Journal of Glaciology, Vol. 14, No. 70, 1975

A MEASUREMENT OF SURFACE-PERPENDICULAR STRAIN-RATE IN A GLACIER

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Abstract. The surface-perpendicular component of velocity and strain-rate have been determined at one site in the ablation area of Blue Glacier, Washington, U.S.A., where the total depth is about 250 m. The strain-rate is near zero at the surface but increases with depth to about 4% a^{-1} at 175 m. The results were obtained with the help of a finite deformation theory from the measured stretch of cables frozen into the ice.

Résumé. *Une mesure de la composante perpendiculaire à la surface des contraintes dans un glacier.* La composante perpendiculaire à la surface de la vitesse et la contrainte a été déterminée en un point de la zone d'ablation du Blue Glacier, Washington, U.S.A., où la profondeur totale est d'environ 250 m. Cette contrainte est à peu près nulle en surface, mais croît avec la profondeur jusqu'à 4% a^{-1} à 175 m. Les résultats ont été obtenus avec l'aide de la théorie des déformations finies à partir de la mesure de l'allongement de câbles gelés dans la glace.

Zusammenfassung. *Eine Messung der zur Oberfläche normalen Verformungsgeschwindigkeit in einem Gletscher.* Die zur Oberfläche normale Komponente der Geschwindigkeit und der Verformungsgeschwindigkeit wurde an einer Stelle in der Ablationszone des Blue Glaciers (Washington, U.S.A.) bestimmt, wo die Gesamttiefe etwa 250 m beträgt. Die Verformungsgeschwindigkeit ist beinahe Null an der Oberfläche, doch steigt sie mit der Tiefe auf etwa 4% pro Jahr bei 175 m an. Die Ergebnisse wurden mit Hilfe einer finiten Deformationstheorie aus der gemessenen Dehnung von Kabeln, die im Eis festgefroren waren, gewonnen.

Introduction

In glacier bore-hole experiments the component of deformation parallel to the hole is normally not measured, although at least one method of doing so has been devised recently (Rogers and LaChapelle, 1974). This component can also be estimated using a suitable bore-hole array and the assumption of incompressibility (Raymond, 1971[a]). The measurement described here was obtained from a study of the stretching of cables frozen into deep holes to measure temperature. The study was carried out to aid the interpretation of the temperatures, which are reported separately (Harrison, 1975). The work was done on Blue Glacier, a temperate glacier on Mt Olympus, Washington, U.S.A., near the site of a previous bore-hole deformation experiment by Shreve and Sharp (1970, p. 67). The site is about 50 m up-glacier from the point designated S2 on their map, and is in the ablation area at an elevation of about 1 590 m.

Procedure

Nine cables were set to different depths in separate, uncased bore holes about 0.7 m apart. Freeze-in time was about one week. Electrical cable, capable of at least 10% strain before failure, was used. Cables were placed in August 1971 and removed 13 months later by melting them free with a small electric current. Total cable stretch was the sum of both elastic and permanent contributions. The elastic part was found by measuring the amount of cable pulled into the hole on melting free. The permanent part was found by tape-measuring the cable before placement and after removal. More experimental details are available (Harrison, 1975).

The method involves two basic assumptions. The first is that should a cable contract from the bottom on melting free, its lower end slides to the bottom of its hole before the length pulled in is measured. The second, which requires justification, is that a cable does not move significantly with respect to the ice.

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The ice can exert a shear stress τ of roughly 1 bar on the surface of a cable, either through a direct bond to the ice, or if there is an intervening water film, by a mechanism analagous to glacier sliding with cable roughness provided by the observed permanent crushing of the cable sheath into space around the electrical conductors. If cable tension T is zero at the end, this shear will cause T to increase linearly from there according to

$$T = 2\pi a\tau l$$

where a is the cable radius (2 mm), and l is distance from the end. T reaches the yield strength of the cable (roughly 700N) about 0.5 m from the end. Motion of the cable with respect to the ice should occur over this length, but further from the end the cable should yield and follow the motion of the ice. Relative motion over 0.5 m would be unimportant. Final results indicate that the cables never were in compression in this particular experiment.

