SOME ORDER PROPERTIES OF COVERINGS OF FINITE-DIMENSIONAL SPACES†

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1. Definitions and introduction. Let $\mathfrak{U} = \{U_i \mid i \in I\}$ be a system of subsets of a normal topological space R; i.e. a mapping from the index set I into the set of all subsets of R. The order of a point x is the number of distinct member sets of $\mathfrak{U}$ which contain x, and is denoted by $x : \mathfrak{U}$; the sets U_i are here considered distinct if they have distinct indices. Thus $x : \mathfrak{U}$ is the number of indices i for which $x \in U_i$; $\nu(\mathfrak{U}) = \max\{x : \mathfrak{U} \mid x \in R\}$ is called the order of the system $\mathfrak{U}$. If every point has an (open) neighbourhood meeting only finitely many members of $\mathfrak{U}$, then $\mathfrak{U}$ is said to be locally finite.

We shall call $\mathfrak{U}$ a k-covering of R if $x : \mathfrak{U} \geqq k$ for some positive integer k and all points x. The covering $\mathfrak{V} = \{V_j \mid j \in J\}$ is said to be a refinement of the covering $\mathfrak{U}$ if, for each j, there is an index $i = \sigma(j)$ such that $V_j \subset U_i$. Moreover, the refinement $\mathfrak{V}$ is called finite-to-one, one-to-one, or strict according as the mapping $\sigma : J \to I$ can be chosen such that σ is finite-to-one, σ is one-to-one, or σ is one-to-one and $\bar{V}_j \subset U_i$.

Theorem 1 of § 2 shows that if the dimension of R is at most n then every finite open covering admits a finite open k-refinement of order at most $n+k$, and conversely ($k = 1, 2, \ldots$); when $k = 1$ this is merely the definition of $\dim R \leqq n$. The class of all finite open coverings involved here may be replaced by the class of all locally finite open coverings or by a certain type of subclass of the latter. Thus, if $\dim R \leqq n$, then a locally finite open covering admits a locally finite open k-refinement $\mathfrak{V}$ say, of order at most $n+k$. We show in Theorem 2 that $\mathfrak{V}$ may be chosen as a strict refinement.

In § 3 it is shown that if $\dim R \geqq n$ then, for any locally finite open (or closed) refinement $\mathfrak{U}$ of some suitably chosen finite open covering, there is a member set of $\mathfrak{U}$ on which the function $x : \mathfrak{U}$ assumes at least $n+1$ distinct values. This is a sharper result than the converse part of Theorem 1. If in addition R is paracompact then there is some point in each neighbourhood of which $x : \mathfrak{U}$ assumes at least $n+1$ values.

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2. The order of k-coverings. Two systems of subsets $\mathfrak{F}$ and $\mathfrak{G}$ are said to be similar if there is some one-to-one correspondence between their index sets such that any finite subsystem of $\mathfrak{F}$ has an empty intersection if and only if the corresponding subsystem of $\mathfrak{G}$ has an empty intersection. Hereafter we identify the index set of a system with a section of the ordinals $0, 1, \ldots, i, \ldots$ $(i < a)$ for some appropriate ordinal a. Also the underlying space is always understood to be normal.

LEMMA 1. *If $\{F_i \mid i < a\}$ and $\{U_i \mid i < a\}$ are locally finite systems such that F_i is closed, U_i is open and F_i lies in U_i, then there exists an open system $\{G_i \mid i < a\}$ such that $F_i \subset G_i$, $\bar{G}_i \subset U_i$ and $\{\bar{G}_i \mid i < a\}$ is similar to $\{F_i \mid i < a\}$.*

For a proof of this see [**4**].

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LEMMA 2. (An extension of a theorem due to Dieudonné [**2**]). *If* $\mathfrak{V} = \{V_i \mid i < a\}$ *is a locally finite open k-covering (of a normal space) then there exists an open k-refinement*

$$\mathfrak{W} = \{W_i \mid i < a\}$$

of $\mathfrak{V}$ *such that* $\overline{W}_i \subset V_i$.

Proof. Suppose that for all ordinals $i < j < j_0$, open sets W_i are defined such that

$$\overline{W}_i \subset V_i \quad (i < j)$$

and

$$\mathfrak{X}_j = \{W_i, V_h \mid i < j, h \geqq j\} \text{ is a } k\text{-covering.}$$

These conditions hold initially with $\mathfrak{X}_0 = \mathfrak{V}$ and $j_0 = 1$. In order to define W_j we consider first the set H_j of all points x such that

$$x : \{W_i, V_h \mid i < j, h > j\} < k.$$

From the induction hypothesis and the fact that $\mathfrak{X}_j$ is locally finite it follows easily that H_j is closed and lies in V_j, and so by normality we can define W_j to be an open set such that $H_j \subset W_j$, $\overline{W}_j \subset V_j$.

Since the systems $\mathfrak{X}_j$ and $\mathfrak{X}_{j+1}$ differ only in their j-th members it follows that $\mathfrak{X}_{j+1}$ is at least a $(k-1)$-covering. Now if x fails to belong to H_j, then $x : \mathfrak{X}_{j+1} \geqq k$; if otherwise, then x belongs to W_j and again $x : \mathfrak{X}_{j+1} \geqq k$.

