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UNIQUE EXTREMALITY, LOCAL EXTREMALITY AND EXTREMAL NON-DECREASABLE DILATATIONS

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Given a quasi-symmetric self-homeomorphism h of the unit circle S^1, let $Q(h)$ be the set of all quasiconformal mappings with the boundary correspondence h. In this paper, it is shown that there exists certain quasi-symmetric homeomorphism h, such that $Q(h)$ satisfies either of the conditions,

(1) $Q(h)$ admits a quasiconformal mapping that is both uniquely locally-extremal and uniquely extremal-non-decreasable instead of being uniquely extremal;

(2) $Q(h)$ *contains infinitely many quasiconformal mappings each of which* has an extremal non-decreasable dilatation.

An infinitesimal version of this result is also obtained.

1. Introduction

Let Δ be the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. Given a quasisymmetric homeomorphism h of the unit disk S^1 onto itself, we denote by $Q(h)$ the class of all quasiconformal mappings from Δ onto itself with the boundary correspondence h. A quasiconformal mapping $f_0 \in Q(h)$ is said to be an extremal mapping for the boundary correspondence h if it minimises the maximal dilatations of $Q(h)$, that is,

$$K[f_0] = K[h] := \inf\{K[f] : f \in Q(h)\},$$

where $K[f]$ is the maximal dilatation of f. f is uniquely extremal if it is extremal and if there are no other extremal mappings for its boundary values; the alternative is that f is non-uniquely extremal.

The notion of non-decreasable was first introduced by Reich in [7] to investigate the unique extremality of quasiconformal mappings between the unit disks with given

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boundary values. An element f in $Q(h)$ has a *non-descreasable dilatation* (or f is called *non-decreasable*), if the hypothesis that g is also in $Q(h)$ together with the condition,

$$(1.1) \qquad |\nu(z)| \leqslant |\mu(z)| \text{ almost everywhere in } \Delta,$$

imply that $f = g$, where μ and ν are the Beltrami coefficients of f and g, respectively. Obviously, if f is uniquely extremal, then it has non-decreasable dilatation. But the converse is not true. So the conception of quasiconformal mappings with non-decreasable dilatations is a generalisation of uniquely extremal quasiconformal mappings.

In [**8**], Shen and Chen proved that, if $Q(h)$ does not contain a conformal mapping, then it must contain infinitely many elements with non-decreasable dilatations. So it is more interesting to investigate extremal quasiconformal mappings with non-decreasable dilatations; accordingly, such non-decreasable dilatations are called extremal ones. It is still an open problem whether an extremal quasiconformal mapping with non-decreasable dilatation always exists in $Q(h)$.

Following [**9**], a quasiconformal mapping f of Δ is said to be *locally extremal* if for any domain $G \subset \Delta$ the mapping f is extremal in G with respect to its boundary values. The complex dilatation μ of f is then called *locally extremal dilatation.* Generally speaking, both the uniqueness and the existence of locally extremal quasiconformal mappings in $Q(h)$ are not clear. An example due to Reich ([**5**], or see [**11**]) shows that local extremality does not imply unique extremality.

Obviously, if f is uniquely extremal, then f is the quasiconformal mapping in $Q(h)$ that is both uniquely locally-extremal and uniquely extremal-non-decreasable. Conversely, one might ask

PROBLEM 1. If f in $Q(h)$ is the quasiconformal mapping that is both uniquely locally-extremal and uniquely extremal-non-decreasable, is it then uniquely extremal?

REMARK 1. If f has an extremal-non-decreasable dilatation $\mu(z)$ with the property that $|\mu(z)|$ =constant almost everhwyere in Δ, then it is obviously uniquely extremal; the converse is not true, as is a well-known result in [**2**]. There are a lot of examples (see [**8**, Corollary 3.1]) to show that uniqueness of extremal non-decreasable dilatations does not imply unique extremality.

On the other hand, it is natural to pose the following problem.

PROBLEM 2. Does there exist h such that $Q(h)$ contains infinitely many extremal quasiconformal mappings with non-decreasable dilatations?

Our main result Theorem 1 gives a negative answer to Problem 1 and a positive one to Problem 2, respectively. Meanwhile, an infinitesimal version is obtained for the tangent space of the universal Teichmüller space.

