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SOME CONTINUED FRACTIONS OF RAMANUJAN AND MEIXNER-POLLACZEK POLYNOMIALS

BY

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ABSTRACT. We examine the convergence and analytic properties of a continued fraction of Ramanujan and its connection to the orthogonal polynomials of Meixner-Pollaczek.

1. **Introduction.** B. Berndt et al. [3] have analysed the entries in Ch. 12 of Ramanujan's second notebook. The majority of entries deal with continued fractions (43 of 49 entries). Of these 43 entries, over half (22/43) are connected with continued fractions of the form

$$CF(z) = z + \operatorname*{K}_{n=1}^{\infty} \left(\frac{-(an^2+bn+c)}{z-dn} \right). \tag{1}$$

That is, continued fractions whose nth partial numerators and denominators are polynomials in n of degree $\leqq 2$ and 1 respectively.

For this class of continued fractions the associated difference equation

$$X_{n+1} - (z-dn)X_n + (an^2+bn+c)X_{n-1} = 0 \tag{2}$$

can be solved exactly in terms of the hypergeometric function ${}_2F_1$ and its limits ${}_1F_1, \Psi, D_\lambda$ and ${}_0F_1$ [7]. Also for a certain range of the parameters, (1) and (2) are related to the orthogonal polynomials of Meixner-Pollaczek [2], [7].

These facts coupled with Pincherle's Theorem [9] allow one to reanalyse many of Ramanujan's continued fractions in greater detail by stating:

1. the precise domain of convergence in the parameter space,
2. the rate of convergence,
3. analytic properties including analytic continuation.

In Sec. 2 we give some background theorems which we apply in Sec. 3 to Ramanujan's Entry 25 ([10], p. 147). See also [3], p. 268 for references to Euler, Stieltjes and Perron.

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2. **Background.** Since Pincherle's Theorem is such a key ingredient linking a continued fraction and its associated difference equation, we repeat a version of it here.

THEOREM (Pincherle [9]): *Let* $a_n \neq 0$, $n \geqq 1$. *Then*

$$\operatorname*{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$$

converges with approximants

$$\operatorname*{K}_{n=1}^{N}\left(\frac{a_n}{b_n}\right) = -X_1^{(s)}/X_0^{(s)} + O(X_{N+1}^{(s)}/X_{N+1}^{(d)}) \tag{3}$$

iff there exists linearly independent solutions $X_n^{(s)}, X_n^{(d)}$ *(subdominant and dominant respectively) to the difference equation*

$$X_{n+1} - b_n X_n - a_n X_{n-1} = 0$$

with the property

$$\lim_{n\to\infty} X_n^{(s)}/X_n^{(d)} = 0.$$

Thus the existence of a subdominant solution yields a necessary and sufficient condition for the convergence of the association continued fraction, an estimate on its rate of convergence and its value in terms of a ratio of subdominant terms. For accessible proofs of the above see [5], [6].

Although a subdominant solution is numerically elusive and explicit examples are rare, one does have the exact analytic result given below [7].

THEOREM 1. *Let* $a, d^2 - 4a \neq 0$. *Then*

$$X_{n+1} - (z - dn)X_n + (an^2 + bn + c)X_{n-1} = 0$$

has: (a) linearly independent solutions

$$X_{n-1}^{\pm}\begin{pmatrix} a, b, c \\ d, z \end{pmatrix} = \left(\pm\frac{a}{\mu}\right)^n \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+\gamma^{\pm})}\, {}_2F_1(n+\alpha, n+\beta; n+\gamma^{\pm}; \delta^{\pm}) \tag{4}$$

where

$$\begin{aligned} \mu &= \sqrt{d^2 - 4a}, -\pi/2 < \arg\mu \leqq \pi/2 \\ \delta^{\pm} &= \frac{1}{2}(1 \pm d/\mu) \\ \gamma^{\pm} &= \left(\frac{a+b}{a}\right)\delta^{\pm} \pm z/\mu \\ a(n+\alpha)(n+\beta) &= an^2 + bn + c, \end{aligned} \tag{5}$$

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(b) *a subdominant solution iff*

$$|\mathrm{Re}(d/\mu)| + \left|\mathrm{Re}\left(\left(\frac{a+b}{2a}\right)\frac{d}{\mu}+\frac{z}{\mu}\right)\right| \neq 0$$

given by

$$X_n^{(s)} = \begin{cases} X_n^+ & \textit{if } \mathrm{Re}\left(\frac{d}{\mu}\right) < 0 \textit{ or if } \mathrm{Re}\left(\frac{d}{\mu}\right) = 0 \textit{ and } \mathrm{Re}(\gamma^+ - \gamma^-) > 0 \\ X_n^- & \textit{if } \mathrm{Re}\left(\frac{d}{\mu}\right) > 0 \textit{ or if } \mathrm{Re}\left(\frac{d}{\mu}\right) = 0 \textit{ and } \mathrm{Re}(\gamma^+ - \gamma^-) < 0 \end{cases}$$

(6) $$|X_n^{(s)}/X_n^{(d)}| = \text{const.}\left[\frac{\left(1-\left|\mathrm{Re}\left(\frac{d}{\mu}\right)\right|\right)^2+\left(\mathrm{Im}\left(\frac{d}{\mu}\right)\right)^2}{\left(1+\left|\mathrm{Re}\left(\frac{d}{\mu}\right)\right|\right)^2+\left(\mathrm{Im}\left(\frac{d}{\mu}\right)\right)^2}\right]^{n/2} \times n^{-|\mathrm{Re}(\gamma^+-\gamma^-)|}\left(1+O\left(\frac{1}{n}\right)\right).$$

For a proof of Theorem 1 and the special case $d = 0$ see [7], [8].