If j_0 is a limit ordinal, then the open sets W_i $(i < j_0)$ are defined by the induction hypothesis and it is easily verified that $\mathfrak{X}_{j_0}$ is a k-covering. Thus the induction is complete and $\mathfrak{W} = \mathfrak{X}_a$ is a strict open k-refinement of $\mathfrak{V}$ as required.

We proceed to determine the dimension of a space in terms of its open k-coverings for each fixed value of k. Let $\{\mathfrak{U}\}$ denote a class of locally finite open coverings of a space R with the properties that each finite open covering of R admits a member covering as a refinement and each finite-to-one open refinement of a member is again a member.

THEOREM 1. $\dim R \leqq n$ *if and only if every covering* $\mathfrak{U}$ *admits a k-refinement* $\mathfrak{U}'$ *of order at most* $n+k$ $(\mathfrak{U}, \mathfrak{U}' \in \{\mathfrak{U}\}, k = 1, 2, \ldots)$.

COROLLARY. $\dim R \leqq n$ *if and only if every locally finite open covering of R admits a locally finite open k-refinement of order at most* $n+k$.

This follows by taking $\{\mathfrak{U}\}$ to be the class of all locally finite open coverings of R. As further examples we may take the class of all star-finite open coverings or the class of all finite open coverings.

Proof by induction over k. In the initial case, if $\dim R \leqq n$, then any locally finite open covering $\mathfrak{U} = \{U_i \mid i < a\}$ admits a locally finite open refinement $\mathfrak{V} = \{V_j \mid j < b\}$ of order at most $n+1$; for the proof of this see [**3**] or [**4**]. For each index j we can choose an index $i = \sigma(j)$ such that $V_j \subset U_i$ and, by putting $U'_i = \bigcup\{V_j \mid \sigma(j) = i\}$, we see that the system $\mathfrak{U}' = \{U'_i \mid i < a\}$ is a one-to-one open refinement of $\mathfrak{U}$ of order at most $n+1$. Thus if $\mathfrak{U}$ belongs to $\{\mathfrak{U}\}$ so does $\mathfrak{U}'$.

Conversely, if $\mathfrak{U}_0$ is any finite open covering then there exists a refinement $\mathfrak{V}$ of order at most $n+1$, which is also a member of $\{\mathfrak{U}\}$. The above process of uniting member sets of $\mathfrak{V}$ produces a finite open refinement of $\mathfrak{U}_0$ of order at most $n+1$. Hence $\dim R \leqq n$ and the case where $k = 1$ is established. The following lemma gives the inductive step and clearly suffices to prove the theorem.

LEMMA 3. *A locally finite open covering* $\mathfrak{U}$ *admits a finite-to-one open k-refinement of order at most p if and only if it admits a finite-to-one open* $(k+1)$*-refinement of order at most* $p+1$.

Proof. Let $\mathfrak{V} = \{V_i \mid i < a\}$ be a finite-to-one open k-refinement of $\mathfrak{U}$ such that $\nu(\mathfrak{V}) \leqq p$. We consider the following system of which a typical member set is

$$F_{i_1 \dots i_k} = \{x \mid x \in V_{i_1}, \dots, V_{i_k} \text{ only}\} \quad (i_1 < \dots < i_k < a).$$

Clearly this system consists of mutually disjoint closed sets and the neighbourhoods $V_{i_1} \cap \dots \cap V_{i_k}$ of $F_{i_1 \dots i_k}$ form a locally finite open system. Hence we may apply Lemma 1 to give the existence of mutually disjoint open sets $G_{i_1 \dots i_k}$ such that

$$F_{i_1 \dots i_k} \subset G_{i_1 \dots i_k} \subset V_{i_1} \cap \dots \cap V_{i_k}.$$

We now define a system $\mathfrak{G}$ consisting of the mutually disjoint open sets

$$G_i = \bigcup\{G_{i_1 \dots i_k} \mid i = i_1 < \dots < i_k\} \quad (i < a).$$

Since G_i lies in V_i, we see that the systems $\mathfrak{G}$ and $\mathfrak{V}$ taken together form a finite-to-one open $(k+1)$-refinement of $\mathfrak{U}$ of order at most $p+1$.

To prove the reverse implication of the lemma let us now take $\mathfrak{V}$ to be a finite-to-one open $(k+1)$-refinement of $\mathfrak{U}$ of order at most $p+1$. By Lemma 2 there exists a strict open $(k+1)$-refinement $\mathfrak{W} = \{W_i \mid i < a\}$ of $\mathfrak{V}$. Thus the system

$$\mathfrak{X} = \{\overline{W}_{i_0} \cap \dots \cap \overline{W}_{i_p} \mid i_0 < i_1 < \dots < i_p\}$$

is locally finite and consists of mutually disjoint sets. We now put

$$W_i' = W_i - \bigcup\{\overline{W}_{i_0} \cap \dots \cap \overline{W}_{i_p} \mid i = i_0 < \dots < i_p\},$$
$$\mathfrak{W}' = \{W_i' \mid i < a\}$$

and show that $\mathfrak{W}'$ is a suitable open k-refinement of $\mathfrak{U}$.

