PSEUDONILPOTENT GROUPS

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The purpose of this note is to introduce some "canonical" characteristic subgroups and to observe some of their more obvious properties.

For any group G, let $P(G)$ stand for the intersection $\cap \{C(a)^G : a \in G\}$ of the normal closures of all centralizers of elements of G. It is not unreasonable to call this the *pseudocentre* of G. The pseudocentre is obviously a characteristic subgroup, and contains the centre of G. What is somewhat less obvious is that nontrivial finite groups have non-trivial pseudocentres; more precisely:

LEMMA. *For every non-trivial group G and any elements* $a_1, a_2, \cdots, a_n$ *of* G, $\bigcap_{i=1}^{n} C(a_i)^G \neq 1$.

PROOF. Proceed by induction on n. For $n = 1$ the result is obvious, so suppose it true for n centralizers, and that $\bigcap_{i=1}^{n+1} C(a_i)^G = 1$ for some elements $a_1, a_2, \cdots, a_{n+1}$ of a group G. For brevity, we let D stand for the intersection on the left of this equation. For any elements $x_1, \cdots, x_n$ in $C(a_1)^G, \cdots, C(a_n)^G$ respectively, the $(n+1)$-fold commutator $[x_1, x_2, \cdots, x_n, a_{n+1}]$ is in D, and so is 1. In other words $[x_1, x_2, \cdots, x_n]$ is in $C(a_{n+1})$ and so is trivial. Thus $[x_1, x_2, \cdots, x_{n-1}] \in C(a_n)$; but then $[x_1, x_2, \cdots, x_{n-1}, a_{n+1}] \in D$, so that $[x_1, x_2, \cdots, x_{n-1}] \in C(a_n) \cap C(a_{n+1})$. This means that $[x_1, x_2, \cdots, x_{n-1}]$ is in D and is therefore trivial. The argument continues in this way to show that $[x_1, x_2] = 1$. Thus $C(a_2) \subseteq C(a_1)$ so that $D = \bigcap_{i=2}^{n+1} C(a_i)^G$. The inductive hypothesis now gives that $D \neq 1$, and the contradiction completes the induction.

COROLLARY. *For every non-trivial finite group* G, $P(G) \neq 1$.

The pseudocentre of a group can be as little as the centre, as it is in nilpotent groups of class 2, or as much as the whole group, as it is for instance in characteristically simple groups and many other cases. It seems to have no very obvious other connection with the known canonical characteristic subgroups.

The pseudocentral series of a group G is defined in the obvious way:

$$1 = P_0(G) \subseteq P_1(G) \subseteq \cdots \subseteq P_n(G) \subseteq \cdots,$$

where $P_{n+1}(G)/P_n(G) = P(G/P_n(G))$ for all n. These terms are all characteristic subgroups, of course, A group G is *pseudonilpotent* if the pseudocentral series reaches G, so that (alas!) all finite groups are pseudonilpotent. The least n with $P_n(G) = G$ is the *pseudonilpotency class* of G, and it is fairly obvious that there exist finite p-groups of arbitrarily high pseudonilpotency class (suitable wreath products, for instance). The nilpotency class is usually much greater than the pseudonilpotency class; for example, dihedral groups are pseudonilpotent of class 2.

There do exist infinite groups with trivial pseudocentres: non-abelian free groups, and non-abelian free soluble groups, for example. Since every group is a subgroup of some simple group, this shows that pseudonilpotency is not inherited by subgroups. However, it is preserved under direct products with finitely many factors, and more generally under nilpotent products; and also under homomorphic images.

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