KELVIN-WAVE DIFFRACTION BY A GAP

Dedicated to the memory of Hanna Neumann

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The problem of diffraction of a Kelvin wave by a narrow opening in a barrier separating two semi-infinite sheets of water is formulated as an integral equation. An approximate solution is obtained on the hypothesis that the gap is small compared with the length of the incident wave. The scattered disturbance comprises both Kelvin and Poincaré waves, and asymptotic formulate for their amplitudes are obtained. The results, which are relevant to tidal diffraction by a strait, reveal that the phase shift in the diffracted Kelvin wave is much larger than the corresponding shift induced by either a narrow channel or an inland sea opening into a semi-infinite ocean.

1. Introduction

Tidal predictions at individual stations are based on harmonic analysis of local observations, but surprisingly little is known about the actual nature of the tides. Munk, Snodgrass and Wimbush [6] have shown that local features are very important, and for this reason it is desirable to gather a collection of solutions to theoretical problems concerned with the diffraction of Kelvin waves. Closed solutions have been obtained by Crease [3] for a semi-infinite barrier, and by Buchwald [1], and Packham and Williams [7] for corners.

Buchwald [2] and Miles [5] have investigated the effects of a narrow channel and an inland sea opening into a semi-infinite ocean. In this paper we consider the diffraction of a Kelvin wave by a narrow opening separating two semi-infinite sheets of water.

Let ζ and $\{u, v\}$ be the complex amplitudes of the vertical displacement and horizontal particle velocity, such that the vertical displacement is given by

$$z(x, y, t) = R\{\zeta(x, y)e^{i\omega t}\} \tag{1.1}$$

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and similarly for the velocity; R implies *the real part of.* The Euler and continuity equations, in a reference frame rotating about a vertical axis with the angular velocity $\frac{1}{2}f$, then reduce to (Lamb [4] §207)

$$h\kappa^2\{u, v\} = i\omega\{\zeta_x, \zeta_y\} + f\{\zeta_y, -\zeta_x\} \tag{1.2}$$

and

$$\zeta_{xx} + \zeta_{yy} + \kappa^2\zeta = 0, \tag{1.3}$$

where h is the mean depth,

$$\kappa^2 = (\omega^2 - f^2)/c^2 = \kappa^2(1 - \mathfrak{f}^2), \tag{1.4}$$

$$c = (gh)^{\frac{1}{2}}, \qquad \kappa = \omega/c, \qquad \text{and} \qquad \mathfrak{f} = f/\omega. \tag{1.5a, b, c}$$

The boundary conditions are

$$u(0, y) = 0 \qquad (|y| > b), \tag{1.6a}$$

$$\zeta(0-, y) = \zeta(0+, y) \qquad (|y| < b), \tag{1.6b}$$

and

$$u(0-, y) = u(0+, y) \qquad (|y| < b), \tag{1.6c}$$

together with appropriate finiteness and radiation conditions at infinity.

We seek the solution of (1.3) and (1.6) for a prescribed incident wave, say $\zeta^{(0)}(x, y)$, on the assumption that

$$\varepsilon \equiv kb \ll 1. \tag{1.7}$$

For definiteness, we suppose $\zeta^{(0)}$ to be a Kelvin wave in $x \geqq 0$,

$$\zeta^{(0)}(x, y) = A\exp\{(i\omega y - fx)/c\}, \qquad (x \geqq 0). \tag{1.8}$$

Setting $x = 0$ and invoking (1.7), we obtain

$$\zeta^{(0)}(0, y) = A\{1 + iky + O(\varepsilon^2)\} \qquad (|y| < b) \tag{1.9}$$

for a point in the gap. Similar approximations to $\zeta^{(0)}(0, y)$ hold for a prescribed Poincaré wave or a prescribed tidal potential and differ from (1.9) only in the coefficient of ky.

2. Integral equation for aperture velocity

We pose the solution of (1.3) and (1.6) in the form

$$\zeta(x, y) = \zeta^{(0)}(x, y)H(x) + (\omega/g)\operatorname{sgn} x \int_{-b}^{b} G[|x|, (y - \eta)\operatorname{sgn} x]u(0, \eta)d\eta, \tag{2.1}$$

where $H(x)$ is Heaviside's step function and $G(x, y)$, the Green's function for a point on the boundary of the half-space, satisfies (1.3),

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(2.2) $$iG_x + \mathfrak{f}G_y = (1 - \mathfrak{f}^2)\delta(y) \qquad (x = 0),$$

and radiation and finiteness conditions at infinity. Substituting (2.1) into (1.6b), we obtain the integral equation

(2.3) $$(\omega/g)\int_{-b}^{b}\{G(0, y - \eta) + G(0, \eta - y)\}u(0,\eta)d\eta = -\zeta^{(0)}(0, y) \qquad (|y| < b).$$

The required approximation to $G(0, y)$ is (Buchwald [2])

(2.4)
$$G(0,y) = \tfrac{1}{2}\left[1 - \frac{2i}{\pi}\log(\tfrac{1}{2}\gamma k|y|) - \mathfrak{f}\{\operatorname{sgn} y + \frac{i}{\pi}\log\left(\frac{1+\mathfrak{f}}{1-\mathfrak{f}}\right) - iky\} + O(\varepsilon^2\log\varepsilon)\right],$$

where $\log\gamma$ is Euler's constant; if $\mathfrak{f} > 1$, the logarithms must be evaluated according to

