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COHOMOLOGICAL LENGTH FUNCTIONS

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Abstract. We study certain integer valued length functions on triangulated categories, and establish a correspondence between such functions and cohomological functors taking values in the category of finite length modules over some ring. The irreducible cohomological functions form a topological space. We discuss its basic properties, and include explicit calculations for the category of perfect complexes over some specific rings.

§1. Introduction

Let C be a triangulated category with suspension $\Sigma : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$. Given a cohomological functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Mod}\, k$ into the category of modules over some (not necessarily commutative) ring k such that $H(C)$ has finite length for each object C, we consider the function

$$\chi : \mathsf{Ob}\, \mathsf{C} \longrightarrow \mathbb{N}, \qquad C \mapsto \mathsf{length}_k\, H(C),$$

and ask the following questions.

– What are the characteristic properties of such a function $\mathsf{Ob}\, \mathsf{C} \longrightarrow \mathbb{N}$?
– Can we recover H from χ?

Somewhat surprisingly, we can offer fairly complete answers to both questions.

It should be noted that similar questions arise in Boij–Söderberg theory when cohomology tables are studied; recent progress [8, 9] provides some motivation for our work. Further motivation comes from the quest (initiated by Paul Balmer, for instance) for points in the context of triangulated categories.

Typical examples of cohomological functors are the representable functors of the form $\mathsf{Hom}(-, X)$ for some object X in C. We begin with a result that takes care of this case; its proof is based on a theorem of Bongartz [4].

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THEOREM 1.1. (Jensen–Su–Zimmerman [19]) *Let k be a commutative ring, and let* C *be a k-linear triangulated category such that each morphism set in* C *has finite length as a k-module. Suppose also for each pair of objects* X, Y *that* $\mathsf{Hom}(X, \Sigma^n Y) = 0$ *for some* $n \in \mathbb{Z}$. *Then, two objects* X *and* Y *are isomorphic if and only if the lengths of* $\mathsf{Hom}(C, X)$ *and* $\mathsf{Hom}(C, Y)$ *coincide for all* C *in* C. □

Examples of triangulated categories satisfying the assumptions in this theorem arise from bounded derived categories. To be precise, if A is a k-linear exact category such that each extension group in A has finite length as a k-module, then its bounded derived category $\mathsf{D}^b(\mathsf{A})$ satisfies the above assumptions, since for all objects X, Y in A (viewed as complexes concentrated in degree zero)

$$\mathsf{Hom}(X, \Sigma^n Y) = \mathsf{Ext}^n(X, Y) = 0 \quad \text{for all } n < 0.$$

On the other hand, Auslander and Reiten provided in [2, Section 4.4] simple examples of triangulated categories that do not have the property that objects are determined by the lengths of their morphism spaces (see also [3]).

Now, let C be an essentially small triangulated category. Thus, the isomorphism classes of objects in C form a set. Given any additive functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups, we denote by $\mathsf{End}(H)$ the endomorphism ring formed by all natural transformations $H \to H$. It should be noted that $\mathsf{End}(H)$ acts on $H(C)$ for all objects C. Moreover, if a ring k acts on $H(C)$ for all objects C in a way that commutes with all morphisms in C, then this action factors through that of $\mathsf{End}(H)$ via a ring homomorphism $k \to \mathsf{End}(H)$. In particular, when $H(C)$ has finite length over k, then it also has finite length over $\mathsf{End}(H)$. This observation motivates the following definition.

DEFINITION 1.2. A cohomological functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is called *endofinite*[1] provided that for each object C

(1) $H(C)$ has finite length as a module over the ring $\mathsf{End}(H)$, and
(2) $H(\Sigma^n C) = 0$ for some $n \in \mathbb{Z}$.

[1] The term *endofinite* reflects condition (1), while (2) is added for technical reasons.

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If $(H_i)_{i\in I}$ are cohomological functors, then the *direct sum* $\bigoplus_{i\in I} H_i$ is cohomological. A non-zero cohomological functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is *indecomposable* if it cannot be written as a direct sum of two non-zero cohomological functors.

An endofinite cohomological functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ gives rise to a function

$$\chi_H : \mathsf{Ob}\,\mathsf{C} \longrightarrow \mathbb{N}, \qquad C \mapsto \mathsf{length}_{\mathsf{End}(H)} H(C),$$

which is cohomological in the following sense (see Lemma 2.4).

DEFINITION 1.3. A function $\chi : \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ is called *cohomological* provided that

(1) $\chi(C \oplus C') = \chi(C) + \chi(C')$ for each pair of objects C and C',
(2) for each object C there is some $n \in \mathbb{Z}$ such that $\chi(\Sigma^n C) = 0$, and
(3) for each exact triangle $A \to B \to C \to$ in C and each labeling

$$\cdots \to X_{-2} \to X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

of the induced sequence

$$\cdots \to \Sigma^{-1}B \to \Sigma^{-1}C \to A \to B \to C \to \Sigma A \to \Sigma B \to \cdots$$

with $\chi(X_0) = 0$ we have

$$\sum_{i=0}^{n} (-1)^{i+n} \chi(X_i) \geqslant 0 \quad \text{for all } n \in \mathbb{Z},$$

and equality holds when $\chi(X_n) = 0$.

If $(\chi_i)_{i\in I}$ are cohomological functions, and for any C in C the set $\{i \in I \mid \chi_i(C) \neq 0\}$ is finite, then we can define the *locally finite sum* $\sum_{i\in I} \chi_i$. A non-zero cohomological function is *irreducible* if it cannot be written as a sum of two non-zero cohomological functions.

The following theorem is the main result of this paper; it is proved in Section 2 and builds on work of Crawley-Boevey on finite endolength objects (see [6, 7]).

THEOREM 1.4. *Let C be an essentially small triangulated category.*

(1) *Every endofinite cohomological functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ decomposes essentially uniquely into a direct sum of indecomposable endofinite cohomological functors with local endomorphism rings.*

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(2) *Every cohomological function* $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ *can be written uniquely as a locally finite sum of irreducible cohomological functions.*
(3) *The assignment* $H \mapsto \chi_H$ *induces a bijection between the isomorphism classes of indecomposable endofinite cohomological functors* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ *and the irreducible cohomological functions* $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$.

Examples of endofinite cohomological functors arise from representable functors of the form $\mathsf{Hom}(-, X)$ when C is a Hom-finite k-linear category. Thus, Theorem 1.1 can be deduced from Theorem 1.4. The following remark shows that in some appropriate setting each endofinite cohomological functor is representable.

REMARK 1.5. Let T be a compactly generated triangulated category, and let C be the full subcategory formed by all compact objects. Then, each endofinite cohomological functor $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is isomorphic to $\mathsf{Hom}(-, X)|_{\mathsf{C}}$ for some object X in T, which is unique up to an isomorphism and represents the functor[2]

$$\mathsf{T}^{\mathrm{op}} \longrightarrow \mathsf{Ab}, \qquad C \mapsto \mathsf{Hom}(\mathsf{Hom}(-, C)|_{\mathsf{C}}, H).$$

Thus, cohomological functions are "represented" by objects in this setting. For specific examples, see [25].

Next, we consider the set of irreducible cohomological functions $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ and endow it with the Ziegler topology (see Proposition B.5). The quotient

$$\mathsf{Sp}\,\mathsf{C} = \{\chi : \mathsf{Ob}\,\mathsf{C} \to \mathbb{N} \mid \chi \text{ irreducible and cohomological}\}/\Sigma$$

with respect to the action of the suspension is by definition the *space of cohomological functions* on C. Thus, the points of $\mathsf{Sp}\,\mathsf{C}$ are equivalence classes of the form $[\chi] = \{\chi \circ \Sigma^n \mid n \in \mathbb{Z}\}$.

Take as an example the category of perfect complexes $\mathsf{D}^b(\mathsf{proj}\,A)$ over a commutative ring A. A prime ideal $\mathfrak{p} \in \mathsf{Spec}\,A$ with residue field $k(\mathfrak{p})$ yields an irreducible cohomological function

$$\chi_{k(\mathfrak{p})} : \mathsf{Ob}\,\mathsf{D}^b(\mathsf{proj}\,A) \longrightarrow \mathbb{N}, \qquad X \mapsto \mathsf{length}_{k(\mathfrak{p})}\,\mathsf{Hom}(X, k(\mathfrak{p})),$$

where $k(\mathfrak{p})$ is viewed as a complex concentrated in degree zero.

[2] The functor is cohomological since H is an injective object in the category of additive functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ (see the proof of Theorem 1.4). Thus, Brown's representability theorem applies.

THEOREM 1.6. *The map* $\mathsf{Spec}\, A \to \mathsf{Sp}\, \mathsf{D}^b(\mathsf{proj}\, A)$ *sending* $\mathfrak{p}$ *to* $[\chi_{k(\mathfrak{p})}]$ *is injective and closed with respect to the Hochster dual of the Zariski topology on* $\mathsf{Spec}\, A$*.*

We prove this result in Section 4 by analyzing the Ziegler spectrum [24, 34] of the category of perfect complexes. A general method for computing the space $\mathsf{Sp}\, \mathsf{C}$ of cohomological functions is to compute the Krull–Gabriel filtration [11, 15] of the abelianization $\mathsf{Ab}\, \mathsf{C}$ (see [10, 31] or Section 2).

