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DISTINGUISHEDNESS OF WEIGHTED FRÉCHET SPACES OF CONTINUOUS FUNCTIONS

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In this paper, we prove that if $\mathcal{U}$ is an increasing sequence of strictly positive and continuous functions on a locally compact Hausdorff space X such that $\bar{V} \simeq \bar{V} \cap C(X)$, then the Fréchet space $C\mathcal{U}(X)$ is distinguished if and only if it satisfies Heinrich's density condition, or equivalently, if and only if the sequence $\mathcal{U}$ satisfies condition (H) (cf. e.g. [1] for the introduction of (H)). As a consequence, the bidual $\lambda_\infty(A)$ of the distinguished Köthe echelon space $\lambda_0(A)$ is distinguished if and only if the space $\lambda_1(A)$ is distinguished. This gives counterexamples to a problem of Grothendieck in the context of Köthe echelon spaces.

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1. Introduction

A locally convex space E is distinguished if its strong dual is barrelled. All the Köthe echelon spaces $\lambda_p(A)$ of order $p=0$ or $1<p<\infty$ are known to be distinguished; in fact, for $1<p<\infty$, they are reflexive (cf. e.g. [4]), and the strong dual of $\lambda_0(A)$ is topologically isomorphic to the LB-space $\mathrm{ind}_{n\to+\infty}\, l_1(a_n^{-1})$, $A=(a_n)_{n\in\mathbb{N}}$. The situation for $p=1$ or $p=\infty$ is more complicated.

The distinguished spaces $\lambda_1(A)$ were characterized by K.-D. Bierstedt, J. Bonet and R. Meise (also see Vogt [6]): K.-D. Bierstadt and R. Meise [3] introduced the condition (D) on a Köthe matrix A and proved that (D) implies $\lambda_1(A)$ distinguished. Then, K.-D. Bierstedt and J. Bonet [2] proved that in fact (D) is also necessary for the distinguishedness of $\lambda_1(A)$.

Concerning the spaces $\lambda_\infty(A)$, which are the strong biduals of the corresponding spaces $\lambda_0(A)$, the problem of characterizing when $\lambda_\infty(A)$ is distinguished is related to the following question of Grothendieck: "Is the bidual of a distinguished Fréchet space also distinguished?" This question of Grothendieck has already been answered in the negative by J. Bonet, S. Dierolf and C. Fernandez [5]. These authors used Fréchet spaces of Moscatelli type to construct counterexamples. Moreover, they proved that this question is also related to the lifting of bounded sets: they show that if E, F are Fréchet spaces such that $E\subset F\subset E''$ and if F is distinguished, then F/E is distinguished and its bounded sets are liftable (with closure). In our situation, this is a key point which allows us to forget about the dual of $\lambda_\infty(A)$, which is not a sequence space, and hence requires a new approach and other methods.

In the present paper, we characterize the distinguished weighted Fréchet spaces of continuous functions on a locally compact Hausdorff space X in terms of condition (H) (cf. Notation). As a particular case, we obtain a characterization of the distinguished spaces $\lambda_\infty(A)$: this space is distinguished if and only if $\lambda_1(A)$ is. Hence, concerning the preceding question of Grothendieck, we can say that every Köthe matrix A which does not satisfy condition (D) (or equivalently (H), cf. Notation) gives a distinguished Fréchet space $\lambda_0(A)$ such that $(\lambda_0(A))''_{bb} \simeq \lambda_\infty(A)$ is not distinguished.

2. Notation

Let X denote a completely regular and Hausdorff space and $\mathcal{U}=(u_m)_{m\in\mathbb{N}}$ denote a countable increasing system of strictly positive weights on X. We set

$$v_m := u_m^{-1}(m\in\mathbb{N}), \quad \mathcal{V}=(v_m)_{m\in\mathbb{N}}$$

and

$$\bar{V}=\left\{\bar{v}: X\to[0,+\infty[;\ \sup_{qx\in X}|\bar{v}(x)/v_m(x)|<+\infty, \forall m\in\mathbb{N}\right\}.$$

Then $C\mathcal{U}(X)$ denotes the linear space of all the continuous function f on X such that $p_m(f):=\sup_{x\in X}u_m(x)|f(x)|<+\infty \quad \forall m\in\mathbb{N}$ endowed with the locally convex topology defined by the semi-norms p_m, $m\in\mathbb{N}$. The notation $\lambda_\infty(A)$, $A=\mathcal{U}$ is used in case X is discrete. Further, $C\mathcal{U}_0(X)$ denotes the subspace of $C\mathcal{U}(X)$ consisting of all the continuous functions f such that $u_m f$ converges to 0 at infinity for every $m\in\mathbb{N}$; in one case X is discrete, $\lambda_0(A)$ is used instead of $C\mathcal{U}_0(X)$.