The basic assumptions are thought to be fairly well satisfied. Any breakdown will cause the strain-rates to be underestimated.

Theory

Initially vertical bore holes have been observed to decrease in length after drilling (Fletcher and Kamb, 1968), and the effect may be fairly typical. After going through a minimum, the length begins to increase and does so indefinitely. This can be understood easily in the case of a glacier deforming in simple shear parallel to its surface. Any element of the hole will be shortened until it is rotated perpendicular to the plane of shear, after which it will be lengthened. The effect, which may be important in interpreting cable stretch or similar bore-hole strain experiments, cannot be described by infinitesimal strain theory, in which deformation increases linearly with time. Before analyzing the data, we must, therefore, work out a finite deformation theory which includes terms at least to second order in time.

We will use a right-handed coordinate system fixed in space with the x-axis pointing down slope parallel to the surface of the glacier and the y-axis pointing downward perpendicular to the surface. Often it will be convenient to use the notation x_i for the coordinates (x, y, z) and similarly for other vector or tensor quantities. Summation over the ranges of repeated suffixes will be understood.

Our goal is to find an expression for the extension of a length of cable in terms of properties of the velocity field along its initial position and the elapsed time. We first express extension in terms of components of a displacement field U_i. If a particle initially located at x_i moves to X_i, referred to the same fixed coordinate system, U_i are defined by

$$X_i - x_i = U_i(x, y, z).$$

It is straightforward (see, for example, Fung, 1965, chapter 4) to show that during the motion from x_i to X_i a short element with initial direction cosines l_i undergoes an extension e (change in length per unit initial length) given by

$$e = \sqrt{1+2E_{ij}l_il_j}-1 \tag{1}$$

where E_{ij} are components of Green's tensor defined in terms of U_i by

$$E_{ij} = \frac{1}{2}\left[\frac{\partial U_j}{\partial x_i}+\frac{\partial U_i}{\partial x_j}+\frac{\partial U_m}{\partial x_i}\frac{\partial U_m}{\partial x_j}\right]. \tag{2}$$

To use these expressions to find the dependence of the extension on the velocity field u_i and the time t, we need to find the dependence of U_i on u_i and t. This is done by solving

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = u_i(X, Y, Z), \tag{3}$$

the definition of a velocity field which, we assume, is time independent. The deformation begins at time zero, when the coordinates X_i of the short element under consideration have the values x_i. To solve Equations (3), we note that from them higher time derivatives of X_i can be found with the chain rule, thus permitting the Taylor's series solution to be constructed:

$$X_i - x_i \equiv U_i = u_i t + \tfrac{1}{2} u_m \frac{\partial u_i}{\partial x_m} t^2 + \ldots \tag{4}$$

where the velocity field components u_i are evaluated at x_i, the initial position of the short element. Equations (4) describe, in the vicinity of x_i, the streamline through that point.

Now Equations (4) can be used to eliminate U_i in Equations (2), with the following result:

$$E_{ij} = \dot{e}_{ij} t + \tfrac{1}{2} \ddot{e}_{ij} t^2 + \ldots \tag{5}$$

where

$$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

is the usual strain-rate tensor, and

$$\ddot{e}_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial x_i}\left(u_m \frac{\partial u_j}{\partial x_m}\right) + 2\,\frac{\partial u_m}{\partial x_i}\,\frac{\partial u_m}{\partial x_j} + \frac{\partial}{\partial x_j}\left(u_m \frac{\partial u_i}{\partial x_m}\right)\right]$$

is another tensor. Evidently $\dot{e}_{ij}$ and $\ddot{e}_{ij}$ are respectively the first and second time derivatives of E_{ij} evaluated at time zero. Finally Equations (5) can be substituted in a Taylor's series expansion of Equation (1) to give