A specific example is the algebra $k[\varepsilon]$ of dual numbers over a field k. Each complex

$$X_n : \cdots \to 0 \to k[\varepsilon] \xrightarrow{\varepsilon} k[\varepsilon] \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} k[\varepsilon] \to 0 \to \cdots$$

of length n corresponds to an isolated point in $\mathsf{Sp}\, \mathsf{D}^b(\mathsf{proj}\, k[\varepsilon])$, and their closure yields exactly one extra point corresponding to the residue field. Thus,

$$\mathsf{Sp}\, \mathsf{D}^b(\mathsf{proj}\, k[\varepsilon]) = \{[\chi_{X_n}] \mid n \in \mathbb{N}\} \cup \{[\chi_k]\}.$$

This example is of particular interest because the derived category $\mathsf{D}^b(\mathsf{proj}\, k[\varepsilon])$ is discrete in the sense of Vossieck [32]; that is, there are no continuous families of indecomposable objects. On the other hand, there are infinitely many indecomposable objects, even up to shift. The Krull–Gabriel dimension explains this behavior because it measures how far an abelian category is away from being a length category. For instance, a triangulated category C is locally finite (see [26] or Section 4) if and only if the Krull–Gabriel dimension of $\mathsf{Ab}\, \mathsf{C}$ equals at most 0.

PROPOSITION 1.7. *The abelianization* $\mathsf{Ab}\, \mathsf{D}^b(\mathsf{proj}\, k[\varepsilon])$ *has Krull–Gabriel dimension equal to* 1*.*

It should be noted that the Krull–Gabriel dimension of the free abelian category $\mathsf{Ab}\, A$ over an Artin algebra A [14] behaves differently; it equals 0 if and only if A is of finite representation type by a result of Auslander [1], and is greater than 1 otherwise (see [17, 21]).

As a final example, let us describe the cohomological functions for the category $\mathsf{coh}\, \mathbb{P}^1_k$ of coherent sheaves on the projective line over a field k.

PROPOSITION 1.8. *The abelianization* $\mathsf{Ab}\, \mathsf{D}^b(\mathsf{coh}\, \mathbb{P}^1_k)$ *has Krull–Gabriel dimension equal to* 2*, and*

$$\mathsf{Sp}\, \mathsf{D}^b(\mathsf{coh}\, \mathbb{P}^1_k) = \{[\chi_X] \mid X \in \mathsf{coh}\, \mathbb{P}^1_k \text{ indecomposable}\} \cup \{[\chi_{k(t)}]\}.$$

These examples illustrate in the triangulated context the following representation theoretic paradigm:

finite type	$\longleftrightarrow$	Krull–Gabriel dimension $= 0$,
discrete type	$\longleftrightarrow$	Krull–Gabriel dimension $\leqslant 1$,
continuous families exist	$\longleftrightarrow$	Krull–Gabriel dimension > 1.

§2. Cohomological functors and functions

In this section, Theorem 1.4 is proved. We deduce it from work of Crawley-Boevey on endofinite objects in locally finitely presented abelian categories (see [6, 7]). We begin with some preparations.

Let C be an essentially small triangulated category.

The abelianization. Following Freyd [10, Section 3] and Verdier [31, II.3], we consider the *abelianization* $\mathsf{Ab}\,\mathsf{C}$ of C, which is the abelian category of additive functors $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ into the category Ab of abelian groups that admit a copresentation

$$0 \longrightarrow F \longrightarrow \mathsf{Hom}(-, A) \longrightarrow \mathsf{Hom}(-, B).$$

The suspension $\Sigma : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$ extends to an equivalence $\mathsf{Ab}\,\mathsf{C} \xrightarrow{\sim} \mathsf{Ab}\,\mathsf{C}$ sending F to $\Sigma F := F \circ \Sigma^{-1}$.

In addition, we work in the category $\mathsf{Mod}\,\mathsf{C}$ of additive functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. This is a locally finitely presented abelian category, and the abelianization $\mathsf{Ab}\,\mathsf{C}$ identifies with the full subcategory of finitely presented objects of $\mathsf{Mod}\,\mathsf{C}$ (see [6] for details).

Additive functions. Let us introduce the analogue of a cohomological function for the abelianization of C.

Definition 2.1. A function $\chi : \mathrm{Ob}\,\mathsf{Ab}\,\mathsf{C} \to \mathbb{N}$ is called *additive*[3] provided that

(1) $\chi(F) = \chi(F') + \chi(F'')$ if $0 \to F' \to F \to F'' \to 0$ is an exact sequence, and

(2) for each object F there is some $n \in \mathbb{Z}$ such that $\chi(\Sigma^n F) = 0$.

We show that additive and cohomological functions are closely related.

[3] The term *additive* reflects condition (1), while (2) is added for technical reasons.

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LEMMA 2.2. *Restricting a function* $\chi : \mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C} \to \mathbb{N}$ *to* $\mathsf{Ob}\,\mathsf{C}$ *by setting* $\chi(C) = \chi(\mathsf{Hom}(-, C))$ *gives a natural bijection between*

– *the additive functions* $\mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C} \to \mathbb{N}$ *and*
– *the cohomological functions* $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$.

Proof. Let $\chi : \mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C} \to \mathbb{N}$ be an additive function. An exact triangle $A \to B \to C \to$ in C yields in $\mathsf{Ab}\,\mathsf{C}$ an exact sequence

$$\cdots \to \mathsf{Hom}(-, A) \to \mathsf{Hom}(-, B) \to \mathsf{Hom}(-, C) \to \mathsf{Hom}(-, \Sigma A) \to \cdots .$$

Using this sequence, it is easily checked that the restriction of χ to C is a cohomological function.

Conversely, given a cohomological function $\chi : \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$, we extend it to a function $\mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C} \to \mathbb{N}$, which again we denote by χ. We fix F in $\mathsf{Ab}\,\mathsf{C}$ with copresentation as above induced by an exact triangle $A \to B \to C \to$ in C, and choose $n \in \mathbb{Z}$ such that $\chi(\Sigma^n(A \oplus B \oplus C)) = 0$. Then, we define

$$\chi(F) = \begin{cases} \sum_{i=0}^{n} \big((-1)^i \chi(\Sigma^i A) - (-1)^i \chi(\Sigma^i B) \\ \quad + (-1)^i \chi(\Sigma^i C)\big), & \text{if } n \geqslant 0; \\ \sum_{i=n}^{-1} \big((-1)^{i+1} \chi(\Sigma^i A) - (-1)^{i+1} \chi(\Sigma^i B) \\ \quad + (-1)^{i+1} \chi(\Sigma^i C)\big), & \text{if } n < 0. \end{cases}$$

This gives a non-negative integer, and does not depend on n since χ is a cohomological function. Moreover, $\chi(F)$ does not depend on the choice of the exact triangle that presents F by a variant of Schanuel's lemma (see Lemma A.1). It should be noted that $\chi(\mathsf{Hom}(-, C)) = \chi(C)$ for each C in C. Standard arguments involving resolutions in $\mathsf{Ab}\,\mathsf{C}$ show that χ is additive.

Clearly, restricting from $\mathsf{Ab}\,\mathsf{C}$ to C and extending from C to $\mathsf{Ab}\,\mathsf{C}$ are mutually inverse operations. ∎

There is a parallel between functions and functors. The analogue of Lemma 2.2 for functors is due to Freyd.

LEMMA 2.3. (Freyd [10]) *Restricting a functor* $F : (\mathsf{Ab}\,\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$ *to* C *by setting* $F(C) = F(\mathsf{Hom}(-, C))$ *gives a natural bijection between*

– *the exact functors* $(\mathsf{Ab}\,\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$ *and*
– *the cohomological functors* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$.

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Proof. The inverse map sends a cohomological functor $H:\mathsf{C}^{\mathrm{op}}\to\mathsf{Ab}$ to the exact functor $\mathsf{Hom}(-,H):(\mathsf{Ab}\,\mathsf{C})^{\mathrm{op}}\to\mathsf{Ab}$. ∎

An endofinite cohomological functor $H:\mathsf{C}^{\mathrm{op}}\to\mathsf{Ab}$ induces two functions:

$$\chi_H:\mathsf{Ob}\,\mathsf{C}\longrightarrow\mathbb{N},\qquad C\mapsto\mathsf{length}_{\mathsf{End}(H)}\,H(C),$$
$$\hat{\chi}_H:\mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C}\longrightarrow\mathbb{N},\qquad F\mapsto\mathsf{length}_{\mathsf{End}(H)}\,\mathsf{Hom}(F,H).$$

LEMMA 2.4. *The function χ_H is cohomological, and $\hat{\chi}_H$ is additive.*

Proof. The functor $\mathsf{Hom}(-,H):(\mathsf{Ab}\,\mathsf{C})^{\mathrm{op}}\to\mathsf{Ab}$ is exact since H is cohomological. It follows that $\hat{\chi}_H$ is additive. The restriction of $\hat{\chi}_H$ to C equals χ_H. Thus, χ_H is cohomological by Lemma 2.2. ∎

Proof of the main theorem. The proof of our main result is based on work of Crawley-Boevey, but we provide some complementary material in Appendix B.