We will also use the following notation:

Q for the quotient map $C\mathcal{U}(X)\to C\mathcal{U}(X)/CU_0(X)$,

b_m for the neighbourhood $\{f\in\lambda_\infty(A): \sup_{x\in X}|u_m(x)f(x)|\leqq 1\}$ in $\lambda_\infty(A)$,

$D(X)$ for the space of all the continuous functions on X with compact support ($D(X,[0,1])$ denotes then the set of the elements of $D(X)$ with values in $[0,1]$),

if $\bar{v}\in\bar{V}$, then $\bar{v}(l_\infty)_1$ is the set $\{f\in\lambda_\infty(A): |f(x)|\leqq\bar{v}(x), \forall x\in X\}$

$(=\{f\in\lambda_\infty(A): \exists g\in l_\infty, |g(x)|\leqq 1 \ \forall x\in X: f=\bar{v}g\})$.

Let us also recall the expressions of conditions (D), (H), (H^{**}) and (ND), as well as the relations between them, cf. [1] (these expressions are given in terms of $\mathcal{V}$ or $\mathcal{U}$):

(D) $\quad \exists J=(X_m)_{m\in\mathbb{N}}, \varnothing\neq X_m\subset X_{m+1} \forall m$:

$(N,J)\ \forall n, \exists m(n): \inf_{x\in X_n} v_k(x)/v_{m(n)}(x)>0 \ \forall k\in\mathbb{N}$

$(M,J)\ \forall n$ and $Y, (Y\not\subset X_m, \forall m)\exists n': \inf_{y\in Y} v_{n'}(y)/v_n(y)=0$;

(H) $\forall \lambda_m > 0\,(m \in \mathbb{N}), \forall n \in \mathbb{N}, \exists \bar{v} \in \bar{V}$ and $M \in \mathbb{N}$:

$$\forall x \in X(\inf_{1 \leqq m \leqq M} \lambda_m v_m(x) \geqq v_n(x) \Rightarrow \bar{v}(x) \geqq v_n(x));$$

(H^{**}) $\forall \lambda_m > 0\,(m \in \mathbb{N}) \exists \bar{v} \in \bar{V}: \forall n \in \mathbb{N}, \forall C > 0, \exists M \in \mathbb{N}$:

$$\forall x \in X(\inf_{1 \leqq m \leqq M} \lambda_m v_m(x) \geqq C v_n(x) \Rightarrow \bar{v}(x) \geqq C v_n(x));$$

(ND) $\exists n \in \mathbb{N}$ and a decreasing sequence $J_k (k \in \mathbb{N})$ of non void subsets of X such that, $\forall k \geqq n$:

(i) $\inf_{x \in J_k} v_k(x)/v_n(x) > 0$; (ii) $\exists l(k) > k: \inf_{x \in J_k} v_{l(k)}(x)/v_n(x) = 0$.

It is known that $(D) \Leftrightarrow (H) \Leftrightarrow (H^{**}) \Leftrightarrow \neg(ND)$.

3. Main results

As will be proved in Proposition 2, under some continuity assumption, the possibility of lifting the bounded sets of $C\mathcal{U}(X)/C\mathcal{U}_0(X)$ is equivalent to (H) (or to $\neg(ND)$). To obtain this result, we need some more information about the sets J_k $(k \in \mathbb{N})$ appearing in (ND):

Lemma 1. *If $\mathcal{V} \subset C(X)$, then in condition (ND), we can assume that the sets J_k's are such that $(J_{k+1})^- \subset J_k = (J_k)^0$ for every $k \in \mathbb{N}$, i.e.*

(ND) *$\exists n \in \mathbb{N}$ and a decreasing sequence $J_k (k \in \mathbb{N})$ of non void subsets of X such that*

$\forall k \in \mathbb{N}$: $(J_{k+1})^- \subset J_k = (J_k)^0$,

$\forall k \geqq n$: (i) $\inf_{x \in J_k} v_k(x)/v_n(x) > 0$,

(ii) $\exists l(k) > k: \inf_{x \in J_k} v_{l(k)}(x)/v_n(x) = 0$.

Proof. It is known that (ND) is equivalent to $\neg(H)$. To obtain the result here, we just change the proof of $\neg(H) \Rightarrow (ND)$ of 1.2.7 of [1] slightly as follows.