2. Preliminaries

Let $\mathfrak{D}$ be a domain in the complex plane $\mathbb{C}$ with at least two boundary points and let $M(\mathfrak{D})$ be the open unit ball of $L^\infty(\mathfrak{D})$. Every element $\mu \in M(\mathfrak{D})$ can be regarded as an element in $L^\infty(\mathbb{C})$ by putting μ equal to zero in the outside of $\mathfrak{D}$. Every $\mu \in M(\mathfrak{D})$ induces a global quasiconformal self-mapping f of the plane which solves the Beltrami equation [**1**],

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{2.1}$$

and f is defined uniquely up to postcomposition by a complex affine map of the plane. Conversely, any quasiconformal mapping f defined on $\mathfrak{D}$ has a Beltrami coefficient $\mu(z) = f_{\bar{z}}(z)/f_z(z)$ in $M(\mathfrak{D})$.

Two Beltrami coefficients $\mu, \nu \in M(\mathfrak{D})$ are equivalent if they induce quasiconformal mappings f and g by (2.1) such that there is a conformal map c from $f(\mathfrak{D})$ to $g(\mathfrak{D})$ and an isotopy through quasiconformal mappings h_t, $0 \leqslant t \leqslant 1$, from $\mathfrak{D}$ to $\mathfrak{D}$ which extend continuously to the boundary of $\mathfrak{D}$ such that

1. $h_0(z)$ is identically equal to z on $\mathfrak{D}$,
2. h_1 is identically to $g^{-1} \circ c \circ f$, and
3. $h_t(p) = g^{-1} \circ c \circ f(p)$ for any $p \in \partial\mathfrak{D}$.

The equivalence relation partitions $M(\mathfrak{D})$ into equivalence classes and the space of equivalence classes is by definition the Teichmüller space $T(\mathfrak{D})$ of $\mathfrak{D}$.

Given $\mu \in M(\mathfrak{D})$, we denote by $[\mu]$ the set of all elements $\nu \in M(\mathfrak{D})$ equivalent to μ, and set

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

We say that μ is extremal (in $[\mu]$) if $\|\mu\|_\infty = k_0([\mu])$, μ is uniquely extremal if $\|\nu\|_\infty > k_0([\mu])$ for any other $\nu \in [\mu]$; the alternative is that μ is non-uniquely extremal. We say that μ is non-decreasable if for any other $\nu \in [\mu]$, the set on which $|\nu(z)| > |\mu(z)|$ has positive measure. Obviously, μ is non-decreasable if it is uniquely extremal.

For any μ, define $h^*(\mu)$ to be the infimum over all compact subsets F contained in $\mathfrak{D}$ of the essential supremum norm of the Beltrami coefficient $\mu(z)$ as z varies over $\mathfrak{D}\backslash F$. Define $h([\mu])$ to be the infimum of $h^*(\mu)$ taken over all representatives μ of the class $[\mu]$. It is obvious that $h([\mu]) \leqslant k_0([\mu])$. Following [**3**], we call a point $[\mu]$ in $T(\mathfrak{D})$ a Strebel point if $h([\mu]) < k_0([\mu])$.

Let $A(\mathfrak{D})$ be the space of integrable holomorphic quadratic differentials φ on $\mathfrak{D}$ and let $A_1(\mathfrak{D})$ be the unit sphere of $A(\mathfrak{D})$. By Strebel's frame mapping theorem, every Strebel point $[\mu]$ is represented by the unique Beltrami differential of the form $k|\varphi|/\varphi$, where $k = k_0([\mu]) \in (0,1)$ and φ is a unit vector in $A_1(\mathfrak{D})$.

Two elements μ and ν in $L^\infty(\mathfrak{D})$ are infinitesimally equivalent, which is denoted by $\mu \approx \nu$, if $\iint_{\mathfrak{D}} \mu\phi\,dx\,dy = \iint_{\Delta} \nu\phi\,dx\,dy$ for all $\phi \in A(\Delta)$. Denote by $N(\mathfrak{D})$ the set of all the elements in $L^\infty(\mathfrak{D})$ which are infinitesimally equivalent to zero. Then $B(\mathfrak{D}) = L^\infty(\mathfrak{D})/N(\mathfrak{D})$ is the tangent space of the space $T(\mathfrak{D})$ at the basepoint.