$$e = (\dot{e}_{ij} l_i l_j) t + \tfrac{1}{2}[\ddot{e}_{ij} l_i l_j - (\dot{e}_{ij} l_i l_j)^2] t^2 + \ldots \tag{6}$$

where $\dot{e}_{ij}$, $\ddot{e}_{ij}$ and the direction cosines l_i are evaluated at the initial position x_i of the short element which undergoes the extension e. The first-order term is a familiar result of small deformation theory. Evidently $\dot{e}_{ij} l_i l_j$ and $[\ddot{e}_{ij} l_i l_j - (\dot{e}_{ij} l_i l_j)^2]$ are respectively the first and second time derivatives of e, evaluated at time zero. Equation (6) is the desired result. It expresses the change in length per unit initial length of a short element of cable in terms of the velocity field at its initial position and the elapsed time.

Equation (6) contains many terms but most of them are negligible for (1) reasonable assumptions about the velocity field consistent with the surface measurements of Meier and others (1974) and the bore-hole measurements of Shreve and Sharp (1970); (2) small glacier surface slope α. The direction cosines l_i for an initially vertical bore hole* are the constants ($\sin\alpha$, $\cos\alpha$, 0), and since α is about 4.8°, the latter assumption is valid. We can also set $\cos\alpha \approx 1$. Equation (6) reduces to

$$e \approx \dot{e}_{yy} t + 2\,[\sin\alpha\,(\dot{e}_{xy} t) + \dot{e}_{xy}^2\, t^2]. \tag{7}$$

A study of the case of simple shear indicates that terms of third and fourth order in time are negligible, and that Equation (7) reduces to this case as it should when $\dot{e}_{yy} = 0$. Equation (7) may not be particularly accurate because of many simplifications, an example being $\dot{e}_{xy} \approx \frac{1}{2}\,\partial u/\partial y$. But its purpose is to determine surface-perpendicular velocity and strain-rate from the observed cable stretch, and we shall see later that the errors are dominated by uncertainty in the stretch. Equation (7) is therefore adequate, at least for this experiment.

The change in length ΔL of a finite length of cable is found by integrating the extension e along the initial position of the cable:

$$\Delta L = \int e(x, y, z)\, \mathrm{d}s$$

* The assumption that the bore hole is initially vertical seems to be justified. See the discussion by Raymond (1971[b], p. 64) who used a similar drilling technique.

where ds is a length element along the cable. However, in this experiment only the y-dependence of e is significant along the cable and $\cos \alpha \approx 1$, so

$$\Delta L = \int_0^y e(y')\, dy'$$

where y is the initial length of the cable. Using Equation (7) and rearranging terms, we finally get a useful relation between surface-perpendicular strain-rate $\dot{e}_{yy}$ and cable stretch ΔL:

$$\left(\int_0^y \dot{e}_{yy}\, dy'\right) t = \Delta L - S \tag{8}$$

where

$$S = 2\left[\sin \alpha \left(\int_0^y \dot{e}_{xy}\, dy'\right) t + \left(\int_0^y \dot{e}_{xy}^2\, dy'\right) t^2\right] \tag{9}$$

is the shear contribution to the total cable stretch ΔL, and can be evaluated from the data of Shreve and Sharp (1970).

Equation (8) is all that is needed for the strain-rate determination. However, to interpret the temperature measurements for which the cables were originally placed, it is important to work out the change in depth of the lower end of a cable, because the temperature sensing element was there. This change Δd is given by

$$\Delta d = V - V_s + b$$

where V and V_s are the y-displacements, given by Equation (4), of respectively the ice at the bottom of the cable and the ice at the surface, and b is the thickness of ice added to the surface. To an adequate approximation

$$\Delta d = (v - v_s)\, t + b \tag{10}$$

where v and v_s are the corresponding ice velocities, and t is the elapsed time. But

$$v - v_s = \int_0^y \dot{e}_{yy}\, dy' \tag{11}$$

where y is the length of the cable. Hence Equation (10) can be written

$$\Delta d = \left(\int_0^y \dot{e}_{yy}\, dy'\right) t + b \tag{12}$$

and the integral can be evaluated from Equation (8).