As (H) does not hold, there are $n \in \mathbb{N}$ and a sequence $\lambda_m > 0$ $(m \in \mathbb{N})$ such that

$$\forall \bar{v} \in \bar{V}, \forall M \in \mathbb{N}, \exists x \in X: \inf_{1 \leqq m \leqq M} \lambda_m v_m(x) \geqq v_n(x) \quad \text{and} \quad \bar{v}(x) < v_n(x). \tag{1}$$

For every $k \in \mathbb{N}$, define

$$J_k := \left\{ x \in X: \inf_{1 \leqq m \leqq k} \lambda_m v_m(x) > (1 - 2^{-k}) v_n(x) \right\}.$$

For every k, as (1) holds, the set J_k is non void; and as the functions v_m's are continuous, we also have

$$(J_{k+1})^- \subset J_k = (J_k)^0 \; \forall k \in \mathbb{N}.$$

Moreover, for every $k \in \mathbb{N}$, one gets

$$\inf_{x \in J_k} \frac{v_k(x)}{v_n(x)} \geqq \frac{1-2^{-k}}{\lambda_k},$$

hence (i) of (ND) is satisfied.

So, to conclude, we just have to prove (ii). If (ii) is not satisfied, there is $k \geqq n$ such that

$$\forall l > k, \delta_l := \inf_{x \in J_k} \frac{v_l(x)}{v_n(x)} > 0.$$

For $m > k$, let $\alpha_m := \delta_m^{-1}$ and for $m = 1, \ldots, k$, let $\alpha_m := \lambda_m$. Then define $\bar{v} := \inf_{m \in \mathbb{N}} \alpha_m v_m$. Since (1) holds, there exists $x \in X$ such that

$$\inf_{1 \leqq m \leqq k} \lambda_m v_m(x) \geqq v_n(x) \text{ and } 2\bar{v} < v_n(x).$$

The first inequality implies $x \in J_k$. Moreover, by construction, $\forall y \in J_k$ we have

$$\lambda_m v_m(y) > (1-2^{-k}) v_n(y) \geqq 2^{-1} v_n(y) \quad \text{for } m = 1, \ldots, k,$$
$$\delta_m^{-1} v_m(y) \geqq v_n(y) \geqq 2^{-1} v_n(y) \quad \text{for } m > k;$$

hence also $2\bar{v}(y) \geqq v_n(y)$. But this contradicts $x \in J_k$ and $2\bar{v}(x) < v_n(x)$. □

Now we can prove the main result of this paper, i.e., the characterization of the lifting of the bounded sets of $C\mathcal{U}(X)/C\mathcal{U}_0(X)$ (with or without closure) in terms of condition (H).

Proposition 2. *Let X be locally compact, $\mathcal{V} \subset C(X)$ and consider the following properties:*

(1) *$\mathcal{V}$ satisfies condition (H) (or equivalently (H^{**}));*
(2) *$\forall B$ bounded subset of $C\mathcal{U}(X)/C\mathcal{U}_0(X)$, $\exists C$ bounded subset of $C\mathcal{U}(X)$ such that $B \subset Q(C)$;*
(3) *$\forall B$ bounded subset of $C\mathcal{U}(X)/C\mathcal{U}_0(X)$, $\exists C$ bounded subset of $C\mathcal{U}(X)$ such that $B \subset (Q(C))^{-C\mathcal{U}(X)/C\mathcal{U}_0(X)}$.*

Then (1)⇒(3) and (2)⇒(3). Moreover, if in addition we have $\bar{V} \simeq \bar{V} \cap C(X)$, then (3)⇒(1) and (1)⇒(2).

Proof. Of course, (2)⇒(3).

(1)⇒(3). Given B, there is a sequence $\lambda_m > 0$ ($m \in \mathbb{N}$) such that

$$B \subset \bigcap_{m \in \mathbb{N}} ((\lambda_m b_m \cap C(X)) + C\mathcal{U}_0(X)).$$

Then condition (H^{**}) gives $\bar{v} \in \bar{V}$; we define

$$\bar{u} := \sup_{M \in \mathbb{N}} \inf \{4M\lambda_M \bar{v}, \lambda_1 v_1, \ldots, \lambda_M v_M\}.$$

As $\bar{u}$ belongs also to $\bar{V}$, the set

$$B' := 2\bar{u}(l_\infty)_1 \cap C(X)$$

is a bounded subset of $C\mathcal{U}(X)$. We claim that

$$B \subset \bigcap_{n \in \mathbb{N}} \left(C\mathcal{U}_0(X) + B' + \left(\frac{1}{n} b_n \cap C(X) \right) \right).$$

Indeed, fix $n \in \mathbb{N}$ and take $f \in B$. Define the sets

$$F := \{x \in X : |u_n(x) f(x)| \leqq 1/2n\}; \; F' := \{x \in X : |u_n(x) f(x)| \geqq 1/n\}.$$

Then F and F' are disjoint zero sets of continuous functions; so there is $g \in C(X, [0,1])$ such that $g = 0$ on F and $g = 1$ on F'. As we certainly have $f = gf + (1-g)f$ and $(1-g)f \in (1/n)b_n \cap C(X)$, to conclude it remains to prove that gf belongs to $B' + C\mathcal{U}_0(X)$.