Given $\mu \in L^\infty(\mathfrak{D})$, we denote by $[\mu]_B$ the set of all elements $\nu \in L^\infty(\mathfrak{D})$ infinitesimally equivalent to μ, and set

$$\|\mu\| = \inf\{\|\nu\|_\infty : \nu \in [\mu]_B\}.$$

We say that μ is infinitesimally extremal (in $[\mu]_B$) if $\|\mu\|_\infty = \|\mu\|$, uniquely infinitesimally extremal if $\|\nu\|_\infty > \|\mu\|$ for any other $\nu \in [\mu]_B$. We say that μ is infinitesimally non-decreasable if for any other $\nu \in [\mu]_B$, the set on which $|\nu(z)| > |\mu(z)|$ has positive measure. Then μ is non-decreasable if it is uniquely extremal.

In a parallel manner we can define the boundary dilatation for the infinitesimal Teichmüller class $[\mu]_B$. The boundary dilatation $b([\mu]_B)$ is the infimum over all elements in the equivalence class $[\mu]_B$ of the quantity $b^*(\nu)$. Here $b^*(\nu)$ is the infimum over all compact subsets F contained in $\mathfrak{D}$ of the essential supremum of the Beltrami coefficient ν as z varies over $\mathfrak{D} - F$.

An infinitesimally equivalent class $[\mu]_B$ is called an infinitesimal Strebel point if $\|\mu\| > b([\mu]_B)$. It follows from the infinitesimal frame mapping theorem (see [**4**, Theorem 2.4]) that if $[\mu]_B$ is an infinitesimal Strebel point, then there exists a unique vector φ in $A_1(\mathfrak{D})$ such that μ and $\|\mu\||\varphi|/\varphi$ are infinitesimally equivalent.

3. Some preparations

For $\mu \in L^\infty(\Delta)$, $\phi \in A(\Delta)$, let

$$\lambda_\mu[\phi] = Re \iint_\Delta \mu(z)\phi(z)dxdy.$$

As is well known, a Beltrami coefficient μ is extremal if and only if it has a so-called Hamilton sequence, namely, a sequence $\{\phi_n \in A(\Delta) : \|\phi_n\| = 1,\ n \in \mathbb{N}\}$, such that

$$\lim_{n\to\infty} \lambda_\mu[\phi_n] = \lim_{n\to\infty} Re \iint_\Delta \mu\phi_n(z)dxdy = \|\mu\|_\infty. \tag{3.1}$$

Given $\mu \in M(\Delta)$, let $f = f^\mu$ be the uniquely determined quasiconformal mapping of Δ onto itself with Beltrami coefficients μ and normalised to fix 1, -1 and i.

Suppose that μ and ν are two equivalent Beltrami coefficients in $T(\Delta)$. Let $\widetilde{\mu}$ and $\widetilde{\nu}$ be the Beltrami coefficients of the quasiconformal mappings f^{-1} and g^{-1}, respectively, where $f = f^\mu$ and $g = f^\nu$. Let $\mathfrak{J} \subset \Delta$ be a Jordan domain with $\overline{\mathfrak{J}} \subset \Delta$.

Lemma 1. *Let μ and ν be two equivalent Beltrami coefficients in $T(\Delta)$. In addition, suppose $\mu(z) = \nu(z)$ for almost every $z \in \Delta\backslash\overline{\mathfrak{J}}$. Then, $f^\mu(z) = f^\nu(z)$ for all z in $\Delta\backslash\mathfrak{J}$ and hence $\widetilde{\mu}(w) = \widetilde{\nu}(w)$ for almost all w in $f(\Delta\backslash\mathfrak{J})$.*

PROOF: For the sake of convenience, let $f = f^{\mu}$ and $g = f^{\nu}$. Let $\mu_{g\circ f^{-1}}(w)$ denote the Beltrami coefficient of $g \circ f^{-1}$. By a simple computation, we have

$$\mu_{g\circ f^{-1}} \circ f(z) = \frac{1}{\tau}\frac{\mu(z) - \nu(z)}{1 - \overline{\mu(z)}\nu(z)},$$

where $\tau = \overline{f_z}/f_z$.