Proof of Theorem 1.4. An endofinite cohomological functor $H:\mathsf{C}^{\mathrm{op}}\to\mathsf{Ab}$ is a finite endolength injective object in $\mathsf{Mod}\,\mathsf{C}$, as defined in [6]. The term "finite endolength" refers to the fact that $\mathsf{Hom}(F,H)$ has finite length as $\mathsf{End}(H)$-module for each F in $\mathsf{Ab}\,\mathsf{C}$ (see Lemma 2.4). The injectivity follows from the proof of [23, Theorem 1.2], using that $\mathsf{Ext}^1(-,H)$ vanishes on $\mathsf{Ab}\,\mathsf{C}$ for any cohomological functor H.

In [6, Theorem 3.5.2], it is shown that each finite endolength object decomposes into a direct sum of indecomposable objects with local endomorphism rings. (See [23, Theorem 1.2] for an alternative proof.) This yields part (1).

In [7], additive functions on locally finitely presented abelian categories are studied. In particular, there it is shown that every additive function on $\mathsf{Ab}\,\mathsf{C}$ can be written uniquely as a locally finite sum of irreducible additive functions. This proves part (2), in view of Lemma 2.2. For an alternative proof, see Proposition B.1.

Finally, part (3) follows from the main theorem in [7], which establishes the bijection between isomorphism classes of indecomposable finite endolength injective objects in $\mathsf{Mod}\,\mathsf{C}$ and irreducible additive functions $\mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C}\to\mathbb{N}$. For an alternative proof, see Proposition B.2, using the bijection between cohomological and exact functors from Lemma 2.3. ∎

§3. Properties of cohomological functions

The correspondence between functors and functions. The assignment $H \mapsto \chi_H$ between endofinite cohomological functors and cohomological functions satisfies some weighted additivity. For instance, $\chi_{H\oplus H'} = \chi_H + \chi_{H'}$ provided that H and H' have no common indecomposable summand, but $\chi_{H\oplus H} = \chi_H$. We have the following concise formula.

PROPOSITION 3.1. *Let $H = \bigoplus_{i\in I} H_i$ be the decomposition of an endofinite cohomological functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ into indecomposables. If $J \subseteq I$ is a subset such that $(H_i)_{i\in J}$ contains each isomorphism class from $(H_i)_{i\in I}$ exactly once, then $\chi_H = \sum_{i\in J} \chi_{H_i}$.*

Proof. We adapt the proofs of [25, Propositions 4.5 and 4.6]. Alternatively, we use Remark B.4. ∎

REMARK 3.2. Let C be a k-linear category such that each morphism set in C has finite length as a k-module. Then, we have two maps $\mathsf{Ob}\,\mathsf{C} \times \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$, taking (X, Y) either to $\mathsf{length}_k \mathsf{Hom}(X, Y)$ or to $\mathsf{length}_{\mathsf{End}(Y)} \mathsf{Hom}(X, Y)$. While the first map preserves sums in both arguments, the second one does not in the second argument, but it satisfies the above "weighted additivity".

Duality. The correspondence in Theorem 1.4 yields a remarkable duality between cohomological functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ and cohomological functors $\mathsf{C} \to \mathsf{Ab}$. This follows from the fact that the definition of a cohomological function $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ is self-dual; it is an analogue of the *elementary duality* between left and right modules over a ring studied by Herzog [16].

The duality links indecomposable endofinite cohomological functors $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ and $H' : \mathsf{C} \to \mathsf{Ab}$ when $\chi_H = \chi_{H'}$. In that case,

$$(\mathsf{End}(H)/\mathsf{rad}\,\mathsf{End}(H))^{\mathrm{op}} \cong \mathsf{End}(H')/\mathsf{rad}\,\mathsf{End}(H'),$$

where $\mathsf{rad}\,A$ denotes the Jacobson radical of a ring A. This follows from Remark B.3.

The duality specializes to Serre duality when C is a Hom-finite k-linear category with k a field. More precisely, if $F : \mathsf{C} \to \mathsf{C}$ is a Serre functor [29] and $D = \mathsf{Hom}(-, k)$, then

$$\mathsf{Hom}(-, FX) \cong D\,\mathsf{Hom}(X, -)$$

for each object X, and therefore $\chi_{\mathsf{Hom}(-,FX)} = \chi_{\mathsf{Hom}(X,-)}$.

The space of cohomological functions. Consider the set of irreducible cohomological functions $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$, and identify this via Lemma 2.2 with a subspace of $\mathsf{Sp}\,\mathsf{Ab}\,\mathsf{C}$, endowed with the Ziegler topology, as explained in Proposition B.5. The quotient

$$\mathsf{Sp}\,\mathsf{C} = \{\chi : \mathsf{Ob}\,\mathsf{C} \to \mathbb{N} \mid \chi \text{ irreducible and cohomological}\}/\Sigma$$

with respect to the action of the suspension is by definition the *space of cohomological functions* on C. Thus, the points of $\mathsf{Sp}\,\mathsf{C}$ are equivalence classes of the form $[\chi] = \{\chi \circ \Sigma^n \mid n \in \mathbb{Z}\}$.

The construction of this space is functorial with respect to certain functors. Let $f : \mathsf{C} \to \mathsf{D}$ be an exact functor between triangulated categories. Given $[\chi]$ in $\mathsf{Sp}\,\mathsf{D}$, the composite $\chi \circ f$ is cohomological but need not be irreducible. Thus, f induces a continuous map $\mathsf{Sp}\,\mathsf{D} \to \mathsf{Sp}\,\mathsf{C}$ provided that irreducibility is preserved. For instance, a quotient functor $\mathsf{C} \to \mathsf{C}/\mathsf{B}$ with respect to a triangulated subcategory $\mathsf{B} \subseteq \mathsf{C}$ has this property; it induces a homeomorphism

$$\mathsf{Sp}\,\mathsf{C}/\mathsf{B} \xrightarrow{\sim} \{[\chi] \in \mathsf{Sp}\,\mathsf{C} \mid \chi(\mathsf{B}) = 0\}.$$

Before we discuss specific examples, let us give one general result. Let k be a field, and let C be a k-linear triangulated category such that for each pair of objects X, Y we have $\mathsf{Hom}(X, \Sigma^n Y) = 0$ for some $n \in \mathbb{Z}$. Suppose that all morphism spaces are finite dimensional, and that C is idempotent complete. Suppose also that C admits a Serre functor [29]. Denote for each object X by χ_X the cohomological function corresponding to $\mathsf{Hom}(-, X)$.

PROPOSITION 3.3. *A point in* $\mathsf{Sp}\,\mathsf{C}$ *is isolated if and only if it equals* $[\chi_X]$ *for some indecomposable object* X*. Moreover, the isolated points form a dense subset of* $\mathsf{Sp}\,\mathsf{C}$*.*

Proof. Each indecomposable object X fits into an Auslander–Reiten triangle $X \to Y \to Z \to$ in C, by [29, Theorem I.2.4]. Such a triangle provides in $\mathsf{Ab}\,\mathsf{C}$ the following copresentation of a simple object S_X:

$$0 \longrightarrow S_X \longrightarrow \mathsf{Hom}(-, X) \longrightarrow \mathsf{Hom}(-, Y).$$

Thus, $(S_X) = \{\chi_X\}$ is a basic open set for each indecomposable object X.

Now, let (F) be a non-empty basic open set with $F \in \mathsf{Ab}\,\mathsf{C}$. The functor F admits a copresentation

$$0 \longrightarrow F \longrightarrow \mathsf{Hom}(-, X) \longrightarrow \mathsf{Hom}(-, Y)$$

with an indecomposable direct summand $X' \subseteq X$ such that

$$\mathsf{Hom}(F, \mathsf{Hom}(-, X')) \neq 0.$$

Thus, $\chi_{X'}$ belongs to (F). It follows that each non-empty open subset of $\mathsf{Sp}\,\mathsf{C}$ contains a point of the form $[\chi_X]$. ∎

The space of cohomological functions may be empty, as the following example shows.

Example 3.4. Let A be a ring without invariant basis number, for example the endomorphism ring of an infinite dimensional vector space. Then, there is no non-zero endofinite cohomological functor $H : \mathsf{D}^b(\mathsf{proj}\,A)^{\mathrm{op}} \to \mathsf{Ab}$. To see this, observe that $H(\Sigma^n A)$ is a finite endolength A-module for all $n \in \mathbb{Z}$, and therefore the zero module (see [5, Section 4.7]).