Using (H^{**}) with n and $C = 1/4n$, we get $M = M(n)$ such that

$$\forall x \in X \left(\inf_{1 \leqq m \leqq M} \lambda_m v_m(x) \geqq \frac{v_n(x)}{4n} \Rightarrow \bar{v}(x) \geqq \frac{v_n(x)}{4n} \right).$$

We can write (recall that f belongs to B)

$$f = f^{(m)} + g^{(m)}, \; m = 1, \ldots, M$$

with $f^{(m)} \in \lambda_m b_m \cap C(X)$ and $g^{(m)} \in C\mathcal{U}_0(X)$ for every $m = 1, \ldots, M$. Then there is a compact subset K of X such that

$$|u_n(x) f(x)| \leqq \lambda_m u_n(x) v_m(x) + \frac{1}{4n} \tag{2}$$

for every $x \in X \backslash K$ and $1 \leqq m \leqq M$. It follows that every $x \in X \backslash (K \cup F)$ satisfies

$$\frac{1}{4n} v_n(x) \leqq \inf_{1 \leqq m \leqq M} \lambda_m v_m(x); \tag{3}$$

hence also (use (H^{**}))

$$\frac{1}{4n} v_n(x) \leqq \bar{v}(x).$$

A look at the definition of $\bar{u}$ shows that the previous inequality implies

$$\bar{u}(x) \geqq \inf\{4M\lambda_M \bar{v}(x), \lambda_1 v_1(x), \ldots, \lambda_M v_M(x)\} = \inf_{1 \leqq m \leqq M} \lambda_m v_m(x)$$

for every $x \in X \backslash (K \cup F)$. Moreover, (2) and (3) implies also

$$|f(x)| \leqq 2 \inf_{1 \leqq m \leqq M} \lambda_m v_m(x) \ (\leqq 2\bar{u}(x)) \tag{4}$$

for every $x \in X \backslash (K \cup F)$. Taking now $\phi \in D(X, [0,1])$, $\phi = 1$ on K, we get $gf = \phi gf + (1-\phi)gf$, with $\phi gf \in D(X) \subset C\mathscr{U}_0(X)$. Finally, by construction and by (4), we obtain

$$(1-\phi)g|f| \leqq 2\bar{u} \text{ on } X$$

hence $(1-\phi)gf$ belongs to B' and we are done.

Now, assume that in addition, every element of $\bar{V}$ is dominated by a continuous element of $\bar{V}$.

(3)⇒(1). We proceed by contradiction. If (H) does not hold, condition (ND) is satisfied and we can assume that it is satisfied with a decreasing sequence of non-void subsets $J_k (k \in \mathbb{N})$ verifying $(J_{k+1})^- \subset J_k = (J_k)^0$ for every $k \in \mathbb{N}$. We can also suppose $n > 1$.

For every $k \geqq n$, we set $\varepsilon_k := \inf_{x \in J_k} u_n(x)/u_k(x)$ (>0, cf (ND)) and we define

$$B := \bigcap_{m \geqq n} ((\varepsilon_m^{-1} b_m \cap C(X)) + C\mathscr{U}_0(X)).$$

which is a bounded subset of $C\mathscr{U}(X)/C\mathscr{U}_0(X)$.

As every bounded subset of $C\mathscr{U}(X)$ is contained in a set of the type $\bar{v}(l_\infty)_1 (\bar{v} \in \bar{V})$, it remains to prove that $\forall \bar{v} = \inf_{m \in \mathbb{N}} \rho_m v_m, (\rho_m > 0 \ \forall m)$, we have

$$B \not\subset C\mathscr{U}_0(X) + (\tfrac{1}{4} b_n \cap C(X)) + (\bar{v}(l_\infty)_1 \cap C(X)) =: C'.$$

Indeed, let $\varepsilon_m := 1$ for $m = 1, \ldots, n-1$ and take $\bar{v} \in \bar{V} \cap C(X)$ and a sequence $r_m > 0 \ \forall m$ such that

$$r_m \geqq \varepsilon_m^{-1} \ \forall m, \ \inf_{m \in \mathbb{N}} r_m v_m \geqq \bar{v} \geqq 2 \inf_{m \in \mathbb{N}} \varepsilon_m^{-1} v_m.$$

Now, we use (ND) and the fact that the J_k's are open to construct sequences $k(j) \in \mathbb{N}$, $x_j \in X$ and $V_j \subset X (j \in \mathbb{N})$ such that, $\forall j \in \mathbb{N}$

$$k(1) = n, \ k(j) < k(j+1);$$

$$V_j = \text{open neighbourhood of } x_j;$$

$$V_j \subset J_{k(j)} \cap \left\{x: u_n(x)v_{k(j+1)}(x) < \inf\left\{\frac{1}{2\rho_{k(j+1)}}, \frac{1}{2r_{k(j+1)}}\right\}\right\}$$