Thus, $\mu_{g\circ f^{-1}}(w) = 0$ for almost all $w \in f(\Delta\backslash\overline{\mathfrak{J}})$ and hence $\Psi = g \circ f^{-1}$ is conformal on $\Delta\backslash\overline{\mathfrak{J}}$. Since $\Psi|_{S^1} = g \circ f^{-1}|_{S^1} = id$, we conclude that $\Psi = id$ in $f(\Delta\backslash\mathfrak{J})$. Thus, $g|_{\Delta\backslash\overline{\mathfrak{J}}} = f|_{\Delta\backslash\overline{\mathfrak{J}}}$. By the continuity of quasiconformal mappings, it follows that $g|_{\Delta\backslash\mathfrak{J}} = f|_{\Delta\backslash\mathfrak{J}}$. In addition, it is evident that $\widetilde{\mu}(w) = \widetilde{\nu}(w)$ for almost all w in $f(\Delta\backslash\mathfrak{J})$. ∎

The following Reich's Construction Theorem is very useful. It was used by the author **[10]** to show that there exists h such that all extremal quasiconformal mappings in $Q(h)$ are not of Teichmüller type.

CONSTRUCTION THEOREM. ([**6**]) *Let A be a compact subset of Δ containing at least two points and such that $\Delta\backslash A$ is doubly connected. There exists a function $\alpha \in L^{\infty}(\Delta)$ and a sequence $\varphi_n \in A(\Delta)$ $(n = 1, 2, \ldots)$ satisfying the following conditions* (3.2)-(3.5)*:*

$$|\alpha(z)| = \begin{cases} 0, & z \in A, \\ 1, & \text{for almost all } z \in \Delta\backslash A, \end{cases} \tag{3.2}$$

$$\lim_{n\to\infty}\{\|\varphi_n\| - \lambda_{\alpha}[\varphi_n]\} = 0, \tag{3.3}$$

$$\lim_{n\to\infty}|\varphi_n(z)| = \infty \quad \text{almost everywhere in } \Delta\backslash A. \tag{3.4}$$

and as $n \to \infty$,

$$\varphi_n(z) \to 0 \text{ uniformly on } A. \tag{3.5}$$

REMARK 2. Equation (3.5) is implied in the proof of Reich's Construction Theorem [**6**].

From Reich's Construction Theorem, we can get

LEMMA 2. *Let $J \subset \Delta$ be a Jordan domain with $A = \overline{J} \subset \Delta$. Let $\alpha(z)$ and the sequence $\varphi_n \in A(\Delta)$ be constructed by Reich's Construction Theorem and let $\mu(z) = k\alpha(z)$ where $k < 1$ is a positive constant. Set*

$$\nu(z) = \begin{cases} \mu(z), & z \in \Delta\backslash A, \\ \beta(z), & z \in A, \end{cases}$$

where $\beta(z)$ is in $M(J)$ with $\|\beta\|_{\infty} \leqslant k$. Then

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(1) $\nu(z)$ *is extremal in* $[\nu]$ *and for any* $\chi(z)$ *extremal in* $[\nu]$, $\chi(z) = \nu(z)$ *for almost all* z *in* $\Delta \backslash A$;

(2) $\nu(z)$ *is extremal in* $[\nu]_B$ *and for any* $\chi(z)$ *extremal in* $[\nu]_B$, $\chi(z) = \nu(z)$ *for almost all* z *in* $\Delta \backslash A$.

PROOF: The proof of the first part of this lemma is the same as that of [**10**, Lemma 4] and the proof of the second part is included in that of [**10**, Theorem 3]. ∎

Recall that a Beltrami coefficient μ in $\mathfrak{D}$ is said to be locally extremal if for any domain $G \subset \mathfrak{D}$ it is extremal in its class in $T(G)$; in other words,

$$\|\mu\|_G := \operatorname*{esssup}_{z\in G} |\mu| = \sup\left\{ \frac{Re \iint_G \mu\phi(z)dxdy}{\iint_G |\phi(z)|dxdy} : \phi \in A(G) \right\}.$$

Obviously, extremality in the whole domain is a prerequisite for a Beltrami coefficient to be locally extremal.

LEMMA 3. *Using the notations of Lemma 2, then* ν *is locally extremal in* Δ *if and only if* β *is locally extremal in* J.