3. **Application.** As an example of the use of the above, we examine Ramanujan's Entry 25 ([10], p. 147) which may be precisely stated as:

ENTRY 25. *One has*

$$\frac{\Gamma\left(\frac{x+k+1}{4}\right)\Gamma\left(\frac{x-k+1}{4}\right)}{\Gamma\left(\frac{x+k+3}{4}\right)\Gamma\left(\frac{x-k+3}{4}\right)} = \frac{4}{x-}\,\frac{k^2-1^2}{2x}\,-\,\frac{k^2-3^2}{2x}\,-\cdots$$

iff $\mathrm{Re}\ x > 0$ *or* $k^2 = 1^2, 3^2, \cdots$.

Although Ramanujan provides no proof and states no conditions on the parameters x, k the Entry 25 above follows from the more detailed statement below concerning the related J-fraction

(7) $$1/CF(z) = \frac{1}{z-}\,\frac{(1^2-k^2)/4}{z}\,-\,\frac{(3^2-k^2)/4}{z}\,-\cdots.$$

THEOREM 2. *If* $\pm\mathrm{Im}\ z > 0$ *then the Nth approximant of* (7) *is*

(8) $$\frac{1}{z-}\,\frac{(1^2-k^2)/4}{z}\,-\cdots-\,\frac{((2N-1)^2-k^2)/4}{z} = f_\pm(z) + O(N^{-|\mathrm{Im} z|})$$

where

(9) $$f_\pm(z) = 2\left[z \pm 4i\left\{\frac{\Gamma\left(\frac{3+k\mp iz}{4}\right)\Gamma\left(\frac{3-k\mp iz}{4}\right)}{\Gamma\left(\frac{1+k\mp iz}{4}\right)\Gamma\left(\frac{1-k\mp iz}{4}\right)}\right\}\right]^{-1}.$$

Furthermore if $k^2 < 1$ then this Nth approximant is a ratio of Meixner-Pollaczek polynomials with the denominator polynomials orthogonal with respect to the real line positive measure $dw(x)$ with

$$\frac{dw(x)}{dx} = \frac{i}{2\pi}(f_+(x) - f_-(x)), x \in (-\infty, \infty). \tag{10}$$

PROOF. By comparing (7) and (1) one has $a = 1, b = -1, c = (1 - k^2)/4$ and $d = 0$. From (5) this yields $\mu = 2i, \delta^{\pm} = \frac{1}{2}, \gamma^{\pm} = \pm z/2i, \alpha = -\frac{1}{2} + \frac{k}{2}$ and $\beta = -\frac{1}{2} - \frac{k}{2}$. From Theorem 1 and Pincherle's Theorem one then obtains (8) and (9) after expressing

$$_2F_1\left(-\frac{1}{2} + \frac{k}{2}, -\frac{1}{2} - \frac{k}{2}; \pm i\frac{z}{2}; \frac{1}{2}\right) \quad \text{and} \quad {}_2F_1\left(\frac{1}{2} + \frac{k}{2}, \frac{1}{2} - \frac{k}{2}; 1 \pm i\frac{z}{2}; \frac{1}{2}\right)$$

in terms of Γ functions using [4], 2.8 (31), (32), (51). The connection between the approximants of (7) and the orthogonal polynomials of Meixner-Pollaczek is detailed in [7] (see also [2]) and follows from the general theory of J-fractions and matrices in [1], [11]. The essential feature is that, for $k^2 < 1$, (7) is a real J-fraction with Cauchy representation

$$1/CF(z) = \int_{-\infty}^{\infty} \frac{dw(x)}{z - x} \tag{11}$$

in terms of a positive measure $dw(x)$. One then has

$$\frac{1}{z-}\,\frac{(1^2 - k^2)/4}{z} \,-\, \cdots \,-\, \frac{((2N-1)^2 - k^2)/4}{z} = \frac{P_N^{\lambda}(z/2, C+1)}{P_{N+1}^{\lambda}(z/2, C)(C+1)}$$

where $P_N^{\lambda}(x, C)$ is a Pollaczek polynomial with $C = (-1+k)/2, \lambda = (1-k)/2$ satisfying $(N+C+1)P_{N+1}^{\lambda}(x, C) - 2xP_N^{\lambda}(x, C) + (N+C+2\lambda-1)P_{N-1}^{\lambda}(x, C) = 0, P_{-1}^{\lambda} = 0, P_0^{\lambda} = 1$ and $\int P_N^{\lambda}(x/2, C)P_M^{\lambda}(x/2, C)dw(x) = 0, N \neq M$. Eq. (10) now follows from (8), (9) and (11). □

One can always express (1) in terms of Γ functions provided that $d = 0$ and $b/a = 0, \pm 1, \pm 2, \cdots$. Entry 25 is a particular case with $b/a = -1$. It yields an interesting example of associated Pollaczek polynomial measures which may be simply expressed in terms of Γ functions.

Note that for this example k^2 can be negative with k then pure imaginary and C, λ complex. The fact that the Pollaczek parameters can be complex seems to have been neglected in the literature (see [2]). Entry 25 provides a simple example of this type.

Note also that $f_{\pm}(z)$ are each meromorphic functions for $z \in \mathbf{C}$. Thus the analytic continuation of (7) from one half plane to the other, yields two related meromorphic functions.

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