(hence $V_j \subset J_{k(j)} \setminus (J_{k(j+1)})^-$, $\forall j \in \mathbb{N}$ and $V_j \cap V_l = \emptyset$ if $j \neq l$). Then, $\forall j \in \mathbb{N}$, let $f_j \in D(X, [0,1])$, $\operatorname{supp}(f_j) \subset V_j$, $f_j(x_j) = 1$ and define

$$f := v_n \sum_{j=1}^{\infty} f_j.$$

We claim that $f \in B \setminus C'$. To prove this, we proceed in several steps.

(a) *f is continuous on X.*
Indeed, take any $x \in X$.
If $x \in \bigcap_{j \in \mathbb{N}} J_j^-$, we obtain $\varepsilon_m^{-1} v_m(x) \geqq v_n(x)\ \forall m \in \mathbb{N}$, hence also

$$\tilde{v}(x) \geqq 2 \inf_{m \in \mathbb{N}} \varepsilon_m^{-1} v_m(x) > v_n(x).$$

Since v_n and $\tilde{v}$ are continuous, the set

$$V := \{y \in X: \tilde{v}(y) > v_n(y)\}$$

is a neighbourhood of x. Moreover, for every $j \in \mathbb{N}$, we have

$$V_j \subset \{y: 2r_{k(j+1)} v_{k(j+1)}(y) < v_n(y)\} \subset \{y: 2\tilde{v}(y) < v_n(y)\}$$

hence $V_j \cap V = \emptyset$ and finally $f = 0$ on V.

If there is j_0 such that $x \notin J_{j_0}^-$, then $V = X \setminus (J_{j_0})^-$ is an open neighbourhood of x which meets only finitely many V_j's (because $V_l \subset J_{k(l)}\ \forall l$ and $J_{k(l)} \subset J_{j_0}$ for $l \geqq j_0$), hence $f|_V$ is a finite sum of continuous functions.

(b) *f belongs to $C\mathcal{U}(X)$.*
Indeed, fix $m \in \mathbb{N}$, $m \geqq n$. We have

$$\sup_{x \in X} u_m(x)|f(x)| = \sup_{j \in \mathbb{N}, x \in V_j} \frac{u_m(x)}{u_n(x)} f_j(x)$$

$$\leqq \sup\left\{\sup_{x \in \bigcup_{j=1}^{m-1} \operatorname{supp}(f_j)} \frac{u_m(x)}{u_n(x)}, \varepsilon_m^{-1}\right\}$$

and hence the required conclusion.

(c) *f belongs to $B = \bigcap_{m \geqq n}((\varepsilon_m^{-1} b_m \cap C(X)) + C\mathcal{U}_0(X))$.*
Indeed, fix $m \in \mathbb{N}$, $m \geqq n$. Then

$$f=f^{(1,m)}+f^{(2,m)},$$

with

$$f^{(1,m)}=v_n\sum_{j=1}^{m-1}f_j;\quad f^{(2,m)}=v_n\sum_{j=m}^{+\infty}f_j$$

and

$$f^{(1,m)}\in D(X)\subset C\mathscr{U}_0(X);\quad f^{(2,m)}\in\varepsilon_m^{-1}b_m\cap C(X).$$

(d) Assume that there are $g\in(1/4)b_n\cap C(X)$, $h\in(l_\infty)_1$ and $w\in C\mathscr{U}_0(X)$ such that $f=g+\bar{v}h+w$. Then, for every $j\in\mathbb{N}$, we have

$$1=u_n(x_j)v_n(x_j)=u_n(x_j)f(x_j)\leqq\tfrac{1}{4}+u_n(x_j)|w(x_j)|+\rho_{k(j+1)}v_{k(j+1)}(x_j)u_n(x_j)$$

$$<\tfrac{1}{4}+u_n(x_j)|w(x_j)|+\tfrac{1}{2}.$$

As w belongs to $C(u_n)_0(X)$, to conclude, it suffices now to prove that the set $\{x_j\colon j\in\mathbb{N}\}$ is not relatively compact.

