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A LOWER BOUND FOR THE PERMANENT ON A SPECIAL CLASS OF MATRICES

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ABSTRACT. Let $U_n(r)$ denote the class of all $n\times n$ (0, 1)-matrices with precisely r-ones, $r\geq 3$, in each row and column. Then

$$\min_{A\in U_n(r)} (\text{per } A) \geq (r-1)!+n(r-2)!+\cdots+n(2!)+n+1.$$

A brief discussion of the main tool of our investigation, the r-nearly decomposable matrix [3], is now given. We define this matrix as follows:

An $n\times n$ (0, 1)-matrix A is r-nearly decomposable if A is fully indecomposable and possess t, $t>0$, ones in different rows and columns so that when any one of these 1's is replaced by 0 yielding A', A' is partly decomposable. Further if all of these 1's are replaced by 0's yielding A'', A'' has precisely r-ones in each row and column. The following lemma concerning this matrix is of particular importance.

LEMMA 1. *Suppose $U_m^*(r)$ denotes the class of all fully indecomposable matrices with precisely r, $r\geq 3$, ones in each row and column. Let $M_m(r)=\min_{A\in U_m^*(r)}(\text{per } A)$. Then if A is $n\times n$ and r-nearly decomposable with $\sum_{i,j} a_{ij}-nr=t$ it follows that* per $A\geq M_m(r)+t(r-1)!$.

Proof. The proof is essentially that of Lemma 2 in [3], making use of a stronger form of Theorem B implied by Hall's inequality, i.e.

THEOREM B′. *If A is an $n\times n$ fully indecomposable* (0, 1)*-matrix with at least k ones in each row and column, then each* 1 *is on at least* $(k-1)!$ *positive diagonals.*

This then provides the impetus for the result of the paper.

THEOREM. *If $A\in U_n(r)$, $r\geq 3$, then*

$$\text{per } A \geq (r-1)!+n(r-2)!+\cdots+n(2!)+n+1.$$

Proof. The proof is by induction on r. For $r=3$ the result is that of Hartfiel [4], hence suppose the theorem holds for all r, $3\leq r<h$. Now let A have h ones in each row and column. As $\min_{A\in U_n(h)}(\text{per } A)$ is achieved on a fully indecomposable matrix [5] we may assume A is fully indecomposable. Let Π be any positive diagonal in A. Replace as many ones of Π as possible with 0's, say $n-t$ in number,

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yielding a matrix A' so that:

Case I. If $t=0$, A' is fully indecomposable.

Case II. If $t>0$, A' is $(h-1)$-nearly decomposable.

Note that $t<n$ is a consequence of the special form for h-nearly decomposable matrices as $h\geq 3$ [3]. In either case by Lemma 1

$$\begin{aligned}\text{per } A' &\geq M_n(h-1)+t(h-2)! \\ &\geq [(h-2)!+n(h-3)!+\cdots+n(2!)+n+1]+t(h-2)!.\end{aligned}$$

Replace $n-t-1$ of the removed 1's on Π and note that by Hall's inequality each of these 1's is on $(h-2)!$ positive diagonals. By replacing the remaining 1 on Π yields $(h-1)!$ positive diagonals. Hence

$$\begin{aligned}\text{per } A &\geq \text{per } A'+(n-t-1)(h-2)!+(h-1)! \\ &= (h-1)!+n(h-2)!+\cdots+n(2!)+n+1.\end{aligned}$$

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