PROOF: The necessary part is a fortiori. Now let β is locally extremal in J. For given domain $G \subset \Delta$ with $G \backslash J \neq \emptyset$, by

$$k \iint_{G\backslash J} |\varphi_n(z)|dxdy - Re \iint_{G\backslash J} \mu(z)\varphi_n(z)dxdy \leqslant \|\varphi_n\| - \lambda_\alpha[\varphi_n],$$

and Reich' Construction Theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(k \iint_G |\varphi_n(z)|dxdy - Re \iint_G \nu(z)\varphi_n(z)dxdy\right) \\
&\qquad \leqslant \lim_{n\to\infty}\left(k \iint_{G\backslash J} |\varphi_n(z)|dxdy - Re \iint_{G\backslash J} \mu(z)\varphi_n(z)dxdy\right) \\
&\qquad\qquad + \lim_{n\to\infty}\left(k \iint_J |\varphi_n(z)|dxdy - Re \iint_J \beta(z)\varphi_n(z)dxdy\right) = 0.
\end{aligned}$$

Moreover, by equation (3.4) and Fatou's lemma,

$$\lim_{n\to\infty} \iint_G |\varphi_n(z)|dxdy \geqslant \lim_{n\to\infty} \iint_{G\backslash J} |\varphi_n(z)|dxdy = \infty,$$

where the fact that $(G\backslash J)^o \neq \emptyset$ is needed. Thus,

$$k - \frac{Re \iint_G \nu(z)\varphi_n(z)dxdy}{\iint_G |\varphi_n(z)|dxdy} \to 0, \; n \to \infty,$$

which indicates that $\nu(z)$ is extremal in its class in $T(G)$. Thus, ν is locally extremal in Δ. ∎

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4. Main theorem

By definition, the following lemma is evident.

Lemma 4. *μ is an extremal-non-decreasable Beltrami coefficient in $[\mu]$ if and only if for any other η extremal in $[\mu]$, the set on which $|\eta(z)| > |\mu(z)|$ has positive measure.*

Let $\Delta_r = \{z : |z| < r\}$ for $r \in (0,1)$. Choose $s = \frac{1}{4}$, $t = \frac{1}{2}$ and $A = \overline{\Delta_t}$.

Lemma 5. *Let $\chi(z)$ be defined as follows,*

$$\chi(z) = \begin{cases} 0, & z \in A - \Delta_s, \\ \widetilde{k} & z \in \Delta_s, \end{cases}$$

where $\widetilde{k} < 1$ is a positive constant. Then $[\chi]$ as a point of the Teichmüller space $T(\Delta_t)$ of Δ_t contains infinitely many non-decreasable Beltrami coefficients η with $\|\eta\|_\infty < \widetilde{k}$.

Proof: Let $s < r < t$. Note that $\chi(z) = 0$ in $A \backslash \Delta_s$. When restricted to Δ_r, $[\chi]$ as a point of $T(\Delta_r)$ has the property $h([\chi]) = 0$ and hence is a Strebel point in $T(\Delta_r)$. Thus, by Strebel's frame mapping theorem, there exist $k_r \in (0,1)$ and a unit vector $\varphi_r \in A_1(\Delta_r)$ such that $k_r|\varphi_r|/\varphi_r$ and χ are equivalent in $T(\Delta_r)$. In addition, it is clear that $k_r < \widetilde{k}$. Put

$$\chi_r(z) = \begin{cases} 0, & z \in A - \Delta_r, \\ k_r \dfrac{|\varphi_r(z)|}{\varphi_r(z)} & z \in \Delta_r. \end{cases}$$

Then χ_r and χ are equivalent in $T(\Delta_t)$. Applying Lemma 1, it is easy to see that χ_{r_1} and χ_{r_2} restricted to Δ_{r_2} are equivalent in $T(\Delta_{r_2})$ whenever $s < r_1 < r_2 < t$. Thus, k_r is a strictly decreasing function as $r \in (s,t)$. Furthermore, we claim that χ_r is non-decreasable in $[\chi]$. Suppose to the contrary. Then there would exist η in $[\chi]$ such that $|\eta(z)| \leqslant |\chi_r(z)|$ for almost all $z \in \Delta_t$. Obviously, $\eta(z) = \chi_r(z) = 0$ on $A - \Delta_r$. Applying Lemma 1 again, we see that η and χ_r restricted to Δ_r are equivalent in $T(\Delta_r)$. This happens if and only if $\eta = \chi_r$, which implies our claim. Thus, this lemma follows. ∎