The subset W_i' of W_i is open because the set union occurring in its definition is taken over a subsystem of $\mathfrak{X}$. Also $\nu(\mathfrak{W}') \leqq p$ since, in defining $\mathfrak{W}'$, each point of order $p+1$ with respect to $\mathfrak{W}$ has been removed from just one of the member sets of $\mathfrak{W}$ to which it belongs. Finally $\mathfrak{W}'$ is a k-covering; for if $W_{i_0}, \dots, W_{i_k}$ are some $k+1$ members of $\mathfrak{W}$ containing a given point x, then x fails to belong to at most one of the sets $W'_{i_0}, \dots, W'_{i_k}$ by virtue of belonging to at most one member set of $\mathfrak{X}$. This proves Lemma 3. We remark that " finite-to-one " may be replaced by " locally finite " throughout the lemma and proof.

Suppose now that $\mathfrak{G}$ is a locally finite open k-covering of an at most n dimensional space. By Theorem 1 we know that a locally finite open k-refinement $\mathfrak{U}$ of order at most $n+k$ exists; (in fact $\mathfrak{V}$ may be chosen as a finite-to-one refinement). The process of uniting member sets of $\mathfrak{V}$ in order to construct a one-to-one refinement of $\mathfrak{G}$ (as described in the proof of Theorem 1) will in general produce a covering which fails to be a k-covering. In the next theorem a strict open k-refinement of $\mathfrak{G}$ of order at most $n+k$ will be constructed without the existence of the k-refinement $\mathfrak{V}$ being assumed. The necessary connection with the dimension number will be supplied by the following result which in the form quoted below is due to K. Morita [4].

If $\{X_i \mid i < a\}, \{Y_i \mid i < a\}$ *are two locally finite open systems of an at most n dimensional space, such that* $\overline{X}_i \subset Y_i$, *then there exist open systems* $\{U_i \mid i < a\}, \{V_i \mid i < a\}$ *such that* $\overline{X}_i \subset U_i, \overline{U}_i \subset V_i, \overline{V}_i \subset Y_i$ *and the order of the system* $\{\overline{V}_i - U_i \mid i < a\}$ *is at most n.*

THEOREM 2. *If* $\mathfrak{G} = \{G_i \mid i < a\}$ *is a locally finite open k-covering of an at most n dimensional space R, then* $\mathfrak{G}$ *admits a strict open k-refinement of order at most* $n+k$.

Proof. Let $\mathfrak{F} = \{F_i \mid i < a\}$ be a strict closed k-refinement of $\mathfrak{G}$ as given by Lemma 2. We suppose that for all ordinals $i < j < j_0$ open sets U_{hi}, V_{hi} $(h < a, i < j)$ have been defined by induction and that, together with the further definitions

$$X_{hj} = \bigcup\{U_{hi} \mid i < j\}, \qquad Y_{hj} = \bigcap\{V_{hi} \mid i < j\},$$

$$\mathfrak{X}_j = \{X_{hj} \mid h < j\}, \qquad \overline{\mathfrak{Y}}_j = \{\overline{Y}_{hj} \mid h < j\},$$

and

$$\mathfrak{F}_j = \{F_i \mid i < j\},$$

the following conditions hold :

$$\left.\begin{array}{l} U_{hi} = V_{hi} = \phi \quad \text{for all } h > i, \\ \overline{U}_{hi'} \subset U_{hi}, \quad \overline{U}_{hi} \subset V_{hi}, \quad \overline{V}_{hi} \subset V_{hi'}, \quad \overline{V}_{hi'} \subset G_h \end{array}\right\} \quad \ldots\ldots(1.j)$$

(and) whenever $i' < i < j$;

$$x : \mathfrak{X}_j \geqq \min(k, x : \mathfrak{F}_j) \quad (x \in R) ; \quad \ldots\ldots(2.j)$$

$$x : \overline{\mathfrak{Y}}_j \leqq n+k \quad (x \in R). \quad \ldots\ldots(3.j)$$

When $j_0 = 1$ this hypothesis is vacuous. From $(1.j)$ it follows that

$$X_{hj} = U_{h\, j-1}, \quad \overline{U}_{h\, j-1} \subset V_{h\, j-1} = Y_{hj}$$

when j is not a limit ordinal. Hence

$$\overline{X}_{hj} \subset Y_{hj}, \overline{Y}_{hj} \subset G_h. \quad \ldots\ldots(4.j)$$

This is also true if j is a limit ordinal, because in that case $U_{hi'} \subset V_{hi}$ for all $i, i' < j$ and moreover

$$\overline{X}_{hj} \subset \overline{V}_{hi+1} \subset V_{hi} \subset \overline{V}_{h0} \subset G_h \quad \text{for all } i < j.$$

