

It remains only to show that $K_{>v}$ is homotopy equivalent to a wedge of $(n-v(G)-3)$-spheres. To do this, we need the following combinatorial theorem.

Let E be a finite set, with P and $\Sigma(E)$ defined as in Section 2. A *grouping* $\mathscr{C}=\{C_0, \ldots, C_k\}$ of E is a partition of E into subsets of C_i (called *clusters*), each containing at least two elements. We define $P\mathscr{C} \subset P$ to be the set of partitions of E into two sets S and $\bar{S}$, each with at least two elements, such that some cluster in $\mathscr{C}$ is "split", i.e. for some i, there are elements e and f in C_i such that $e \in S$ and $f \in \bar{S}$. Note that $P-P\mathscr{C}$ consists of partitions into sets of the form $\bigcup_{i \in I} C_i$, where I is some subset of $\{0, 1, \ldots, k\}$. Define $\Sigma(E, \mathscr{C})$ to be the subcomplex of $\Sigma(E)$ spanned by all elements of $P\mathscr{C}$.

Theorem 3.1. *Let E be a set with m elements, and let $\mathscr{C}=\{C_0, \ldots, C_k\}$ be a grouping of E. Then $\Sigma(E, \mathscr{C})$ is homotopy equivalent to a wedge of $(m-4)$-spheres.*

Proof. If $k=0$, then $\Sigma(E, \mathscr{C})=\Sigma(E)$ which is $(m-4)$-spherical by Theorem 2.4. Thus we may assume that $k \geqq 1$.

The proof proceeds by induction on m. In order to make the induction work, we need a slightly more complex induction hypothesis than the statement of the theorem. The extra complexity involves putting a filtration $\mathscr{D}$ on C_1:

$$C_1 = D_1 \supset D_2 \supset \cdots \supset D_l \supset D_{l+1} = \emptyset, \qquad l \geqq 1.$$

If $l=1$, we will say the filtration is *trivial*. We define $\Sigma(E, \mathscr{C}, \mathscr{D})$ to be the subcomplex of $\Sigma(E, \mathscr{C})$ spanned by all elements of $P\mathscr{C}$ except those of the form $\{S, \bar{S}\}$ with S or $\bar{S}$ equal to

$$D_j \cup \bigcup_{i \in I} C_i$$

for some $1 \leqq j \leqq l$ and some subset $I \subset \{2, \ldots, k\}$. If the filtration $\mathscr{D}$ is trivial, we have $\Sigma(E, \mathscr{C}, \mathscr{D})=\Sigma(E, \mathscr{C})$.

Induction Hypothesis. *$\Sigma(E', \mathscr{C}', \mathscr{D}')$ is homotopy equivalent to a wedge of spheres of*

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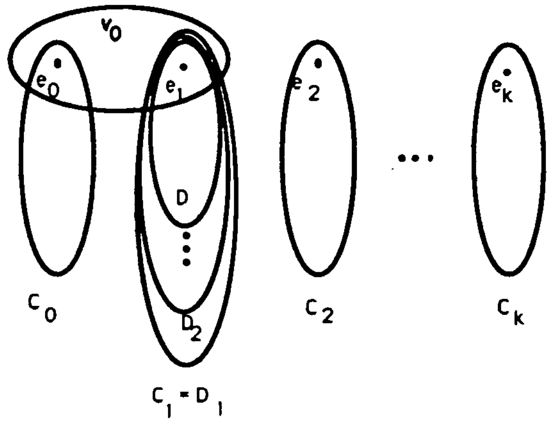


FIGURE 5. The vertex v_0.

dimension $\#E'-4$ *for any* E' *with* $4\leqq \#E'<m$, *any grouping* $\mathscr{C}'$ *of* E' *and any filtration* $\mathscr{D}'$ *of* C_1'.

If $m=4$ then there are exactly two clusters (since $k\geqq 1$). The only possible filtrations on C_1 are the trivial filtration and the two maximal filtrations. For any of these filtrations, there are only two allowable partitions, which are not compatible. Therefore $\Sigma(E,\mathscr{C},\mathscr{D})$ consists of two distinct points, i.e. $\Sigma(E,\mathscr{C},\mathscr{D})\simeq S^0$.