§4. Examples: perfect complexes

We compute the space $\mathsf{Sp}\,\mathsf{C}$ of cohomological functions in some examples, for instance when C is the triangulated category of perfect complexes over some ring. It is convenient to view $\mathsf{Sp}\,\mathsf{C}$ as a subspace of the spectrum $\mathsf{Zsp}\,\mathsf{C}$, as defined in Appendix C.

The problem of computing the space of cohomological functions is reduced to the study of the Krull–Gabriel filtration of the abelianization $\mathsf{Ab}\,\mathsf{C}$. This filtration yields a dimension. For an abelian category A, the Krull–Gabriel dimension $\mathsf{KGdim}\,\mathsf{A}$ is an invariant, which measures how far A is away from being a length category (see Appendix C).

Modules. Let A be a ring. We write $\mathsf{Mod}\,A$ for the category of A-modules, $\mathsf{mod}\,A$ for the full subcategory of finitely presented ones, and $\mathsf{proj}\,A$ for the full subcategory of finitely generated projectives.

Following [14], we consider the *free abelian category*[4] over A, and denote it by $\mathsf{Ab}\,A$. Thus, the category of A-modules identifies with the category of exact functors $(\mathsf{Ab}\,A)^{\mathrm{op}} \to \mathsf{Ab}$.

We consider the derived category $\mathsf{D}(\mathsf{Mod}\,A)$, and write $\mathsf{per}\,A$ for the full subcategory $\mathsf{D}^b(\mathsf{proj}\,A)$ of *perfect complexes*.

[4]The category $\mathsf{Ab}\,A$ is the opposite of the category of functors $F : \mathsf{mod}\,A \to \mathsf{Ab}$ that admit a presentation $\mathsf{Hom}(Y, -) \to \mathsf{Hom}(X, -) \to F \to 0$, and A (viewed as a category with a single object) embeds via $A \mapsto \mathsf{Hom}(A, -)$. Any additive functor $A \to \mathsf{A}$ to an abelian category A extends uniquely to an exact functor $\mathsf{Ab}\,A \to \mathsf{A}$.

The *Ziegler spectrum* $\mathsf{Zsp}\, A$ of A is by definition the set of isomorphism classes of indecomposable pure-injective A-modules with the topology introduced by Ziegler [28, 34]. We identify $\mathsf{Zsp}\, A$ with the spectrum $\mathsf{Zsp}\,\mathsf{Ab}\, A$ of the abelian category $\mathsf{Ab}\, A$ (see Appendix C).

It should be noted that the spectrum $\mathsf{Zsp}\,\mathsf{per}\, A$ identifies with a quotient of the Ziegler spectrum of the compactly generated triangulated category $\mathsf{D}(\mathsf{Mod}\, A)$ introduced in [24] and further investigated in [12]. The identification takes an object X in $\mathsf{D}(\mathsf{Mod}\, A)$ to $\mathsf{Hom}(-, X)|_{\mathsf{per}\, A}$.

Let us compare $\mathsf{Zsp}\, A$ and $\mathsf{Zsp}\,\mathsf{per}\, A$. We consider the natural inclusion $i : A \to \mathsf{per}\, A$, which extends to an exact functor $i^* : \mathsf{Ab}\, A \to \mathsf{Ab}\,\mathsf{per}\, A$ by the universal property of $\mathsf{Ab}\, A$.

LEMMA 4.1. *Let $\mathsf{S}_0 \subseteq \mathsf{Ab}\,\mathsf{per}\, A$ be the Serre subcategory generated by the representable functors $\mathsf{Hom}(-, \Sigma^n A)$ with $n \neq 0$. Then, the composite*

$$\mathsf{Ab}\, A \xrightarrow{i^*} \mathsf{Ab}\,\mathsf{per}\, A \twoheadrightarrow (\mathsf{Ab}\,\mathsf{per}\, A)/\mathsf{S}_0$$

is an equivalence.

Proof. The cohomological functors $H : (\mathsf{per}\, A)^{\mathrm{op}} \to \mathsf{Ab}$ annihilating $\Sigma^n A$ for all $n \neq 0$ identify with $\mathsf{Mod}\, A$, by taking H to $H(A)$. Thus, the exact functors $(\mathsf{Ab}\,\mathsf{per}\, A)^{\mathrm{op}} \to \mathsf{Ab}$ annihilating S_0 identify with $\mathsf{Mod}\, A$, by Lemma 2.3. It remains to observe that an exact functor $f : \mathsf{C} \to \mathsf{D}$ between abelian categories is an equivalence if precomposition with f induces an equivalence between the categories of exact functors $\mathsf{D}^{\mathrm{op}} \to \mathsf{Ab}$ and $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. ∎

We view an A-module X as a complex concentrated in degree zero, and denote by H_X the corresponding cohomological functor $\mathsf{Hom}(-, X) : (\mathsf{per}\, A)^{\mathrm{op}} \to \mathsf{Ab}$. It is convenient to identify H_X with the exact functor

$$\mathsf{Hom}(-, H_X) : (\mathsf{Ab}\,\mathsf{per}\, A)^{\mathrm{op}} \longrightarrow \mathsf{Ab}.$$

PROPOSITION 4.2. *The assignment $X \mapsto [H_X]$ induces an injective and continuous map $\phi : \mathsf{Zsp}\, A \to \mathsf{Zsp}\,\mathsf{per}\, A$; its image is a closed subset.*

Proof. We apply Lemma 4.1. The composite

$$f : \mathsf{Ab}\,\mathsf{per}\, A \twoheadrightarrow (\mathsf{Ab}\,\mathsf{per}\, A)/\mathsf{S}_0 \xrightarrow{\sim} \mathsf{Ab}\, A$$

identifies $\mathsf{Zsp}\, A$ with a closed subset of $\mathsf{Zsp}\,\mathsf{Ab}\,\mathsf{per}\, A$, which we denote by U. Viewing an A-module X as an exact functor $(\mathsf{Ab}\, A)^{\mathrm{op}} \to \mathsf{Ab}$, we have

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$X \circ f = H_X$. It should be noted that the subsets $\Sigma^n \mathsf{U}$, $n \in \mathbb{Z}$, are pairwise disjoint. It follows that ϕ is injective.

Next, we observe that

$$\bigcup_{n \in \mathbb{Z}} \Sigma^n \mathsf{U} = (\mathsf{Zsp}\,\mathsf{Ab}\,\mathsf{per}\, A) \Big\backslash \bigcup_{(r,s)} \mathsf{U}_{r,s},$$

where (r, s) runs through all pairs of integers $r \neq s$, and

$$\mathsf{U}_{r,s} = \{F \in \mathsf{Zsp}\,\mathsf{Ab}\,\mathsf{per}\, A \mid F(\Sigma^r A) \neq 0 \neq F(\Sigma^s A)\}.$$

Thus, the image of ϕ is closed.

Now, let $\mathsf{V} \subseteq \mathsf{Zsp}\,\mathsf{per}\, A$ be a closed subset, which corresponds to a Serre subcategory $\mathsf{S} \subseteq \mathsf{Ab}\,\mathsf{per}\, A$ (see Lemma C.1). Thus, V consists of all functors in $\mathsf{Zsp}\,\mathsf{per}\, A$ that vanish on S. Then, $\phi^{-1}(\mathsf{V})$ consists of all functors vanishing on $f(\mathsf{S})$. It follows that ϕ is continuous. □

Hereditary rings. A complex of A-modules can be written as a direct sum of stalk complexes when A is hereditary. This has some useful consequences, which are collected in the following proposition.

PROPOSITION 4.3. *For a hereditary ring A, the following holds.*

(1) *The map $\phi : \mathsf{Zsp}\, A \to \mathsf{Zsp}\,\mathsf{per}\, A$ taking X to $[H_X]$ is a homeomorphism.*
(2) $\mathsf{KGdim}\,\mathsf{Ab}\,\mathsf{per}\, A = \mathsf{KGdim}\,\mathsf{Ab}\, A$.

Proof. (1) We apply Proposition 4.2. Each indecomposable complex of A-modules is concentrated in a single degree since A is hereditary. This observation yields a disjoint union

$$\mathsf{Zsp}\,\mathsf{Ab}\,\mathsf{per}\, A = \bigcup_{n \in \mathbb{Z}} \mathsf{U}_n \quad \text{with } \mathsf{U}_n = \{F \in \mathsf{Zsp}\,\mathsf{Ab}\,\mathsf{per}\, A \mid F(\Sigma^n A) \neq 0\},$$

and each U_n is homeomorphic to $\mathsf{Zsp}\, A$. An open subset $\mathsf{V} \subseteq \mathsf{Zsp}\, A$ identifies with an open subset $\mathsf{V}_n \subseteq \mathsf{U}_n$, and therefore with an open subset of $\mathsf{Zsp}\,\mathsf{per}\, A$ via ϕ. It follows that ϕ is open and therefore a homeomorphism by Proposition 4.2.