Now if j_0 is a limit ordinal then the open sets U_{hi}, V_{hi} $(h < a, i < j_0)$ are defined and satisfy $(1.j_0)$. From the definitions it is clear that

$$X_{hj} \subset X_{hj_0}, \quad Y_{hj_0} \subset Y_{hj} \quad (h < j < j_0).$$

Thus if $F_{i_1}, \ldots, F_{i_r}$ $(i_1 < \ldots < i_r < j_0)$ are the finitely many member sets of $\mathfrak{F}_{j_0}$ containing a given point x, it follows that for some i_0, $i_r < i_0 < j_0$,

$$x : \mathfrak{X}_{j_0} \geqq x : \mathfrak{X}_{i_0} \geqq \min(k, x : \mathfrak{F}_{i_0}) = \min(k, x : \mathfrak{F}_{j_0})$$

and so $(2.j_0)$ holds. Similarly, by using the local-finiteness of $\overline{\mathfrak{Y}}_{j_0}$, it is easily shown that $(3.j_0)$ holds and so the induction is complete in the case of a limit ordinal.

We now put $j_0 = j+1$, thereby fixing j. In the following construction for the sets U_{hj}, $V_{hj} (h < a)$ the symbol j is sometimes suppressed.

We observe that, by $(4.j)$, the systems $\mathfrak{X}_j$, $\mathfrak{Y}_j$ satisfy the hypothesis of Morita's theorem and accordingly take open systems

$$\mathfrak{U} = \{U_{hj} \mid h < j\}, \quad \mathfrak{V} = \{V_{hj} \mid h < j\}$$

such that

$$\left.\begin{array}{l} \overline{X}_{hj} \subset U_{hj}, \quad \overline{U}_{hj} \subset V_{hj}, \quad \overline{V}_{hj} \subset Y_{hj} \quad (h < j) \\ \nu\{\overline{V}_{hj} - U_{hj} \mid h < j\} \leqq n. \end{array}\right\} \quad \ldots\ldots(5)$$

(and)

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It remains only to define the sets U_{jj}, V_{jj} (and the empty sets U_{hj}, V_{hj}, $h > j$). As preliminaries to this we define

$$F = \{x \mid x : \mathfrak{U} < \min (k, x : \mathfrak{F}_{j+1})\}$$

and
$$G = \{x \mid x : \overline{\mathfrak{D}} < n + k\},$$

where $\overline{\mathfrak{D}} = \{\overline{V}_{hj} \mid h < j\}$. We show that

$$F \subset F_j, \quad F \subset G, \quad F \text{ is closed and } G \text{ is open.} \qquad \text{(6)}$$

Firstly let $x \notin F_j$, so that $x : \mathfrak{F}_j = x : \mathfrak{F}_{j+1}$; it follows, by (5) and (2.j), that

$$x : \mathfrak{U} \geqq x : \mathfrak{X}_j \geqq \min (k, x : \mathfrak{F}_j)$$

and therefore x fails to belong to F.

Secondly let $x \in F$ so that, in particular, $x : \mathfrak{U} < k$. Now, by (5), we have that

$$x : \{\overline{V}_{hj} \mid h < j\} \leqq x : \{\overline{V}_{hj} - U_{hj} \mid h < j\} + x : \{U_{hj} \mid h < j\} ;$$

i.e. $x : \overline{\mathfrak{D}} < n + k$. Therefore F lies in G as required.

Thirdly, since both the open system $\mathfrak{U}$ and the closed system $\mathfrak{F}_{j+1}$ are locally finite, a given point x has some small neighbourhood of which any point y satisfies the relations

$$y : \mathfrak{U} \geqq x : \mathfrak{U} \quad \text{and} \quad x : \mathfrak{F}_{j+1} \geqq y : \mathfrak{F}_{j+1}.$$

Thus $x \notin F$ implies $y \notin F$ for all y and therefore F is closed. Similarly it can be shown that G is open.

Since F lies in both G and G_j we can define U_{jj} and V_{jj} as open sets such that

$$F \subset U_{jj}, \quad \overline{U}_{jj} \subset V_{jj}, \quad \overline{V}_{jj} \subset G \cap G_j. \qquad \text{(7)}$$

This completes the construction of the sets U_{hj}, V_{hj} ($h < a$).

From conditions (5) and (7) it is clear that (1.$j+1$) holds. From the definitions it also follows that $X_{h\,j+1} = U_{hj}$, $Y_{h\,j+1} = V_{hj}$ ($h \leqq j$). Thus

$$\mathfrak{X}_{j+1} = \{\mathfrak{U}, U_{jj}\}, \quad \mathfrak{Y}_{j+1} = \{\mathfrak{D}, V_{jj}\}$$

and in particular $x : \mathfrak{X}_{j+1} \geqq x : \mathfrak{U}$ for all x. In proving (2.$j+1$) we may therefore assume that $x : \mathfrak{U} < \min (k, x : \mathfrak{F}_{j+1})$ i.e. $x \in F$. Since x necessarily belongs to U_{jj} and F_j, we have by (2.j) that

$$(x : \mathfrak{X}_{j+1}) - 1 \geqq x : \mathfrak{U} \geqq x : \mathfrak{X}_j \geqq \min (k, x : \mathfrak{F}_j) \geqq \min (k, x : \mathfrak{F}_{j+1}) - 1.$$

This verifies (2.$j+1$).