Theorem 1. *Let $A = \overline{\Delta_t}$ and let $\alpha(z)$ be constructed by Reich's Construction Theorem. Put $\mu(z) = k\alpha(z)$, where $k \in (0,1)$ is a constant. Set*

$$\nu(z) = \begin{cases} \mu(z), & z \in \Delta \backslash A, \\ 0, & z \in A - \Delta_s, \\ \widetilde{k} & z \in \Delta_s, \end{cases}$$

where $\widetilde{k} \in [0,k]$ is a constant. Then,

(1) *when $\widetilde{k} > 0$, $[\nu]$ contains infinitely many extremal non-decreasable Beltrami coefficients;*

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(2) *if $\widetilde{k} = 0$, then ν is the Beltrami coefficient in $[\nu]$ that is both uniquely locally-extremal (obviously, non-uniquely extremal) and uniquely extremal-non-decreasable.*

And hence, if we set $h = f^{\nu}$, then either $Q(h)$ contains infinitely many extremal quasiconformal mappings with non-decreasable dilatations (when $\widetilde{k} > 0$) or admits an extremal quasiconformal mapping (but not uniquely extremal) that is both uniquely locally-extremal and uniquely extremal-non-decreasable (when $\widetilde{k} = 0$).

PROOF: First, let $0 < \widetilde{k} \leqslant k$. By Lemma 2, for any η extremal in $[\nu]$, $\eta(z) = \nu(z)$ almost everywhere on $\Delta \backslash A$. Then by Lemma 1, $\eta(z)$ and $\nu(z)$ restricted to Δ_t are equivalent in $T(\Delta_t)$. Therefore, by Lemma 4, if η restricted to Δ_t is non-decreasable in its equivalence class $[\chi]$ (defined in Lemma 5), then it is non-decreasable in $[\nu]$ in $T(\Delta)$.

For $s < r < t$, put

$$\nu_r(z) = \begin{cases} \mu(z), & z \in \Delta \backslash A, \\ 0, & z \in A - \Delta_r, \\ k_r \dfrac{|\varphi_r(z)|}{\varphi_r(z)} & z \in \Delta_r. \end{cases}$$

where k_r and φ_r are from Lemma 5. Then ν_r is an extremal non-decreasable dilatation in $[\nu]$ by Lemma 5. Thus, (1) of Theorem 1 is proved.

Now, let $\widetilde{k} = 0$. It follows directly from Lemmas 2, 4 that ν is the element in $[\nu]$ that is uniquely extremal-non-decreasable. Since $\beta \equiv 0$ on A, as an immediate consequence of Lemma 3, ν is locally-extremal in $[\nu]$. On the other hand, the uniqueness of local extremal follows clearly from Lemma 2 and the definition of local extremality. □

REMARK 3. The example of local extremal (of course, instead of being uniquely extremal) given by Reich [5] has a constant modulus, whereas our example does not. The modulus of certain extremal Beltrami coefficients was discussed in a recent paper [12] of the author (joint with Yi Qi).

5. INFINITESIMAL VERSION

We have the infinitesimal version of Lemma 4 as follows.

LEMMA 6. *μ is an infinitesimally extremal-non-decreasable Beltrami coefficient in $[\mu]_B$ if and only if for any other η extremal in $[\mu]_B$, the set on which $|\eta(z)| > |\mu(z)|$ has positive measure.*

LEMMA 7. *Let $\chi(z)$ be defined as in Lemma 5. Then $[\chi]_B$ as a point of the space $B(\Delta_t)$ of Δ_t contains infinitely many non-decreasable extremals η with $\|\eta\|_\infty < \widetilde{k}$.*

The proof of Lemma 7 is a suitable modification from that of Lemma 5 except that the infinitesimal frame mapping criterion is used here.

Theorem 2. *Let ν be the same as in Theorem 1. Then either $[\nu]_B$ contains infinitely many infinitesimally non-decreasable extremals when $0 < \widetilde{k} \leqslant k$, or ν is the element in $[\nu]_B$ that is both uniquely locally-extremal (obviously, non-uniquely infinitesimally extremal) and uniquely infinitesimally extremal-non-decreasable if $\widetilde{k} = 0$.*

Proof: By Lemmas 2, 6, 7, the proof almost takes word by word from that of Theorem 1 and so is skipped. □

At last, we end this paper with an open problem.

Problem 3. Does there exist h such that each extremal quasiconformal mapping (of course, non-uniquely extremal) in $Q(h)$ has a non-decreasable dilatation?

If the answer is positive, then each extremal quasiconformal mapping in such $Q(h)$ is also locally extremal.

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