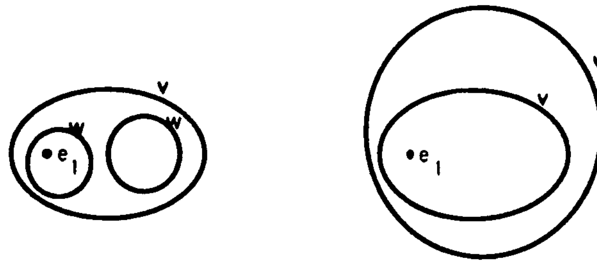
Now assume $m>4$. Choose an element $e_0\in C_0$ and $e_1\in D_l\subset C_1$. Set $S_0=\{e_0,e_1\}$ and $v_0=\{S_0,\bar{S}_0\}$ (see Figure 5). Let $\Sigma_0\subset\Sigma(E,\mathscr{C},\mathscr{D})$ be the subcomplex spanned by partitions compatible with v_0. Then Σ_0 is a cone on the vertex v_0, and is hence contractible.

For any vertex v of $\Sigma(E,\mathscr{C},\mathscr{D})$, define the *inside* of v to be the subset of the partition which contains e_1, and the *size* of v to be the number of elements in the inside of v.

Let $v=\{S,\bar{S}\}$ be a vertex of $\Sigma(E,\mathscr{C},\mathscr{D})-\Sigma_0$, with $e_1\in S$. If $p(v)=\{S-\{e_1\},S\cup\{e_1\}\}$ is a vertex of $\Sigma(E,\mathscr{C},\mathscr{D})$, we say v is *pushable*. If v is pushable, then $p(v)$ is in Σ_0 and is compatible with both v and v_0. In fact, any vertex which is compatible with both v and v_0 is compatible with $p(v)$.

Let Σ_1 denote the subcomplex of $\Sigma(E,\mathscr{C},\mathscr{D})$ spanned by Σ_0 and all pushable vertices. We will now show that Σ_1 is contractible. To do this, we adjoin the pushable vertices in $\Sigma(E,\mathscr{C},\mathscr{D})-\Sigma_0$ to Σ_0 in order of decreasing size, and claim that the complex remains contractible after each such adjunction.

Let v be a pushable vertex of size r, and suppose we have adjoined all pushable vertices of size bigger than r, as well as possibly some size r, to Σ_0 to obtain a subcomplex B of $\Sigma(E,\mathscr{C},\mathscr{D})$ which is contractible. Let J_v be the subcomplex of B spanned by vertices w which are compatible with v. Note that $p(v)$ is in J_v. In fact J_v is a cone on $p(v)$: if w is in J_v, w is compatible with v and either compatible with v_0 (in which case it's compatible with $p(v)$) or pushable of size at least r (and hence compatible with $p(v)$). Since J_v is contractible, $\langle B,v\rangle$ is contractible by the Coning Lemma.



w inside *v* *w* outside *v*

FIGURE 6

Note that any vertex with inside of the form $\{e_1, x\}$, $x \neq e_0$, is neither compatible with v_0 nor pushable, so is not in Σ_1. We now adjoin these vertices to Σ_1 to obtain a larger subcomplex Σ_2 of $\Sigma(E, \mathscr{C}, \mathscr{D})$. Let v be any such vertex, and let E' be the set obtained from E by identifying the points e_1 and x. We will define a grouping $\mathscr{C}'$ and filtration $\mathscr{D}'$ on E' which depends on the location of x in E. In each case, one checks that the quotient map $q': E \to E'$ induces an isomorphism $\mathrm{lk}(v) \cap \Sigma_1 \to \Sigma(E', \mathscr{C}', \mathscr{D}')$. Since $\#E' = m-1$, this shows by induction that $\mathrm{lk}(v) \cap \Sigma_1$ is homotopy equivalent to a wedge of $(m-5)$-spheres. This implies that Σ_2 is homotopy equivalent to a wedge of $(m-4)$-spheres.

Case 1. $x \in C_0$.
In this case we take $\mathscr{C}' = \{C'_0, \ldots, C'_{k-1}\}$, where $C'_0 = q'(C_0 \cup C_1)$ and $C'_i = q'(C_{i+1}) = C_{i+1}$ for $1 \leqq i \leqq k-1$. The filtration $\mathscr{D}'$ of C'_1 is trivial.