Lastly, let $x \notin \overline{V}_{jj}$; together with (3.j) this implies that

$$x : \overline{\mathfrak{Y}}_{j+1} = x : \overline{\mathfrak{D}} \leqq x : \overline{\mathfrak{Y}}_j \leqq n + k.$$

On the other hand, if $x \in \overline{V}_{jj}$, then $x \in G$ and consequently

$$x : \overline{\mathfrak{Y}}_{j+1} \leqq 1 + x : \overline{\mathfrak{D}} \leqq n + k.$$

In either case (3.$j+1$) holds and the induction is complete.

Open systems $\mathfrak{X}_a$, $\mathfrak{Y}_a$ exist satisfying (2.a), (3.a) and (4.a) ; (2.a) implies that $\mathfrak{X}_a$ is a k-covering because $\mathfrak{F}_a$ ($= \mathfrak{F}$) was chosen as a k-refinement of $\mathfrak{G}$ at the outset ; (3.a) and (4.a) imply that $\mathfrak{X}_a$ and $\mathfrak{Y}_a$ are strict k-refinements of $\mathfrak{Y}_a$ and $\mathfrak{G}$ respectively, each having order at most $n + k$. Thus either k-refinement serves to prove the theorem.

3. The values assumed by the functions $x : \mathfrak{V}$. Let dim $R \geqq n$. From the corollary to Theorem 1 we deduce that for each k there exists a locally finite open covering $\mathfrak{U}$ of which every locally finite open k-refinement has order at least $n+k$. In view of Lemma 3 it is clear that one fixed covering $\mathfrak{U}$ serves for all values of k. Now let $\mathfrak{V}$ be any locally finite open refinement of $\mathfrak{U}$ and consider the values which the function $x : \mathfrak{V}$ may assume. If k denotes the least such value, then the greatest value is at least $n+k$. We generalise this by showing that on some member set of $\mathfrak{V}$ at least $n+1$ distinct values are assumed. Moreover $\mathfrak{U}$ may be chosen as a finite open covering and a similar property holds for locally finite closed refinements of $\mathfrak{U}$. These results are corollaries to the proof of the following

THEOREM 3. *If R is a paracompact space of dimension at least n, then there exists a finite open covering $\mathfrak{U}_0$ such that for every locally finite open or closed refinement $\mathfrak{U}$ there is some point in every neighbourhood of which $x : \mathfrak{U}$ assumes at least $n+1$ distinct values.*

We take $\mathfrak{U}_0 = \{U_j \mid j < b\}$ to be a finite open covering of which every finite open (or closed) refinement has order at least $n+1$. The case of the closed refinements and that of the open refinements are considered separately as the methods of proof differ. For brevity we shall write $X_I = X_{i_1} \cap \ldots \cap X_{i_m}$ and $\overline{X}_I = \overline{X}_{i_1} \cap \ldots \cap \overline{X}_{i_m}$, where $\{X_i \mid i < a\}$ is any system of subsets, I is any finite set of ordinals $i_1, \ldots, i_m < a$ and $|I| = m$.

We mention a result allied to Theorem 3 which is given in [1]. In our terminology it states that if R is a compact metric space of dimension at least n then, for any finite open or closed refinement $\{X_i \mid i < a\}$ of the covering $\mathfrak{U}_0$ (chosen as above), there exist subsets $I_0, \ldots, I_n$ such that $\phi \subset X_{I_0} \subset \ldots \subset X_{I_n}$, the inclusions being proper.

Proof of Theorem 3 (closed case). Suppose that $\mathfrak{F} = \{F_i \mid i < a\}$ is a locally finite closed refinement of $\mathfrak{U}_0$ such that each point x admits a neighbourhood $U(x)$ in which the required order property fails. By paracompactness the open covering $\{U(x) \mid x \in R\}$ has a locally finite open refinement and by Lemma 2 there exists a further strict closed refinement $\mathfrak{K}$. Thus $\mathfrak{K}$ has the property that

$$x : \mathfrak{F} = m_1(K), \ldots, \text{or } m_n(K) \quad (x \in K \in \mathfrak{K}), \qquad \ldots\ldots(8)$$

where $m_1 > \ldots > m_n$ are some n positive integers chosen for each K.

Proceeding by induction we suppose that for each integer $r < s \leqq n+1$ a finite system $\{G_{rj} \mid j < b\}$ of mutually disjoint open sets has been constructed such that

$$\left.\begin{aligned} G_{rj} &\subset U_j \quad (j < b) \\ \text{and} \qquad x \in G_{s-1} &= \bigcup\{G_{rj} \mid r = 0, \ldots, s-1\,;\ j < b\}, \end{aligned}\right\} \qquad \ldots\ldots(9)$$

whenever $x : \mathfrak{F} \geqq m_{s-1}(K)$ $(x \in K \in \mathfrak{K})$.