(2) We apply Lemma C.4 and use the family of quotient functors

$$(\mathsf{Ab}\,\mathsf{per}\, A \twoheadrightarrow (\mathsf{Ab}\,\mathsf{per}\, A)/\mathsf{S}_n \xrightarrow{\sim} \mathsf{Ab}\, A)_{n \in \mathbb{Z}}$$

from Lemma 4.1, where S_n is by definition the Serre subcategory generated by the representable functors $\mathsf{Hom}(-, \Sigma^p A)$ with $p \neq n$. □

Commutative rings. Let A be a commutative ring, and let $\mathsf{Spec}\, A$ be the set of prime ideals. We endow $\mathsf{Spec}\, A$ with the dual of the Zariski topology in the sense of Hochster [18]. A prime ideal $\mathfrak{p}$ with residue field $k(\mathfrak{p})$ yields an irreducible cohomological function

$$\chi_{k(\mathfrak{p})} : \mathsf{Ob}\, \mathsf{per}\, A \longrightarrow \mathbb{N}, \qquad X \mapsto \mathsf{length}_{k(\mathfrak{p})} \mathsf{Hom}(X, k(\mathfrak{p})).$$

As before, we identify $\chi_{k(\mathfrak{p})}$ with $k(\mathfrak{p})$. In particular, $\chi_{k(\mathfrak{p})}$ is irreducible since $k(\mathfrak{p})$ is indecomposable.

THEOREM 4.4. *The map $\rho : \mathsf{Spec}\, A \to \mathsf{Sp}\, \mathsf{per}\, A$ sending $\mathfrak{p}$ to $[\chi_{k(\mathfrak{p})}]$ is injective and closed with respect to the Hochster dual of the Zariski topology on $\mathsf{Spec}\, A$.*

Proof. The injectivity is clear since different primes $\mathfrak{p}, \mathfrak{q}$ yield non-isomorphic functors $\mathsf{Hom}(-, k(\mathfrak{p}))$ and $\mathsf{Hom}(-, k(\mathfrak{q}))$.

To show that the image $\mathsf{Im}\, \rho$ is closed, observe first that

$$\mathsf{U} = \{[\chi] \in \mathsf{Sp}\, \mathsf{per}\, A \mid \chi(\Sigma^n A) \neq 0 \text{ for at most one } n \in \mathbb{Z}\}$$

is closed by Proposition 4.2. Moreover,

$$\mathsf{U}_1 = \{[\chi] \in \mathsf{Sp}\, \mathsf{per}\, A \mid \chi(\Sigma^n A) \leqslant 1 \text{ for all } n \in \mathbb{Z}\}$$

is closed by Lemma B.6. The indecomposable A-modules X with $\mathsf{length}_{\mathsf{End}(X)} X \leqslant 1$ are precisely the residue fields $k(\mathfrak{p})$ (see [5, Section 4.7]). Thus, $\mathsf{Im}\, \rho = \mathsf{U} \cap \mathsf{U}_1$ is closed.

Given a closed subset $\mathsf{V} \subseteq \mathsf{Spec}\, A$, we need to show that $\rho(\mathsf{V})$ is closed. It follows from Thomason's classification of thick subcategories [30, Theorem 3.15] that there is a thick subcategory C of $\mathsf{per}\, A$ with $\mathfrak{p} \in \mathsf{V}$ if and only if $\mathsf{Hom}(X, k(\mathfrak{p})) = 0$ for all $X \in \mathsf{C}$. Here, one uses that $X^* \otimes^{\mathbf{L}}_A k(\mathfrak{p}) \cong \mathsf{RHom}_A(X, k(\mathfrak{p}))$ when X is perfect.

The latter condition means that $\chi_{k(\mathfrak{p})}(X) = 0$ for all $X \in \mathsf{C}$. Thus, $\{[\chi_{k(\mathfrak{p})}] \mid \mathfrak{p} \in \mathsf{V}\}$ is Ziegler closed. ∎

REMARK 4.5. This result generalizes to schemes that are quasi-compact and quasi-separated.

Krull–Gabriel dimension zero. Following [26], a triangulated category C is *locally finite* if its abelianization $\mathsf{Ab}\, \mathsf{C}$ is a length category, which means that $\mathsf{KGdim}\, \mathsf{Ab}\, \mathsf{C} \leqslant 0$. We set $\chi_X = \chi_{\mathsf{Hom}(-,X)}$ for $X \in \mathsf{C}$ when $\mathsf{Hom}(-, X)$ is endofinite.

PROPOSITION 4.6. *Let* C *be a locally finite and idempotent complete triangulated category. Suppose for each pair of objects* X, Y *that* $\mathsf{Hom}(X, \Sigma^n Y) = 0$ *for some* $n \in \mathbb{Z}$. *Then,*

$$\mathsf{Sp}\,\mathsf{C} = \{[\chi_X] \mid X \in \mathsf{C} \textit{ indecomposable}\}.$$

Proof. Let X be an object in C. Then, we have for each object C

$$\mathsf{length}_{\mathsf{End}(X)} \mathsf{Hom}(C, X) \leqslant \mathsf{length}_{\mathsf{Ab}\,\mathsf{C}} \mathsf{Hom}(-, C) < \infty$$

by [1, Theorem 2.12] (see also Remark B.4). Thus, $\mathsf{Hom}(-, X)$ is endofinite.

Let $\chi : \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ be an irreducible cohomological function, and let $\hat{\chi} : \mathsf{Ob}\,\mathsf{Ab}\,\mathsf{C} \to \mathbb{N}$ be its extension to $\mathsf{Ab}\,\mathsf{C}$. Then, $\hat{\chi}(S) \neq 0$ for some simple object S. There is an indecomposable object X in C with $\mathsf{Hom}(S, \mathsf{Hom}(-, X)) \neq 0$, and it follows that $\mathsf{Hom}(-, X)$ is an injective envelope. Thus, $\chi = \chi_X$. □

EXAMPLE 4.7. Let k be a field, and let Γ be a quiver with underlying diagram of Dynkin type. Denote by $\mathsf{rep}(\Gamma, k)$ the category of finite dimensional k-linear representations of Γ. The indecomposable endofinite cohomological functors $\mathsf{D}^b(\mathsf{rep}(\Gamma, k))^{\mathrm{op}} \to \mathsf{Ab}$ are precisely (up to isomorphism) the representable functors $\mathsf{Hom}(-, \Sigma^n X)$, with X an indecomposable object in $\mathsf{rep}(\Gamma, k)$ (viewed as complex concentrated in degree zero) and $n \in \mathbb{Z}$.

Examples of triangulated categories C with $\mathsf{KGdim}\,\mathsf{Ab}\,\mathsf{C} = 1$ show that the description of $\mathsf{Sp}\,\mathsf{C}$ in Proposition 4.6 does not hold more generally.

Krull–Gabriel dimension one. Let k be a field. We give an example of a finite dimensional k-algebra A such that the Krull–Gabriel dimension of $\mathsf{Ab}\,\mathsf{per}\,A$ equals 1. This gives some evidence for the following conjecture.

CONJECTURE 4.8. *Let* A *be a finite dimensional* k*-algebra. Then, the Krull–Gabriel dimension of* $\mathsf{Ab}\,\mathsf{per}\,A$ *equals* 0 *or* 1 *if and only if* $\mathsf{per}\,A$ *is a discrete derived category in the sense of [32].*

PROPOSITION 4.9. *Let* $k[\varepsilon]$ *be the algebra of dual numbers. Then,*

$$\mathsf{KGdim}\,\mathsf{Ab}\,\mathsf{per}\,k[\varepsilon] = 1.$$

Proof. Set $\mathsf{C} = \mathsf{per}\,k[\varepsilon]$, and write $H_X = \mathsf{Hom}(-, X)$ for each X in C. The indecomposable objects are the complexes

$$X_{n,r} : \cdots \to 0 \to k[\varepsilon] \xrightarrow{\varepsilon} k[\varepsilon] \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} k[\varepsilon] \to 0 \to \cdots$$

concentrated in degrees $n, n+1, \cdots, n+r$ and parametrized by pairs (n, r) in $\mathbb{Z} \times \mathbb{N}$. We denote by $\tilde{\varepsilon} : X_{n,r} \to X_{n,r}$ the endomorphism given by $k[\varepsilon] \xrightarrow{\varepsilon} k[\varepsilon]$ in degree n, and zero in all other degrees. The Auslander–Reiten triangles are of the form

$$X_{n+1,r} \longrightarrow X_{n+1,r-1} \oplus X_{n,r+1} \longrightarrow X_{n,r} \xrightarrow{\tilde{\varepsilon}} X_{n,r},$$

with $X_{n,-1} = 0$. Such a triangle induces in $\mathsf{Ab}\,\mathsf{C}$ an exact sequence

$$(4.1)\quad 0 \to S_{n+1,r} \to H_{X_{n+1,r}} \to H_{X_{n+1,r-1}} \oplus H_{X_{n,r+1}} \to H_{X_{n,r}} \to S_{n,r} \to 0,$$

with simple end terms.