Case 2. $x \in D_r - D_{r+1}$ for some $1 \leqq r \leqq l$.
Take $\mathscr{C}' = \{C'_0, \ldots, C'_k\}$, where $C'_i = q'(C_i) = C_i$ for all $0 \leqq i \leqq k$, and $\mathscr{D}' = D'_1 \supset \cdots \supset D'_r$, where $D'_i = q'(D_i)$ for $1 \leqq i \leqq r$.

Case 3. $x \in C_r$ for some $r \geqq 2$.
Take $\mathscr{C}' = \{C'_0, \ldots, C'_{k-1}\}$, where $C'_0 = q'(C_0) = C_0$, $C'_1 = q'(C_1 \cup C_r)$, $C'_i = q'(C_i) = C_i$ for $2 \leqq i < r$ and $C'_i = q'(C_{i+1}) = C_{i+1}$ for $r < i \leqq k-1$. $\mathscr{D}' = D'_1 \supset \cdots \supset D'_{l+1}$, where $D'_i = q'(D_i \cup C_r)$ for $1 \leqq i \leqq l$ and $D'_{l+1} = q'(\{e_1\} \cup C_r)$.

The vertices of $\Sigma(E, \mathscr{C}, \mathscr{D}) - \Sigma_2$ are those which are not compatible with v_0, not pushable, and have size bigger than 2. Any such vertex v must be of the form $\{S, \bar{S}\}$ for some $S = \{e_1\} \cup \bigcup_{i \in I} C_i$, where $I = \{i_1, \ldots, i_r\} \subset \{2, \ldots, k\}$. The number r of clusters contained entirely in S will be called the *cluster size* of v.

We adjoin these vertices to Σ_2 in order of increasing cluster size.

Suppose that we have adjoined with cluster size less than r, as well as possibly some with cluster size equal to r, to Σ_2 to obtain a subcomplex B of $\Sigma(E, \mathscr{C}, \mathscr{D})$ which is homotopy equivalent to a wedge of $(m-4)$-spheres. Let J_v be the subcomplex of B spanned by vertices w which are compatible with v. We will say w is *inside* v if either the inside or the outside of w is contained in S. w is *outside* v if the outside of w is contained in $\bar{S}$ (Fig. 6). Since w is compatible with v, it must be either inside or outside v.

Every element of J_v which is inside v is compatible with every element of J_v which is outside v. This is reflected in the simplicial complex J_v by the decomposition

$$J_v = \langle\{w \in J_v : w \text{ inside } v\}\rangle * \langle\{w \in J_v : w \text{ outside } v\}\rangle$$

$$= Interior(v) * Exterior(v).$$

Let E'' be the set obtained from E by identifying all of $\bar{S}$ to a single point. Then the quotient map $q'': E \to E''$ induces an isomorphism $Interior(v) \to \Sigma(E'', \mathscr{C}'', \mathscr{D}'')$; here $\mathscr{C}'' = \{C_0'', \dots, C_r''\}$, where $C_0'' = q''(\bar{S} \cup \{e_1\})$ and $C_j'' = q''(C_{i_j})$ for $1 \leqq j \leqq r$, and the filtration $\mathscr{D}''$ of C_1'' is trivial. If $s = \#S$, then $\#E'' = s + 1 < m$, so by induction we have $Interior(v) \simeq \vee S^{s-3}$.

We now consider $Exterior(v)$. Let E''' be the set obtained from E by identifying S to a single point. Then the quotient map $q''': E \to E'''$ induces an isomorphism $Exterior(v) \to \Sigma(E''', \mathscr{C}''', \mathscr{D}''')$; here $\mathscr{C}''' = \{C_0''', \dots, C_{k-r-2}'''\}$, where $C_0''' = q'''(C_0)$, $C_1''' = q'''(C_1 \cup S)$, and $C_i''' = q'''(C_i) = C_i$ for $i \in \{2, \dots, k\} - I$. The filtration $\mathscr{D}''' = D_1''' \supset \cdots \supset D_{l+1}'''$ is given by $D_i''' = q'''(D_i \cup S)$ for $1 \leqq i \leqq l$, and $D_{l+1}''' = q'''(S)$. Since S has s elements, $\#E''' = m - s + 1 < m$; thus by induction $Exterior(v) \simeq \vee S^{m-s-3}$.