We initiate the construction by putting $G_{0j} = \phi$ $(j < b)$. Let $\mathfrak{F}_s$ be the system of which a typical member set F_{sI} consists of all points x such that

$$x \in F_I - G_{s-1}, \qquad \ldots\ldots(10)$$

$$x \in K \quad \text{and} \quad m_s(K) = |I| \quad \text{for some} \quad K \in \mathfrak{K}, \qquad \ldots\ldots(11)$$

where I is any finite set of indices $i_1, \ldots i_m < a$. We assert that

$$\mathfrak{F}_s \textit{ is a locally finite system of mutually disjoint closed sets.} \qquad \ldots\ldots(12)$$

Firstly, by (10), $\mathfrak{F}_s$ inherits the local-finiteness property of $\mathfrak{F}$. Next let $x \notin F_{sI}$; if (10) fails,

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then $(R - F_I) \cup G_{s-1}$ is an open neighbourhood of x; if (11) fails then, by the local-finiteness of the closed covering $\mathfrak{K}$, we can find a neighbourhood $P(x)$ meeting only those members of $\mathfrak{K}$ which contain x. Thus, whenever y is a point of P, $K \ni y$ implies $K \ni x$ and consequently condition (11) fails. In either case there is some neighbourhood of x disjoint from F_{sI} and hence the latter is closed.

Now let us suppose that for some distinct pair I, I' the sets F_{sI} and $F_{sI'}$, have a common point x; thus, by (10), $x \in F_I \cap F_{I'}$. If there is a (proper) inclusion relation between I and I', say $I \subset I'$, then, $x : \mathfrak{F} > |I|$; the latter is also true when there is no inclusion relation. From (10) and (11) we have that, for some particular K containing x, $m_s(K) = |I|$ and $x \notin G_{s-1}$. Now by (8), $x : \mathfrak{F}$ assumes one of the values $m_1(K), \ldots, m_n(K)$ and, by (9) the first $s-1$ values are excluded. Thus $x : \mathfrak{F} \leqq m_s(K)$ and we have a contradiction from the fact that $m_s(K) = |I|$ and $|I| < x : \mathfrak{F}$. This establishes (12).

Since $\mathfrak{F}$ is a refinement of $\mathfrak{U}_0$, we can choose $j = j(I)$ such that

$$F_{sI} \subset F_I \subset U_j \quad (j < b)$$

and from (12) it follows that the sets $\bigcup\{F_{sI} \mid j(I) = j\}$ $(j < b)$ are mutually disjoint and closed. By Lemma 1, we can find a system of mutually disjoint open sets $\{G_{sj}\}$ such that

$$\bigcup\{F_{sI} \mid j(I) = j\} \subset G_{sj} \subset U_j \quad (j < b),$$

and it only remains to show that the induction hypothesis holds for this system.

Let $x : \mathfrak{F} \geqq m_s(K)$, $(x \in K \in \mathfrak{K})$. We may assume that x does not belong to G_{s-1} as otherwise x belongs to G_s and there is nothing further to prove. Thus $x : \mathfrak{F} = m_s(K)$ because the other possible values are now excluded by (9). Taking F_I to be the intersection of all members of $\mathfrak{F}$ containing x, it is easy to see that, by conditions (10) and (11), x belongs to F_{sI}. Consequently x belongs to G_s as required.

From (8) and (9) it follows that G_n is the whole space. Thus the systems $\{G_{rj} \mid j < b\}$ $(r = 1, 2, \ldots, n)$ of mutually disjoint sets form a finite open refinement of $\mathfrak{U}_0$ of order at most n and this is contrary to the choice of $\mathfrak{U}_0$. This proves the closed case of Theorem 3.

With paracompactness omitted from the hypothesis the following weaker result is possible.

Corollary. $\dim R \geqq n$ *implies that for every locally finite closed refinement* $\mathfrak{F}$ *of* $\mathfrak{U}_0$ *there is some member set on which* $x : \mathfrak{F}$ *assumes at least* $n+1$ *values.*

For if $\mathfrak{F}$ is a refinement for which this is not true, then we can identify $\mathfrak{F}$ with $\mathfrak{K}$ in the above proof and derive a contradiction without reference to paracompactness.

The next lemma is designed to show that, if the open case of Theorem 3 is false, then it is false for some locally finite covering by open F_σ-sets.

Lemma 4. *If* $\mathfrak{K}$ *is a locally finite closed covering and* $\mathfrak{U} = \{U_i \mid i < a\}$ *is a locally finite open covering with the property that* $x : \mathfrak{U} = m_1(K), \ldots,$ *or* $m_n(K)$ *whenever* $x \in K \in \mathfrak{K}$, *then there exists a one-to-one refinement* $\mathfrak{V}$ *of* $\mathfrak{U}$ *by open* F_σ*-sets having the same order property as* $\mathfrak{U}$.