Results

The calculations are outlined in Table I. Column I gives the initial depth of each cable, and column II, the measured total stretch ΔL. Column III gives the shear contribution S to the stretch, as found from Equation (9) using a surface slope α of 4.8° and values of the shear strain-rate $\dot{e}_{xy}$ measured by Shreve and Sharp (1970). For convenience $\dot{e}_{xy}$ was found from the approximation $\dot{e}_{xy} \approx \frac{1}{2}\, \partial u/\partial y$, and the functional form

$$u \approx u_s - c y^{n+1}$$

where u_s is the value of u at the surface, $c = 1.8 \times 10^{-8}\ \text{m}^{-n}\ \text{a}^{-1}$ and $n = 2.9$. This was obtained from Shreve and Sharp's data by Meier and others (1974). It is seen in Column

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TABLE I. CALCULATION OF SURFACE-PERPENDICULAR VELOCITY v, DEPTH, AND STRAIN-RATE $\dot{e}_{yy}$. DETAILS IN TEXT

I	II	III	IV	V	VI	VII	VIII	IX	X
Initial depth August 1971	ΔL *August 1971 to September 1972*	S	$\int_0^y \dot{e}_{yy}\,dy'$	v	*Depth October 1971*	*Depth September 1972*	$\dot{e}_{yy}$	*Depth interval*	*Average depth*
m	m	m	m a⁻¹	m a⁻¹	m	m	% a⁻¹	m	m
6.9	0 ±0.05	0	0 ±0.05	−1.80±0.05	5.6	5.8			
14.5	0 ±0.05	0	0 ±0.05	−1.80±0.05	13.2	13.4			
15.0	0 ±0.05	0	0 ±0.05	−1.80±0.05	13.7	13.9	0 ±0.3	0 – 25.1	12.5
25.1	0 ±0.05	0	0 ±0.05	−1.80±0.05	23.7	24.0			
50.0	0.04±0.1	−0.01	0.05±0.1	−1.75±0.1	48.7	48.9	0.2±0.4	25.1– 50.0	37.6
84.1	0.16±0.1	−0.05	0.19±0.1	−1.61±0.1	82.8	83.2	0.4±0.4	50.0– 84.1	67.0
119.4	0.59±0.2	−0.15	0.68±0.2	−1.12±0.2	118.2	119.1	1.4±0.7	84.1–119.4	101.8
154.5	1.63±0.3	−0.24	1.73±0.3	−0.07±0.3	153.5	155.3	3.0±1.0	119.4–154.5	137.0
189.9	3.26±0.4	+0.07	2.95±0.4	+1.15±0.4	189.1	192.0	3.5±1.5	154.5–189.9	172.2