Indeed, if it was compact, we could find $x_0\in\bigcap_{N\in\mathbb{N}}\{x_j\colon j\geqq N\}^-$. But for every $N\in\mathbb{N}$ and $j\geqq N$, we have $x_j\in V_j\subset J_{k(j)}\subset J_{k(N)}\subset J_N$, which implies $\varepsilon_N^{-1}v_N(x_j)\geqq v_n(x_j)$ and finally the inclusion

$$\{x_j\colon j\geqq N\}^-\subset\{x\in X\colon\varepsilon_N^{-1}v_N(x)\geqq v_n(x)\}.$$

This implies that x_0 satisfies $\inf_{m\in\mathbb{N}}\varepsilon_m^{-1}v_m(x_0)\geqq v_n(x_0)$ hence also $\bar{v}(x_0)>v_n(x_0)$. As in the case (a) above, $V=\{x\in X\colon\tilde{v}(x)>v_n(x)\}$ is then a neighbourhood of x_0 and it follows that there exists M such that $\tilde{v}(x_M)>v_n(x_M)$. But this implies

$$r_{k(M+1)}v_{k(M+1)}(x_M)>v_n(x_M),$$

which is a contradiction because

$$x_M\in V_M\subset\left\{x\colon\frac{u_n(x)}{u_{k(M+1)}(x)}<\frac{1}{2r_{k(M+1)}}\right\}.$$

(1)⇒(2). We improve the proof of (1)⇒(3) in the case $\bar{V}$ that satisfies the continuous domination property (i.e. $\bar{V}\simeq\bar{V}\cap C(X)$).

As B is bounded in $C\mathscr{U}(X)/C\mathscr{U}_0(X)$, there is a sequence $\lambda_m>0$ $(m\in\mathbb{N})$ such that

$$B\subset\bigcap_{m\in\mathbb{N}}((\lambda_m b_m\cap C(X))+C\mathscr{U}_0(X)).$$

Using (H^{**}) (equivalent to (H)), we get $\bar{v}\in\bar{V}$ such that $\forall n\in\mathbb{N}$, $\exists M(n)\geqq n$:

$$\forall x\left(\inf_{1\leqq m\leqq M(n)} \lambda_m v_m(x) \geqq \frac{1}{4n} v_n(x) \Rightarrow \bar{v}(x) \geqq \frac{1}{4n} v_n(x)\right). \tag{5}$$

We define

$$\bar{u}' := \sup_{m\in\mathbb{N}} \inf\{4M\lambda_M \bar{v}, \lambda_1 v_1, \ldots, \lambda_M v_m\}.$$

We have $\bar{u}' \in \bar{V}$. Let $\tilde{u}$ be a strictly positive (the condition (H) implies the existence of a strictly positive element of $\bar{V}$) and continuous element of $\bar{V}$ such that $\tilde{u} \geqq 2\inf_{k\in\mathbb{N}} \lambda_k v_k$ and $\bar{u} \geqq \bar{u}'$. We claim that

$$B' := \bigcap_{m\in\mathbb{N}} (\lambda_m b_m \cap C(X) + C\mathcal{U}_0(X)) \subset 8\tilde{u}(l_\infty)_1 + C\mathcal{U}_0(X).$$

Indeed, let $f \in B'$. For every $m \in \mathbb{N}$, there are $f^{(m)} \in \lambda_m b_m \cap C(X)$ and $g^{(m)} \in C\mathcal{U}_0(X)$ such that $f = f^{(m)} + g^{(m)}$. Hence, for every $n \in \mathbb{N}$, there exists a compact subset $K_n \subset X$ such that

$$u_n(x)|g^{(k)}(x)| \leqq \frac{1}{4n};\ \forall k = 1, \ldots, M(n);\ \forall x \in X \backslash K_n. \tag{6}$$

We set $K_0 := \varnothing$; moreover, in the previous construction (this construction is possible except if X is compact. But then, the property is of course true and we have nothing to prove), we can assume that $\varnothing \neq K_n \subsetneqq (K_{n+1})^0$, $\forall n \in \mathbb{N}$.

We proceed again in several steps.

(a) Let $\phi'_1 \in D(X, [0,1])$ be such that $\phi'_1 = 1$ on K_1, $\mathrm{supp}(\phi_1) \subset (K_2)^0$ and, for every $n \geqq 2$, let $\phi'_n \in D(X, [0,1])$ be such that $\phi'_n = 1$ on $K_n \backslash (K_{n-1})^0$, $\mathrm{supp}(\phi_n) \subset (K_{n+1})^0 \backslash K_{n-2}$. Moreover, as the sets $F := \{x \in X : |f(x)| \leqq \tilde{u}(x)\}$ and $G := \{x \in X : |f(x)| \geqq 2\tilde{u}(x)\}$ are disjoint zero-sets of continuous functions, there is $h \in C(X, [0,1])$ satisfying $h = 0$ on F, $h = 1$ on G.

We show that $1 - h$, $h\phi'_n (n \in \mathbb{N})$ form a locally finite family $\mathcal{F}$ of continuous functions on X such that $\phi'(x) := 1 - h(x) + \sum_{n=1}^{+\infty} h(x)\phi'_n(x) > 0, \forall x \in X$.