Proof. We put

$$\mathscr{I} = \{(K, I) \mid \ |I| \neq m_1(K), \ldots, m_n(K)\},$$

where $K \in \mathfrak{K}$ and I is any finite set $i_1, \ldots, i_m < a$. The order property of $\mathfrak{U}$ is now equivalent to

$$K \cap U_I \subset \bigcup\{U_i \mid i \notin I;\ i < a\} \quad \text{for all} \quad (K, I) \in \mathscr{I}. \qquad \text{.................(13)}$$

We shall prove the lemma by constructing a suitable refinement $\mathfrak{V}$ for which the member sets satisfy the same collection of inclusion relations. The construction consists mainly of establishing a countable sequence of open systems $\mathfrak{V}_p = \{V_{pi} \mid i < a\}$ $(p = 0, 1, \ldots)$ such that

$$\left.\begin{array}{l} \overline{V}_{pi} \subset V_{p+1\,i},\ \overline{V}_{p+1\,i} \subset U_i \quad (i < a) \\ K \cap \overline{V}_{pI} \subset \bigcup\{V_{p+1\,i} \mid i \notin I\ ;\ i < a\} \end{array}\right\} \qquad (14.p)$$

and

for all $(K, I) \in \mathscr{I}$.

By putting $V_{0i} = \phi$ $(i < a)$ and taking a strict open refinement $\mathfrak{V}_1$ of $\mathfrak{U}$ we obtain (14.0). We define $\mathfrak{V}_2$ by a transfinite process which, when iterated, will define $\mathfrak{V}_p$.

We assume that open sets V_{2i} $(i < j < j_0)$ have been defined such that

$$\overline{V}_{1i} \subset V_{2i},\ \overline{V}_{2i} \subset U_i \quad (i < j)$$

and

$$K \cap \overline{V}_{1I} \subset \bigcup\{V_{2i}, U_h \mid i < j, h \geqq j\ ;\ i, h \notin I\} \qquad (15.j)$$

for all $(K, I) \in \mathscr{I}$.

Since $\overline{V}_{1i}$ lies in U_i, (15.0) is given by (13). In order to see how to define V_{2j}, we consider all points which would cause an inclusion relation of $(15.j+1)$ to fail if V_{2j} were the empty set. Formally this is the set H_j of all points x such that for some element (K, I) of $\mathscr{I}$

$$x \in K \cap \overline{V}_{1I} \qquad (16)$$

and

$$x \notin \bigcup\{V_{2i}, U_h \mid i < j, h > j\ ;\ i, h \notin I\}. \qquad (17)$$

It is easily shown that H_j is closed. Moreover H_j lies in U_j; for if x satisfies (16) and (17) for some (K, I), then, by $(15.j)$, x belongs to some member of the system

$$\{V_{2i}, U_h \mid i < j, h \geqq j\ ;\ i, h \notin I\}.$$

Now U_j fails to be a member set or not according as I happens to contain j or not, and by (17) x cannot belong to any member set other than the jth. Hence I does not contain j and x belongs to U_j as required. We define V_{2j} to be an open set such that $H_j \subset V_{2j}$, $\overline{V}_{2j} \subset U_j$ and proceed to verify $(15.j+1)$. Let $x \in K \cap \overline{V}_{1I}$, $(K, I) \in \mathscr{I}$; if $x \notin H_j$ then (17) is not true and it follows that x belongs to

$$\bigcup\{V_{2i}, U_h \mid i < j+1, h \geqq j+1\ ;\ i, h \notin I\}. \qquad (18)$$

On the other hand if $x \in H_j$, then I does not contain j (as shown above) and x belongs to V_{2j}. Hence again x belongs to (18), and thus $(15.j+1)$ holds. The induction is easily completed in the case where j_0 is a limit ordinal by using the local-finiteness of $\mathfrak{U}$. Thus we have an open system $\mathfrak{V}_2$ satisfying (14.1). By repeating the construction we obtain open systems $\mathfrak{V}_p$ satisfying conditions $(14.p)$. Since the system $\mathfrak{V}_1$ was chosen as a refinement of $\mathfrak{U}$ all the subsequent systems are refinements too. We now define

$$V_i = \bigcup\{V_{pi} \mid p = 1, 2, \ldots\} \quad (i < a), \qquad \mathfrak{V} = \{V_i \mid i < a\}$$

and observe that V_i is an open F_σ-set. It is simply verified that the order property of $\mathfrak{U}$ expressed in (13) also holds for the refinement $\mathfrak{V}$ of $\mathfrak{U}$ and the lemma is proved.

Let $\mathfrak{G}$ be a covering of a space R. We denote $\bigcup\{G \mid x \in G \in \mathfrak{G}\}$ by st$(x, \mathfrak{G})$; $\mathfrak{G}$ is called a delta-refinement of a covering $\mathfrak{U}$ if the covering $\{\text{st}(x, \mathfrak{G}) \mid x \in R\}$ is a refinement of $\mathfrak{U}$. It is known that a locally finite open covering (of a normal space) admits an open delta-refinement.