We fix $n \in \mathbb{Z}$. We claim that the Hasse diagram of the lattice of subobjects of $H_{X_{n,0}}$ has the following form:

To prove this, consider the sequence of morphisms

$$\cdots \longrightarrow X_{n,2} \longrightarrow X_{n,1} \longrightarrow X_{n,0}$$

given by the Auslander–Reiten triangles. For each $t \geqslant 0$, the composite $\phi_{n,t} : X_{n,t} \to X_{n,0}$ induces a morphism $H_{\phi_{n,t}}$ in $\mathsf{Ab}\,\mathsf{C}$, and its image is the unique subobject $U \subseteq H_{X_{n,0}}$ such that $H_{X_{n,0}}/U$ has length t. This explains the upper half of the Hasse diagram. The form of the lower half then follows by Serre duality. More precisely, Serre duality yields an equivalence $\mathsf{C}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{C}$, which is the identity on objects. It extends to an equivalence $(\mathsf{Ab}\,\mathsf{C})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Ab}(\mathsf{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathsf{Ab}\,\mathsf{C}$, which induces a bijection between subobjects and quotient objects of $H_{X_{n,0}}$. It remains to show that $H_{X_{n,0}}$ has no further

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subobjects. To see this, let $V \subseteq H_{X_{n,0}}$ be a subobject; it is the image of some morphism $H_\phi : H_X \to H_{X_{n,0}}$. We may assume that $\phi \neq 0$, and that X is indecomposable. The property of the Auslander–Reiten triangle for $X_{n,0}$ implies that the endomorphism $\tilde{\varepsilon} : X_{n,0} \to X_{n,0}$ factors through ϕ via a morphism $\phi' : X_{n,0} \to X$, since $X_{n,0} \to \mathsf{cone}\,\phi$ factors through the left almost split morphism $X_{n,0} \to X_{n,1}$. Thus, ϕ and ϕ' yield in degree n endomorphisms of $k[\varepsilon]$, and exactly one of them is an isomorphism. If ϕ^n is an isomorphism, then $H_{X_{n,0}}/V$ has finite length; otherwise V is of finite length.

The form of the lattice of subobjects implies that $H_{X_{n,0}}$ is a simple object in $(\mathsf{Ab}\,\mathsf{C})/(\mathsf{Ab}\,\mathsf{C})_0$. Using induction on r, the sequence (4.1) shows that $H_{X_{n,r}}$ has length $r+1$. Thus, $(\mathsf{Ab}\,\mathsf{C})_1 = \mathsf{Ab}\,\mathsf{C}$. ∎

COROLLARY 4.10. *We have* $\mathsf{Sp}\,\mathsf{per}\,k[\varepsilon] = \{[\chi_{X_{0,r}}] \mid r \in \mathbb{N}\} \cup \{[\chi_k]\}$.

Proof. Set $\mathsf{C} = \mathsf{per}\,k[\varepsilon]$. The Krull–Gabriel filtration of $\mathsf{Ab}\,\mathsf{C}$ yields a filtration of $\mathsf{Zsp}\,\mathsf{C}$ by Proposition C.2. Thus, the points of $\mathsf{Zsp}\,\mathsf{C}$ correspond to the simple objects in $\mathsf{Ab}\,\mathsf{C}$ and $(\mathsf{Ab}\,\mathsf{C})/(\mathsf{Ab}\,\mathsf{C})_0$. These simple objects are described in the proof of Proposition 4.9. The simples in $\mathsf{Ab}\,\mathsf{C}$ correspond to the indecomposable objects in C, and yield isolated points (see also Proposition 3.3 and its proof). The simples in $(\mathsf{Ab}\,\mathsf{C})/(\mathsf{Ab}\,\mathsf{C})_0$ correspond to the complexes with k concentrated in a single degree. Thus, all points in $\mathsf{Zsp}\,\mathsf{C}$ are endofinite, and this yields the description of $\mathsf{Sp}\,\mathsf{C}$. ∎

REMARK 4.11. It should be noted that $\mathsf{KGdim}\,\mathsf{Ab}\,A \neq 1$ for any Artin algebra A (see [17, 21]).

Krull–Gabriel dimension two. Let k be a field, and consider the Kronecker algebra $A = \left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$. Work of Geigle [13] shows that the Krull–Gabriel dimension of $\mathsf{Ab}\,A$ equals 2. Thus, $\mathsf{KGdim}\,\mathsf{Ab}\,\mathsf{per}\,A = 2$ by Proposition 4.3, since A is hereditary.

Now, let $\mathsf{coh}\,\mathbb{P}^1_k$ be the category of coherent sheaves on the projective line over k. There is a well-known derived equivalence

$$\mathsf{RHom}(T, -) : \mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^1_k) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\,A)$$

given by $T = \mathcal{O}(0) \oplus \mathcal{O}(1)$, and we use this to establish the description of the cohomological functions on $\mathsf{coh}\,\mathbb{P}^1_k$ stated in the introduction.

Proof of Proposition 1.8. We have

$$\mathsf{KGdim}\,\mathsf{Ab}\,\mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^1_k) = \mathsf{KGdim}\,\mathsf{Ab}\,\mathsf{per}\,A = \mathsf{KGdim}\,\mathsf{Ab}\,A = 2,$$

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by Proposition 4.3 and [13, Theorem 4.3]. This yields an explicit description of the points in $\mathsf{Zsp}\,\mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^1_k)$, which is parallel to that given in Corollary 4.10. More explicitly, the indecomposable endofinite cohomological functors $\mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^1_k)^{\mathrm{op}} \to \mathsf{Ab}$ are precisely the representable functors $\mathsf{Hom}(-, \Sigma^n X)$, with X an indecomposable object in $\mathsf{coh}\,\mathbb{P}^1_k$ or $X = k(t)$ the function field, and $n \in \mathbb{Z}$. □

Added in proof. Further material on Conjecture 4.8 can be found in the following subsequent publication: G. Bobiski and H. Krause, *The Krull-Gabriel dimension of discrete derived categories*, Bull. Sci. Math. **139** (2015), 269–282.

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Appendix A. Schanuel's lemma for triangulated categories

Let C be a triangulated category. An exact triangle $A \to B \to C \to$ induces a *presentation* of a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ provided that there exists an exact sequence

$$\mathsf{Hom}(-, B) \longrightarrow \mathsf{Hom}(-, C) \longrightarrow F \longrightarrow 0.$$

Two exact triangles are called *homotopy equivalent*[5] if they induce presentations of the same functor.

LEMMA A.1. *Let $A \to B \to C \to$ and $A' \to B' \to C' \to$ be two homotopy equivalent exact triangles. Then, $A \oplus B' \oplus C \cong A' \oplus B \oplus C'$.*

Proof. The triangles induce exact sequences

$$0 \to \Sigma^{-1}F \to \mathsf{Hom}(-, A) \to \mathsf{Hom}(-, B) \to \mathsf{Hom}(-, C) \to F \to 0$$

and

$$0 \to \Sigma^{-1}F \to \mathsf{Hom}(-, A') \to \mathsf{Hom}(-, B') \to \mathsf{Hom}(-, C') \to F \to 0,$$

[5] This notion is consistent with the homotopy relation introduced in [27, Section 1.3].

which represent the same class in $\mathsf{Ext}^3(F, \Sigma^{-1}F)$, since the presentations induce a morphism between the two triangles. Now, we apply the variant of Schanuel's lemma, which is given below. □

LEMMA A.2. *Let*

$$0 \to M \to P_r \to \cdots \to P_1 \to P_0 \to N \to 0$$

and

$$0 \to M \to Q_r \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

be exact sequences in some abelian category, which represent the same class in $\mathsf{Ext}^{r+1}(N, M)$. *If all* P_i *and* Q_i *are projective, then*

$$\bigoplus_{i \geqslant 0}(P_{2i} \oplus Q_{2i+1}) \cong \bigoplus_{i \geqslant 0}(P_{2i+1} \oplus Q_{2i}).$$

Proof. We use induction on r. The case $r = 0$ is clear, and we suppose that $r > 0$. The pullback of $\eta : P_0 \to N$ and $\theta : Q_0 \to N$ induces an exact sequence

$$\varepsilon : 0 \longrightarrow K \longrightarrow P_0 \oplus Q_0 \longrightarrow N \longrightarrow 0$$

with $Q_0 \oplus \mathsf{Ker}\,\eta \cong K \cong P_0 \oplus \mathsf{Ker}\,\theta$, by Schanuel's lemma. Adding complexes of the form $Q_0 \xrightarrow{\text{id}} Q_0$ and $P_0 \xrightarrow{\text{id}} P_0$ yields two exact sequences

$$0 \to M \to P_r \to \cdots \to P_2 \to P_1 \oplus Q_0 \to K \to 0$$

and

$$0 \to M \to Q_r \to \cdots \to Q_2 \to Q_1 \oplus P_0 \to K \to 0,$$

which represent the same class in $\mathsf{Ext}^r(K, M)$, since multiplication with the class corresponding to ε induces an isomorphism $\mathsf{Ext}^r(K, M) \xrightarrow{\sim} \mathsf{Ext}^{r+1}(N, M)$. Now, the assertion follows from the induction hypothesis. □

Appendix B. Additive functions

Let A be an abelian category. A function $\chi : \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ is called *additive* if $\chi(X) = \chi(X') + \chi(X'')$ for each exact sequence $0 \to X' \to X \to X'' \to 0$.