Indeed, for every $x \in \bigcap_{n\in\mathbb{N}} (X \backslash K_n)$, we have $g^{(k)}(x) = 0\ \forall k \in \mathbb{N}$ (cf. (6)), hence $|f(x)| \leqq \lambda_k v_k(x)\ \forall k \in \mathbb{N}$ and finally $|f(x)| \leqq \inf_{k\in\mathbb{N}} \lambda_k v_k(x) < \tilde{u}(x)$; it follows that F is a neighbourhood of $\bigcap_{n\in\mathbb{N}} (X \backslash K_n)$. Then, as $h = 0$ on F, the family $\mathcal{F}$ is locally finite on $\bigcap_{n\in\mathbb{N}} (X \backslash K_n)$, and, by construction of the functions ϕ'_n, it is also locally finite on $\bigcap_{n\in\mathbb{N}} K_n$.

To prove that $\phi'(x) > 0\,\forall x \in X$, is suffices to remark that

$$h(x) \neq 0 \Rightarrow x \in \bigcup_{m\in\mathbb{N}} K_n \Rightarrow \begin{cases} 3 \geqq \sum_{n=1}^{+\infty} \phi'_n(x) \geqq 1 \\ \phi'(x) = 1 + h(x)(-1 + \sum_{n=1}^{+\infty} \phi'_n(x)) > 0. \end{cases}$$

We set

$$\phi_0 := \frac{1-h}{\phi'}, \quad \phi_n := \frac{h\phi'_n}{\phi'} (n \in \mathbb{N}).$$

(b) For every $n \in \mathbb{N}$, let us define

$$F_1^{(n)} := \{x \in X : u_n(x)|f(x)| \leqq 1/2n\}; \quad F_2^{(n)} := \{x \in X : u_n(x)|f(x)| \leqq 1/n\}$$

and take $h^{(n)} \in C(X, [0,1])$ such that $h^{(n)} = 0$ on $F_1^{(n)}$, $h^{(n)} = 1$ on $F_2^{(n)}$. For every $x \in X \backslash (K_n \cup F_1^{(n)})$, we have (cf. the decompositions $f = f^{(m)} + g^{(m)}$ of f)

$$\frac{1}{2n} < u_n(x)|f(x)| \leqq u_n(x) \inf_{1 \leqq k \leqq M(n)} \lambda_k v_k(x) + \frac{1}{4n},$$

hence

$$\frac{1}{4n} v_n(x) < \inf_{1 \leqq k \leqq M(n)} \lambda_k v_k(x)$$

and (from (5))

$$\frac{1}{4n} v_n(x) \leqq \bar{v}(x).$$

It follows that

$$\inf_{1 \leqq k \leqq n} \lambda_k v_k(x) \leqq \tilde{u}'(x) \leqq \tilde{u}(x)$$

and that

$$\begin{aligned} |f(x)| &\leqq \inf_{1 \leqq k \leqq M(n)} \lambda_k v_k(x) + \frac{v_n(x)}{4n} \\ &\leqq 2 \inf_{1 \leqq k \leqq M(n)} \lambda_k v_k(x) \\ &\leqq 2\tilde{u}(x). \end{aligned}$$

(c) Now, f can be decomposed as follows:

$$f = f\phi_0 + f(\phi_1 + \phi_2) + \sum_{n=3}^{+\infty} f\phi_n h^{(n-2)} + \sum_{n=3}^{+\infty} f\phi_n (1 - h^{(n-2)}).$$

For every $n \geqq 3$, we have

$$f\phi_n(x) h^{(n-2)}(x) \neq 0 \Rightarrow \begin{cases} x \notin F_1^{(n-2)} \\ x \in \operatorname{supp}(\phi_n) \subset (K_{n+1})^0 \backslash K_{n-2} \subset X \backslash K_{n-2}, \end{cases}$$

hence $|f(x)| \leqq 2\tilde{u}(x)$ and finally

$$\left|\sum_{n=3}^{+\infty} f(x)\phi_n(x)h^{(n-2)}(x)\right| \leqq 6\tilde{u}(x) \quad \forall x \in X.$$

Next, let us verify that $\sum_{n=3}^{+\infty} f\phi_n(1-h^{(n-2)})$ belongs to $C\mathscr{U}_0(X)$. First, this function clearly belongs to $C\mathscr{U}(X)$. Now fix $N \in \mathbb{N}$ and $\varepsilon > 0$. For every $n \geqq 3$, we have

$$f\phi_n(x)(1-h^{(n-2)}(x)) \neq 0 \Rightarrow x \in X \setminus F_2^{(n-2)} \Rightarrow u_{n-2}(x)|f(x)| \leqq \frac{1}{n-2}.$$