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Proof of Theorem 3 (open case). Suppose that $\mathfrak{U}$ is a locally finite open refinement of $\mathfrak{U}_0$ admitting neighbourhoods $U(x)$ $(x \in R)$ in each of which there occur points of at most n distinct orders with respect to $\mathfrak{U}$. By paracompactness the covering $\{U(x) \mid x \in R\}$ admits a locally finite open refinement and, by Lemma 2, there exists a further one-to-one closed refinement $\mathfrak{K}$ such that $\{\text{Int}\,(K) \mid K \in \mathfrak{K}\}$ is also a covering. Since $x : \mathfrak{U}$ assumes at most n values on any one member of $\mathfrak{K}$, $\mathfrak{U}$ admits, according to Lemma 4, a one-to-one refinement $\mathfrak{V} = \{V_i \mid i < a\}$ having the same order property as $\mathfrak{U}$. Taking $\mathfrak{G}$ to be an open delta-refinement of $\{\text{Int}\,(K) \mid K \in \mathfrak{K}\}$ we see that

$$\textit{the function } y : \mathfrak{V} \textit{ assumes at most } n \textit{ values on the set } \text{st}\,(x, \mathfrak{G}) \quad (x \in R). \qquad \text{......(19)}$$

Since V_i is an open F_σ-set we can find a continuous real-valued function $f(i\,;\ x)$ which is positive on V_i and zero on $R - V_i$. Let $f(i_1 \ldots i_m\,;\ x)$ denote the sum of $f(i_1\,;\ x), \ldots, f(i_m\,;\ x)$. We define a system $\mathfrak{W}$ of which the typical member $W_J = W_{j_1 \ldots j_m}$ consists of all points x such that

$$x \in V_{j_1} \cap \ldots \cap V_{j_m}, \qquad \text{......(20)}$$

$$y : \mathfrak{V} = m \quad \text{for some} \quad y \in \text{st}\,(x, \mathfrak{G}), \qquad \text{......(21)}$$

$$f(i_1 \ldots i_m\,;\ x) < f(j_1 \ldots j_m\,;\ x) \quad \text{for all} \quad (i_1 \ldots i_m) \neq J, \qquad \text{......(22)}$$

where J is any finite set $j_1, \ldots, j_m < a$.

Firstly, let the members of $\mathfrak{V}$ containing a given point x be $V_{j_1}, \ldots V_{j_m}$; then x belongs to $W_{j_1 \ldots j_m}$ because (20) and (21) are valid (with $y = x$) and (22) follows from the fact that the functions $f(j\,;\ x)$ $(j = j_1, \ldots, j_m)$, and only these functions, are positive. Hence $\mathfrak{W}$ is a covering and refines $\mathfrak{V}$.

Secondly, let x be a point of W_J; by restricting attention to some small neighbourhood of x we see that condition (22) involves in effect only the finitely many functions $f(i\,;\ z)$ that are not everywhere zero. Hence condition (22) is valid for all points in some smaller neighbourhood $P(x)$ say. Now choose a point y and a member set G of $\mathfrak{G}$ as given by (21); it is not difficult to see that the common part of P, G, $V_{j_1}, \ldots, V_{j_m}$ is a neighbourhood of x lying in W_J.

Thirdly, let x belong to $W_{J_1}, \ldots, W_{J_p}$. Condition (22) implies that x belongs to at most one set of the form $W_{j_1 \ldots j_m}$ for each value of m and condition (21) implies that st $(x, \mathfrak{G})$ contains points of orders $|J_1|, \ldots, |J_p|$. Hence these orders are distinct and by (19) are at most n in number. Thus we have that $\mathfrak{W}$ is an open refinement of $\mathfrak{U}_0$ of order at most n.

Finally, by the process of uniting member sets of $\mathfrak{W}$, as described in the proof of Theorem 1, we produce a finite open refinement of $\mathfrak{U}_0$ of order at most n, and this is contrary to the choice of $\mathfrak{U}_0$.

Corollary. *If R is a normal space of dimension at least n (not necessarily paracompact), then for any locally finite open refinement $\mathfrak{U}$ of $\mathfrak{U}_0$ there is some member set of $\mathfrak{U}$ on which $x : \mathfrak{U}$ assumes at least $n+1$ values.*

For if not, then we can choose some member U_x of $\mathfrak{U}$ as a neighbourhood of x and identify the system $\{U(x) \mid x \in R\}$ of the above proof with the covering $\{U_x \mid x \in R\}$; since the latter admits some subsystem of $\mathfrak{U}$ as a locally finite open refinement the above argument may be applied without reference to paracompactness.

REFERENCES

1. P. Alexandroff and A. Kolmogoroff, Endliche Überdeckungen topologischer Raüme. *Fundamenta Math.* **26** (1936), 267.

2. J. Dieudonné, Une généralisation des espaces compacts, *J. Math. Pures Appliquées* **23** (1944), 65–76.

3. C. H. Dowker, Mapping theorems for non-compact spaces, *Amer. J. Math.* **69** (1947), 200–242.

4. K. Morita, On the dimension of normal spaces II. *J. Math. Soc. Japan* **2** (1950), 16–33.

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