We give a quick proof of the following result using the localization theory for abelian categories [11].

PROPOSITION B.1. (Crawley-Boevey [7]) *Every additive function* $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ *can be written uniquely as a locally finite sum of irreducible additive functions.*

Proof. Fix an additive function $\chi : \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$. The objects X satisfying $\chi(X) = 0$ form a Serre subcategory of A, which we denote by S_χ. The quotient category $\mathsf{A}/\mathsf{S}_\chi$ is an abelian length category since the length of each object X is bounded by $\chi(X)$. Let $\mathsf{Sp}\,\chi$ (the *spectrum* of χ) denote a representative set of simple objects in $\mathsf{A}/\mathsf{S}_\chi$. For each S in $\mathsf{Sp}\,\chi$, let S_S denote the Serre subcategory of A formed by all objects X such that a composition series of X in $\mathsf{A}/\mathsf{S}_\chi$ has no factor isomorphic to S. Define $\chi_S : \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ by sending X to the length of X in A/S_S. From the construction, it follows that

$$\chi = \sum_{S \in \mathsf{Sp}\,\chi} \chi(S)\chi_S. \tag{B.1}$$

We claim that each χ_S is irreducible, and that the above expression is unique. To see this, we write $\chi = \chi' + \chi''$ as a sum of two additive functions. This implies that $\mathsf{S}_\chi \subseteq \mathsf{S}_{\chi'}$, and if $\chi' \neq 0$, then for some $S \in \mathsf{Sp}\,\chi$ the object S is non-zero in $\mathsf{A}/\mathsf{S}_{\chi'}$. In that case, χ_S arises as a summand of χ' with multiplicity $\chi'(S)$. ∎

Now, suppose that A is *essentially small*. Thus, the isomorphism classes of objects in A form a set. An exact functor $F : \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ is called *endofinite* if $F(X)$ has finite length as $\mathsf{End}(F)$-module for each object X. An endofinite exact functor F induces an additive function

$$\chi_F : \mathsf{Ob}\,\mathsf{A} \longrightarrow \mathbb{N}, \qquad X \mapsto \mathsf{length}_{\mathsf{End}(F)}\, F(X).$$

PROPOSITION B.2. *The assignment $F \mapsto \chi_F$ induces a bijection between the isomorphism classes of indecomposable endofinite exact functors $\mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ and the irreducible additive functions $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$.*

Proof. We construct the inverse map. Let $\chi : \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ be an irreducible additive function. Following the proof of Proposition B.1, we consider the Serre subcategory S_χ of A consisting of the objects X satisfying $\chi(X) = 0$. The quotient category $\mathsf{B} = \mathsf{A}/\mathsf{S}_\chi$ is an abelian length category, and $\chi(X)$ equals the length of X in B for each object X, since χ is irreducible. Now, consider the abelian category $\mathsf{Lex}(\mathsf{B}^{\mathrm{op}}, \mathsf{Ab})$ of left exact functors $\mathsf{B}^{\mathrm{op}} \to \mathsf{Ab}$ (see [11] for details). The Yoneda functor

$$\mathsf{B} \longrightarrow \mathsf{Lex}(\mathsf{B}^{\mathrm{op}}, \mathsf{Ab}), \qquad X \mapsto H_X = \mathsf{Hom}(-, X)$$

identifies B with the full subcategory of finite length objects. There is a unique simple object in $\mathsf{Lex}(\mathsf{B}^{\mathrm{op}}, \mathsf{Ab})$ since χ is irreducible, and we denote by F its injective envelope. It follows that F is indecomposable, and the injectivity implies that F is exact. For each X in B we have

$$\mathsf{length}_{\mathsf{End}(F)} F(X) = \mathsf{length}_{\mathsf{End}(F)} \mathsf{Hom}(H_X, F) = \mathsf{length}_{\mathsf{B}} X = \chi(X),$$

since each finitely generated $\mathsf{End}(F)$-submodule of $\mathsf{Hom}(H_X, F)$ is of the form $\mathsf{Hom}(H_X/H_{X'}, F)$ for some subobject $X' \subseteq X$. Let $F' : \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be the composite of F with the quotient functor $\mathsf{A} \to \mathsf{B}$, and observe that $\mathsf{End}(F') \cong \mathsf{End}(F)$. Then F' has the desired properties: it is indecomposable endofinite exact, and $\chi_{F'} = \chi$.

It remains to show for an indecomposable endofinite exact functor $F : \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ that the function χ_F is irreducible. Set $\mathsf{B} = \mathsf{A}/\mathsf{S}_{\chi_F}$, and view F as an exact functor $\mathsf{B}^{\mathrm{op}} \to \mathsf{Ab}$. It should be noted that $\mathsf{Hom}(H_S, F) = F(S) \neq 0$ for each simple object S in B. The indecomposability of F implies that all simple objects in B are isomorphic, and the equation (B.1) then implies that χ is irreducible, since for each simple object S

$$\chi_F(S) = \mathsf{length}_{\mathsf{End}(F)} F(S) = \mathsf{length}_{\mathsf{End}(F)} \mathsf{Hom}(H_S, F) = \mathsf{length}_{\mathsf{B}} S = 1. \quad \square$$

Remark B.3. Let $F : \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be an indecomposable endofinite exact functor, and let S be the corresponding simple object in $\mathsf{A}/\mathsf{S}_{\chi_F}$. Then, the endomorphism ring $\mathsf{End}(F)$ is local, and

$$\mathsf{End}(F)/\mathsf{rad}\,\mathsf{End}(F) \cong \mathsf{End}(S),$$

since F identifies with an injective envelope of S. Here, $\mathsf{rad}\, A$ denotes the Jacobson radical of a ring A.

Remark B.4. Let $F : \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be an exact functor, and let $\mathsf{B} = \mathsf{A}/\mathsf{S}_F$, where S_F denotes the Serre subcategory of objects X satisfying $F(X) = 0$. For each object X in A, we have

$$\chi_F(X) = \mathsf{length}_{\mathsf{End}(F)} F(X) = \mathsf{length}_{\mathsf{End}(F)} \mathsf{Hom}(H_X, F) = \mathsf{length}_{\mathsf{B}} X,$$

and this can be used to compute $\sum_i \chi_{F_i}$ for any decomposition $F = \bigoplus_i F_i$ into exact functors.

Let $\mathsf{Sp}\,\mathsf{A}$ denote the set of irreducible additive functions $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$. Following [20, Section 4], we define on $\mathsf{Sp}\,\mathsf{A}$ the *Ziegler topology*. The basic open sets are of the form

$$(X) = \{\chi \in \mathsf{Sp}\,\mathsf{A} \mid \chi(X) \neq 0\}, \quad X \in \mathsf{Ob}\,\mathsf{A}.$$

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PROPOSITION B.5. *The set* $\mathsf{Sp}\,\mathsf{A}$ *of irreducible additive functions* $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ *forms a topological space which satisfies the* T_1*-axiom; that is,* $\{\chi\}$ *is closed for each* $\chi \in \mathsf{Sp}\,\mathsf{A}$.

Proof. We identify each irreducible additive function $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ with an indecomposable injective object in $\mathsf{Lex}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$, as in Proposition B.2 and its proof. Thus, [20, Lemma 4.1] applies, and the argument given there shows that for two objects X_1, X_2 in A, the set $(X_1) \cap (X_2)$ can be written as a union of basic open sets.

A singleton $\{\chi\}$ is closed since χ is the only irreducible function satisfying $\chi(X) = 0$ for all $X \in \mathsf{S}_\chi$. □

The space $\mathsf{Sp}\,\mathsf{A}$ of additive functions identifies via Proposition B.2 with a subspace of $\mathsf{Zsp}\,\mathsf{A}$, which is discussed in the subsequent Appendix C.

LEMMA B.6. *Let* X *be an object in* A, *and let* $n \geqslant 0$. *Then,*

$$\mathsf{U}_{X,n} = \{\chi \in \mathsf{Sp}\,\mathsf{A} \mid \chi(X) \leqslant n\}$$

is a closed subset of $\mathsf{Sp}\,\mathsf{A}$.