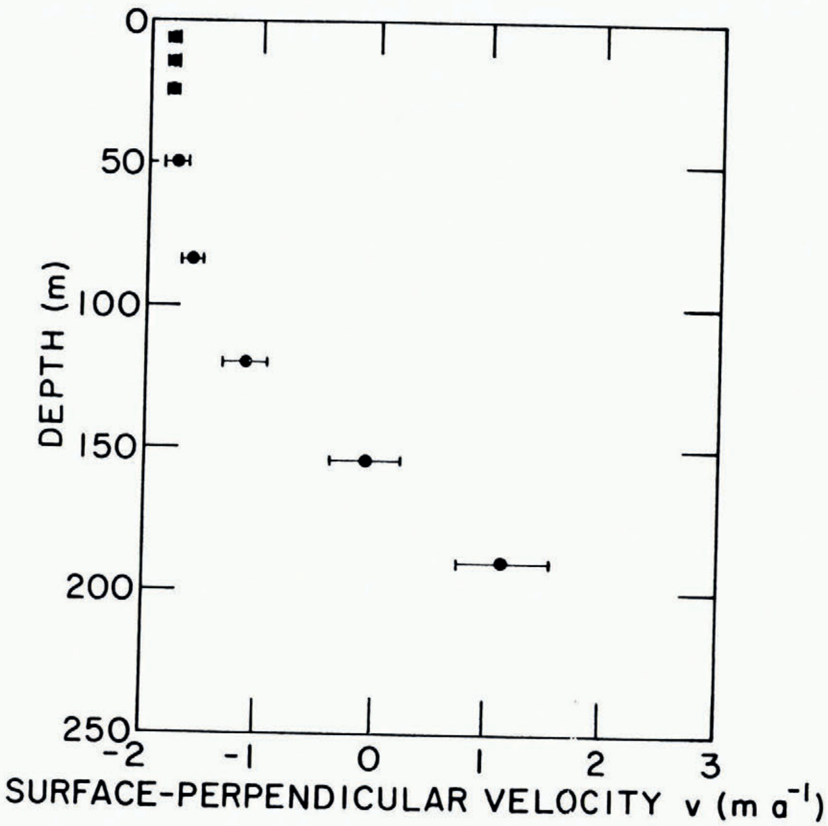

Fig. 1. Dependence of surface-perpendicular velocity on depth. A downward velocity is positive. In addition to the errors shown, there is an uncertainty in the zero position of the velocity scale of roughly 0.7 m a⁻¹. Total depth is about 250 m (Corbató, 1965; Meier and others, 1974).

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III that S is negative except for the deepest cable; only for it does the non-linear term in Equation (9) dominate. Column IV gives the integral of the surface-perpendicular strain-rate from the surface to the depths given in column I, as found from Equation (8). Because $S \ll \Delta L$ for all cables, the result is rather insensitive to errors in S that might be caused by errors in $\dot{e}_{xy}$, α, or the basic relation, Equation (7). Final errors are dominated by uncertainty in cable stretch ΔL.

The integrated strain-rate in column IV is related to the surface-perpendicular velocity v at depth y and that at the surface v_s by Equation (11). Column V gives v assuming $v_s = -1.8$ m a^{-1}, which was obtained by averaging data of Meier and others (1974) over an area with dimensions of the order of the glacier thickness. The errors given do not include the uncertainty in v_s, which is difficult to estimate but may be about 0.7 m a^{-1}. In Figure 1 v is plotted against depth.

The depth of the end of each cable was calculated from the initial depth in column I using Equation (12). Columns VI and VII give the depths after time intervals ending on 9 October 1971 and 11 September 1972 when temperatures were measured. The ice added at the surface b for these intervals was -1.3 m and -1.1 m respectively.

Column VIII gives the surface-perpendicular component of the strain-rate $\dot{e}_{yy}$, found by subtracting successive entries in column IV and dividing by the appropriate depth interval, as obtained from column I. These values of $\dot{e}_{yy}$ are thus averages over the depth intervals given in column IX; the average depth of the interval is given in column X. The result is