Thus

$$J_v = Exterior(v) * Interior(v)$$

$$\simeq \vee S^{m-s-3} * \vee S^{s-3}$$

$$\simeq \vee S^{m-5}.$$

By the Coning Lemma $\langle B, v\rangle$ is homotopy equivalent to a wedge of $(m-4)$-spheres. After adjoining all vertices $v \in \Sigma(E, \mathscr{C}, \mathscr{D}) - \Sigma_2$ to Σ_2 in this way, we have

$$\Sigma(E, \mathscr{C}, \mathscr{D}) \simeq \vee S^{m-4}.$$

Corollary 3.2. *Let (g, G) be a vertex of K. Then $K_{<(g,G)}$ is homotopy equivalent to a wedge of spheres of dimension $2n - v(G) - 3$.*

Proof. $K_{<(g,G)}$ is the subcomplex of $L_{>(g,G)}$ spanned by graphs with no bridges. In Section 2, we identified a vertex of $\mathscr{I}(G) \simeq L_{>(g,G)}$ with an ideal edge at a vertex x of G. We need to understand when this ideal edge corresponds to a bridge.

Form a grouping $\mathscr{C}_x$ of $E(G, x)$ by putting two oriented edges in the same cluster if they are in the same connected component of $G - x$. Since G has no bridges, each cluster in this grouping has at least two elements. An ideal edge at x corresponds to a bridge if and only if the ideal edge is a union of clusters in $\mathscr{C}_x$, i.e. the ideal edge is not in $P\mathscr{C}_x$.

As in the case of the complex L, we have a decomposition

$$K_{<(g,G)} \simeq K_1 * K_2 * \cdots * K_{v(G)}$$

where K_i is the simplicial complex spanned by the ideal edges which lie at the ith vertex x_i of G.

By Theorem 3.1, we have

$$K_i \cong \Sigma(E(G, x_i), \mathscr{C}_{x_i}) \simeq \vee S^{k_i - 4}$$

where k_i is the number of oriented edges terminating at x_i.

Thus

$$\begin{aligned} K_{<(g,G)} &\simeq K_1 * \cdots * K_{v(G)} \\ &\simeq \vee S^{k_1 - 4} * \cdots * \vee S^{k_{v(G)} - 4} \\ &\simeq \vee S^{\Sigma k_i - 4v(G) + (v(G) - 1)} \end{aligned}$$

But $\Sigma k_i = \# E(G) = 2(v(G) + n - 1)$; thus

$$K_{<(g,G)} \simeq \vee S^{2n - v(G) - 3}.$$

Remark. The complexes K and L can be used to study the behavior of $Out(F_n)$ "at infinity". For $n \geqq 3$, it can be shown that any action of $Out(F_n)$ on a tree has a fixed point. Therefore, by a theorem of Stallings, $Out(F_n)$ has only one end. This implies that the complexes K and L are connected at infinity. If they are $(n-5)$-connected at infinity, it can be shown that $Out(F_n)$ is a virtual duality group, i.e. there is a module I and an isomorphism $H^i(Out(F_n), M) \to H_{d-i}(Out(F_n), M \otimes I)$ for any $Out(F_n)$-module M and any $i \leqq d$, where d is the virtual cohomological dimension of $Out(F_n)$.

To try to understand the behavior of K and L at infinity, we define a norm on roses $\rho = (r, R)$ as follows. Let W be any set of cyclic words in F_n. Represent each word in $w \in W$ by a path ω in R_0, and define the length of w to be the edge-path length of the shortest path in R representing $r(\omega)$. The norm of ρ is then the sum of the lengths of the elements of W. Define the ball of radius m to be the union of the stars of roses of norm less than or equal to m. In [1] it is shown that the ball of radius m in K deformation retracts onto the ball of radius $m-1$. If W is chosen appropriately, these balls are compact. In fact, one can choose W so that the ball of minimal radius is the star of a single rose, and the deformation retraction maps the entire exterior of this ball onto the link of that rose. There is a similar retraction for L, but the retraction is not onto. The theorem in this section shows that the link of a rose in K is homotopy equivalent to a wedge of $(2n-4)$-spheres, providing evidence that the space $K = K_n$ may be $(2n-5)$-connected at infinity.

Remark. The homogeneous space $X(n)$ for $Out(F_n)$ has a decomposition as a union of "fat Teichmüller spaces", i.e. contractible $(3n-4)$-dimensional manifolds on which the mapping class groups of surfaces with fundamental group F_n act. The methods of this paper can be easily extended to show that the corresponding subcomplexes of K and L are Cohen–Macauley.

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https://doi.org/10.1017/S0013091500004818 Published online by Cambridge University Press