Proof. For a chain of subobjects

$$\phi\colon 0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n+1} = X,$$

set

$$\mathsf{U}_\phi = \bigcup_{i=0}^{n} \{\chi \in \mathsf{Sp}\,\mathsf{A} \mid \chi(X_{i+1}/X_i) = 0\},$$

and let $\mathsf{U} = \bigcap_\phi \mathsf{U}_\phi$, where $\phi = (X_i)_{0 \leqslant i \leqslant n+1}$ runs through all such chains. This set is closed by construction, and it follows from Remark B.4 that $\mathsf{U} = \mathsf{U}_{X,n}$. □

Appendix C. The spectrum of an abelian category

Let A be an essentially small abelian category. We consider the category of exact functors $\mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$. This category inherits an exact structure from Ab, and we denote by $\mathsf{Zsp}\,\mathsf{A}$ the set of isomorphism classes of indecomposable injective objects. It should be noted that $\mathsf{Zsp}\,\mathsf{A}$ equals the spectrum of the Grothendieck abelian category $\mathsf{Lex}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$ of left exact functors $\mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ in the sense of [11, Chapter IV]. Following [20, Section 4], we define on $\mathsf{Zsp}\,\mathsf{A}$ the *Ziegler topology*. The basic open sets are of the form

$$(X) = \{F \in \mathsf{Zsp}\,\mathsf{A} \mid F(X) \neq 0\}, \quad X \in \mathsf{Ob}\,\mathsf{A}.$$

LEMMA C.1. *The assignment*

$$\mathsf{Zsp}\,\mathsf{A} \supseteq \mathsf{U} \longmapsto \{X \in \mathsf{A} \mid F(X) = 0 \text{ for all } F \in \mathsf{U}\}$$

induces an inclusion reversing bijection between the closed subsets of $\mathsf{Zsp}\,\mathsf{A}$ *and the Serre subcategories of* A. *In particular,* $\mathsf{Zsp}\,\mathsf{A}$ *is quasi-compact if and only if* A *admits a generator; that is, an object not contained in any proper Serre subcategory of* A.

Proof. See [20, Theorem 4.2 and Corollary 4.5]. ∎

The construction of this space is functorial with respect to certain functors. Let $f : \mathsf{A} \to \mathsf{B}$ be an exact functor between abelian categories. Given F in $\mathsf{Zsp}\,\mathsf{B}$, the composite $F \circ f$ is injective, since the left adjoint of restriction along f is exact, and a right adjoint of an exact functor preserves injectivity. However, $F \circ f$ need not be indecomposable. Thus, f induces a continuous map $\mathsf{Zsp}\,\mathsf{B} \to \mathsf{Zsp}\,\mathsf{A}$ provided that indecomposability is preserved. For instance, a quotient functor $\mathsf{A} \to \mathsf{A}/\mathsf{C}$ with respect to a Serre subcategory $\mathsf{C} \subseteq \mathsf{A}$ has this property; it induces a homeomorphism

$$\mathsf{Zsp}\,\mathsf{A}/\mathsf{C} \xrightarrow{\sim} \{F \in \mathsf{Zsp}\,\mathsf{A} \mid F(\mathsf{C}) = 0\}.$$

The Krull–Gabriel filtration. Following [11, Chapter IV] and [15, Section 6], we define a filtration of A recursively as follows.

- A_{-1} is the full subcategory containing only the zero object.
- A_α is the full subcategory of objects of finite length in $\mathsf{A}/\mathsf{A}_\beta$, if $\alpha = \beta + 1$.
- $\mathsf{A}_\alpha = \bigcup_{\gamma<\alpha} \mathsf{A}_\gamma$, if α is a limit ordinal.

If $\mathsf{A} = \bigcup_\alpha \mathsf{A}_\alpha$, then the smallest ordinal α such that $\mathsf{A} = \mathsf{A}_\alpha$ is called the *Krull–Gabriel dimension*, and is denoted $\mathsf{KGdim}\,\mathsf{A}$. In that case, we say that $\mathsf{KGdim}\,\mathsf{A}$ *exists*.

For each ordinal α, let $\mathsf{Zsp}_\alpha\,\mathsf{A}$ denote the set of functors $F \in \mathsf{Zsp}\,\mathsf{A}$ such that $F(\mathsf{A}_\alpha) = 0$ and $F(X) \neq 0$ for some object X that is simple in $\mathsf{A}/\mathsf{A}_\alpha$. This yields a bijection between the isomorphism classes of simple objects in $\mathsf{A}/\mathsf{A}_\alpha$ and the elements in $\mathsf{Zsp}_\alpha\,\mathsf{A}$.

PROPOSITION C.2. *Suppose that* $\mathsf{KGdim}\,\mathsf{A} = \alpha$. *Then,* $\mathsf{Zsp}\,\mathsf{A}$ *equals the disjoint union* $\bigcup_{\beta<\alpha} \mathsf{Zsp}_\beta\,\mathsf{A}$.

Proof. See [22, Theorem 12.7]. ∎

Removing successively from $\mathsf{Zsp}\,\mathsf{A}$ the points in $\mathsf{Zsp}_\beta\,\mathsf{A}$ for $\beta = -1, 0, 1, \ldots$ yields the *Cantor–Bendixson filtration* of $\mathsf{Zsp}\,\mathsf{A}$, provided that $\mathsf{KGdim}\,\mathsf{A}$ exists. This follows from the next lemma.

LEMMA C.3. *Let $F \in \mathsf{Zsp}\,\mathsf{A}$. If $F(X) \neq 0$ for some finite length object X, then F is isolated; that is, $\{F\}$ is open. The converse holds when $\mathsf{KGdim}\,\mathsf{A}$ exists.*

Proof. If $F(X) \neq 0$ for some finite length object X, then we may assume that X is simple. Thus, $\{F\} = (X)$, since F is an injective envelope of $\mathsf{Hom}(-, X)$ in $\mathsf{Lex}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$. For the converse, see [22, Lemma 12.11]. □

LEMMA C.4. *Let $(f_i : \mathsf{A} \to \mathsf{A}_i)_{i \in I}$ be a family of quotient functors, and set $\mathsf{U}_i = \{F \in \mathsf{Zsp}\,\mathsf{A} \mid F \text{ factors through } f_i\}$ for each i. Suppose that $\mathsf{Zsp}\,\mathsf{A} = \bigcup_i \mathsf{U}_i$, and that each U_i is an open subset. If $\mathsf{KGdim}\,\mathsf{A}$ exists, then $\mathsf{KGdim}\,\mathsf{A} = \mathsf{sup}_i\,\mathsf{KGdim}\,\mathsf{A}_i$.*

Proof. The assumption on each U_i to be open implies that $f_i(\mathsf{A}_\alpha) = (\mathsf{A}_i)_\alpha$ for all i and each ordinal α, by Lemma C.3. On the other hand, $(\mathsf{A}_i)_\alpha = \mathsf{A}_i$ for all i implies $\mathsf{A}_\alpha = \mathsf{A}$, since $\mathsf{Zsp}\,\mathsf{A} = \bigcup_i \mathsf{U}_i$. From this the assertion follows. □

Triangulated categories. Let G be a group of automorphisms acting on A. Then, we denote by $\mathsf{Zsp}\,\mathsf{A}/G$ the corresponding orbit space of $\mathsf{Zsp}\,\mathsf{A}$. Thus, the points in $\mathsf{Zsp}\,\mathsf{A}/G$ are the equivalence classes of the form $[F] = \{F \circ \gamma \mid \gamma \in G\}$. The closed subsets correspond to Serre subcategories of A that are G-invariant.

Let C be an essentially small triangulated category with suspension $\Sigma : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$. We identify cohomological functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ with exact functors $(\mathsf{Ab}\,\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$ via Lemma 2.3, and denote by $\mathsf{Zsp}\,\mathsf{C}$ the orbit space $(\mathsf{Zsp}\,\mathsf{Ab}\,\mathsf{C})/\Sigma$ with respect to the action of Σ.

LEMMA C.5. *The space $\mathsf{Zsp}\,\mathsf{C}$ is quasi-compact if and only if C admits a generator; that is, an object not contained in any proper thick subcategory of C.*

Proof. Suppose first that C has a generator, say X. Then, any Σ-invariant Serre subcategory of $\mathsf{Ab}\,\mathsf{C}$ containing $\mathsf{Hom}(-, X)$ equals $\mathsf{Ab}\,\mathsf{C}$. Thus, $\mathsf{Zsp}\,\mathsf{C}$ is quasi-compact by Lemma C.1. To show the converse, consider for each $X \in \mathsf{C}$ the closed subset

$$\mathsf{U}_X = \{[F] \in \mathsf{Zsp}\,\mathsf{C} \mid F(Y) = 0 \text{ for all } Y \in \mathsf{Thick}(X)\}.$$

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Then, $\bigcap_{X \in \mathsf{C}} \mathsf{U}_X = \varnothing$. If $\mathsf{Zsp}\,\mathsf{C}$ is quasi-compact, then there are finitely many objects such that $\mathsf{U}_{X_1} \cap \cdots \cap \mathsf{U}_{X_r} = \varnothing$. This implies $\mathsf{C} = \mathsf{Thick}(X_1, \ldots, X_r)$. ∎

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