Hence, if $N' \in \mathbb{N}$ is such that $N' \geqq \sup\{N+2, 3\varepsilon^{-1}+2\}$, for every $x \notin K_{N'}$, we get

$$\begin{aligned} u_N(x)\left|\sum_{n=3}^{+\infty} f\phi_n(x)(1-h^{(n-2)}(x))\right| &= u_N(x)\left|\sum_{n=N'}^{+\infty} f\phi_n(x)(1-h^{(n-2)}(x))\right| \\ &\leqq \sum_{n=N'}^{+\infty} |f(x)|\phi_n(x)u_{n-2}(x)(1-h^{(n-2)}(x)) \\ &\leqq \frac{3}{N'-2} \leqq \varepsilon. \end{aligned}$$

Finally, as $\phi_0 = 0$ on $G = \{x \in X : |f(x)| \geqq 2\tilde{u}(x)\}$, we also have $|f(x)|\phi_0(x) \leqq 2\tilde{u}(x)$, $\forall x \in X$.

Hence the conclusion: f belongs to the set $8\tilde{u}(l_\infty)_1 + C\mathscr{U}_0(X)$. □

Let us now recall Lemma 1 of [5] which shows how distinguishedness and lifting of bounded sets are connected.

Lemma 3 ([5]). *Let E, F be Fréchet spaces such that $E \subset F \subset E''$, and let q: $F \to F/E$ denote the quotient map. Assume that F is distinguished. Then*

(i) *F/E is distinguished, and*
(ii) *$\forall$ bounded subset B of F/E, $\exists A$ bounded subset of F such that $B \subset (q(A))^-$.*

Proposition 2 and the lemma recalled above lead now to the following result.

Theorem 4. *Let X be locally compact, $\mathscr{V} \subset C(X)$ and $\bar{V} \simeq \bar{V} \cap C(X)$. Then the following properties are equivalent:*

(1) *$C\mathscr{U}(X)$ is distinguished;*
(2) *$C\mathscr{U}(X)$ (resp. $C\mathscr{U}_0(X)$) satisfies S. Heinrich's density condition;*
(3) *$\mathscr{V}$ satisfies (H);*

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(4) $\forall B$ *bounded subset of* $C\mathcal{U}(X)/C\mathcal{U}_0(X)$, $\exists C$ *bounded subset of* $C\mathcal{U}(X)$ *such that* $B\subset Q(C)$;

(5) $\forall B$ *bounded subset of* $C\mathcal{U}(X)/C\mathcal{U}_0(X)$, $\exists C$ *bounded subset of* $C\mathcal{U}(X)$ *such that* $B\subset(Q(C))^{-C\mathcal{U}(X)/C\mathcal{U}_0(X)}$.

Proof. From [1], we know that (2) and (3) are equivalent (this result is valid without the assumption $\bar{V}\simeq\bar{V}\cap C(X)$).

The equivalence between (3), (4) and (5) is proved in the preceding proposition.

As $E=C\mathcal{U}_0(X)$ and $F=C\mathcal{U}(X)$ are Fréchet spaces satisfying $E\subset F\subset E''$, we can apply Lemma 1 of [5] and we get $(1)\Rightarrow(5)$.

As S. Heinrich's density condition for Fréchet spaces implies distinguishedness, the proof is complete. □

Corollary 5. *Let* $A=(a_n)_{n\in\mathbb{N}}$ *be a Köthe matrix on a discrete space X and let q denote the quotient map* $\lambda_\infty(A)\to\lambda_\infty(A)/\lambda_0(A)$. *Then the following properties are equivalent*:

(1) $\lambda_\infty(A)$ *is distinguished*;

(2) $\lambda_1(A)$ *is distinguished*;

(3) $\lambda_\infty(A)$ (*resp.* $\lambda_1(A)$) *satisfies S. Heinrich's density condition*;

(4) $\forall B$ *bounded subset of* $\lambda_\infty(A)/\lambda_0(A)$, $\exists C$ *bounded subset of* $\lambda_\infty(A)$ *such that* $B\subset q(C)$;

(5) $\forall B$ *bounded subset of* $\lambda_\infty(A)/\lambda_0(A)$, $\exists C$ *bounded subset of* $\lambda_\infty(A)$ *such that* $B\subset(q(C))^{-\lambda_\infty(A)/\lambda_0(A)}$.

Remark. Completely independently from this paper, E. Shalück (Universität-GH-Paderborn) obtained results about the distinguishedness of weighted spaces $CV_0(X)$. He proved that if V is an increasing sequence of strictly positive and continuous functions on a locally compact Hausdorff space X such that every lower semi-continuous $v: X\to[0,+\infty[\cup\{\infty\}$ satisfying $\sup_{x\in X}v_n(x)/v(x)<\infty$ $\forall n\in\mathbb{N}$ is dominated by a continuous function of the same type, then $CV_0(X)$ is distinguished.

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