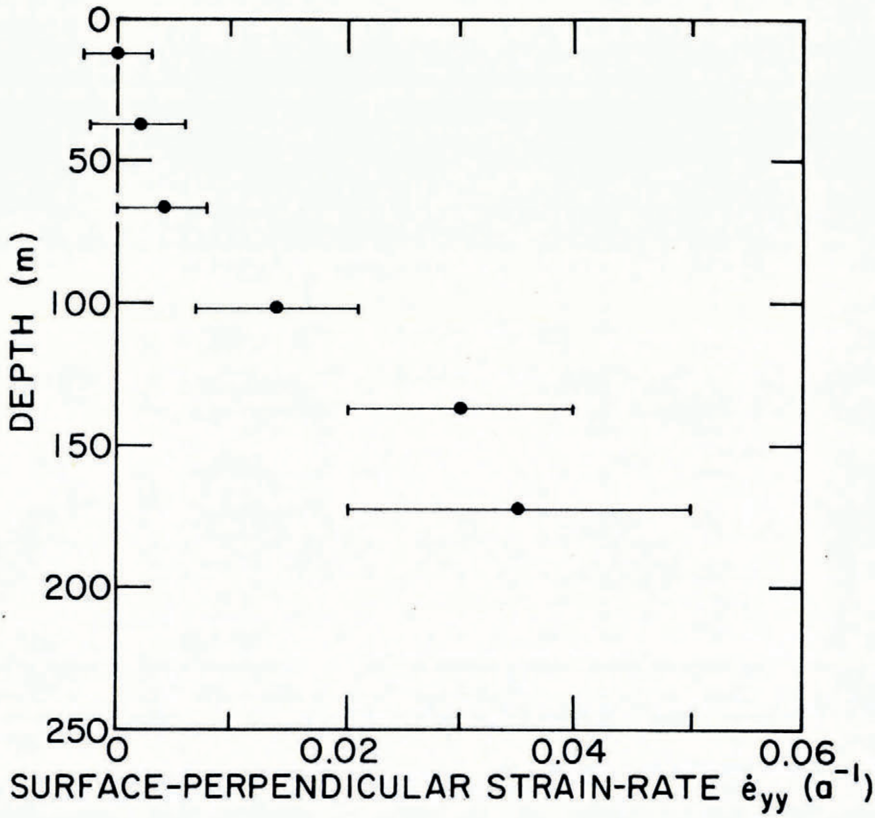

Fig. 2. Dependence of surface-perpendicular strain-rate on depth. Total depth is about 250 m.

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plotted in Figure 2. Evidently the surface-perpendicular strain-rate is positive (extensional) and increases with depth in this region of Blue Glacier.

The positive mean value of $\dot{e}_{yy}$ is expected. Because the site is in the ablation area, the surface-perpendicular velocity component at the surface v_s should be upward, or $v_s < 0$. Because the glacier thickens down-stream (see, for example, Shreve and Sharp, 1970, p. 82), the same component at the bed v_b should be downward, or $v_b > 0$. Figure 1 is consistent with these features. Therefore the mean value of $\dot{e}_{yy}$, which is given by $(v_b - v_s)/h$, where h is the glacier thickness, should indeed be positive.

The reason for the increase of $\dot{e}_{yy}$ with depth is not so obvious. In this context the longitudinal strain-rate $\dot{e}_{xx}$ determined by Shreve and Sharp (1970, p. 81), which shows the opposite behaviour, is of interest. A calculation based on a uniform bending model (Raymond, 1971[b], p. 76) using surface velocity measurements of Meier and others (1974) indicates that bending may be able to account qualitatively for the behaviour of $\dot{e}_{xx}$. If the transverse strain-rate $\dot{e}_{zz}$ were zero, by incompressibility $\dot{e}_{yy} = -\dot{e}_{xx}$, and bending might also describe the behaviour of $\dot{e}_{yy}$. But probably the picture is more complicated. For example, although the data show that $\dot{e}_{yy}$ and $\dot{e}_{xx}$ do tend to be equal and opposite in the upper part of the glacier, there is some indication that this may break down at greater depth.

Acknowledgements

I am indebted to Professor R. L. Shreve for providing unpublished strain-rate data and other information, and to an anonymous referee for his comments. William Glude and David Bell helped with the field work, and the National Park Service gave permission for work on the glacier. The National Science Foundation provided the principal financial support under grants GU 2655 and GA 28544. The analysis was supported by State of Alaska funds.

MS. received 19 March 1974 and in revised form 21 May 1974

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