(2.5) $$\log(1 - \mathfrak{f}) = \log(\mathfrak{f} - 1) - i\pi \qquad (\mathfrak{f} > 1).$$

Substituting (1.9) and (2.4) into (2.3), we obtain

(2.6) $$\frac{\omega}{g}\int_{-b}^{b}\left[Z - \frac{2i}{\pi}\log(2|y - \eta|/b)\right]u(0,\eta)d\eta = -A\{1 + iky\}, \qquad (\mathfrak{f}|y| < b),$$

where

(2.7) $$Z = 1 - \frac{i}{\pi}\left\{2\ln(\tfrac{1}{4}\gamma kb) + \mathfrak{f}\ln\left(\frac{1+\mathfrak{f}}{1-\mathfrak{f}}\right)\right\},$$

The integral equation (2.6) is closely related to one that arises in potential theory and admits a solution of the form

(2.8) $$u(0,y) = (g/\omega)(b^2 - y^2)^{-\frac{1}{2}}\{A_0 + A_1 ky + O(\varepsilon^2)\}..$$

Substituting (2.8) into (2.6) and introducing

(2.9a, b) $$y = b\cos\theta, \qquad \eta = b\cos\phi,$$

and

$$\log(2|y - \eta|/b) = -2\sum_{n=1}^{\infty} n^{-1}\cos n\theta\cos n\phi,$$

we obtain

(2.11a, b) $$A_0 = -(\pi Z)^{-1}A \qquad \text{and} \qquad A_1 = -\tfrac{1}{2}A.$$

3. The diffracted field

We restrict explicit calculation of the diffracted field to those waves whose total energy does not vanish as $kr \to \infty$, where $r = \sqrt{(x^2 + y^2)}$. It follows from (2.1), (2.8) and (2.11) that

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$$\zeta(x,y) \sim \zeta^{(0)}(x,y)H(x) - (A/Z)\tilde{G}(|x|, y \operatorname{sgn} x) \operatorname{sgn} x + O(\varepsilon^2), \tag{3.1}$$

where

$$\tilde{G} = G + O(k^{-1}r^{-1}), \qquad kr \to \infty, \tag{3.2}$$

the asymptotic approximation to the Green's function, is given by Buchwald [2] as

$$\tilde{G}(|x|, y \operatorname{sgn} x) = L_k + L_p. \tag{3.3}$$

In (3.3), L_k represents Kelvin waves, in the form

$$L_k = \mathfrak{f} \exp\{(i\omega y - fx) \operatorname{sgn} x / c\} H(-\theta - \theta_0), \tag{3.4}$$

where $\theta = \tan^{-1}(y/x)$, and

$$\theta_0 = \cos^{-1}\mathfrak{f} \qquad (\mathfrak{f} < 1, x > 0), \tag{3.5a}$$

$$= \cos^{-1}\mathfrak{f} - \pi \qquad (\mathfrak{f} < 1, x > 0), \tag{3.5b}$$

$$= 0 \qquad (\mathfrak{f} > 1). \tag{3.5c}$$

Diffracted Poincare waves are represented by L_p, in the form

$$L_p = (1 - \mathfrak{f}^2)(2\pi kt)^{-\frac{1}{2}}(1 - i\mathfrak{f}\tan\theta)^{-1} \exp\{-i(kr - \tfrac{1}{4}\pi)\} \qquad (\mathfrak{f} < 1) \tag{3.6a}$$

$$= 0. \qquad (\mathfrak{f} > 1) \tag{3.6b}$$

The diffracted Kelvin and Poincaré waves are illustrated in Figure 1.

The diffracted Kelvin wave along $x = 0+$ is given by

$$\zeta(0+, y) \sim \zeta^{(0)}(0,y)\{1 - (\mathfrak{f}/Z) + O(\varepsilon^2)\} \qquad (ky \to -\infty). \tag{3.7}$$

An interesting consequence of (3.7) is that, as $\varepsilon \to 0$, the phase change in the Kelvin wave across the opening is $O[(\log \varepsilon)^{-1}]$, compared with phase changes of $O(\varepsilon)$ predicted in the cases where the opening leads to a narrow channel (Buchwald [2]) or an inland sea (Miles [5]). This difference is due to the fact that waves propagating through the gap convey energy away in two dimensions, as opposed to either one dimension or none in the other two cases.

Consider, for example a gap in a western boundary at a latitude of 36°N (which implies $\mathfrak{f} = 0.6$ for the diurnal tide). Choosing $2b = 1.6$ km and $h = 3.6$ km, we obtain a time delay of 13 min. for the Kelvin wave moving northward along the boundary. The corresponding delay for an inland sea which has the same gap width with an area of 1200 km^2, corresponding to San Francisco Bay, is 3 sec (Miles [5]). (Munk, Snodgrass & Wimbush [6]) report an anomalous phase delay of the order of an hour in the tide at Crescent City, north of San Francisco; however, there appears to be little justification for regarding San Francisco Bay and it tributaries as equivalent to a semi-infinite ocean, and we cite the example primarily to illustrate expected orders of magnitude.)

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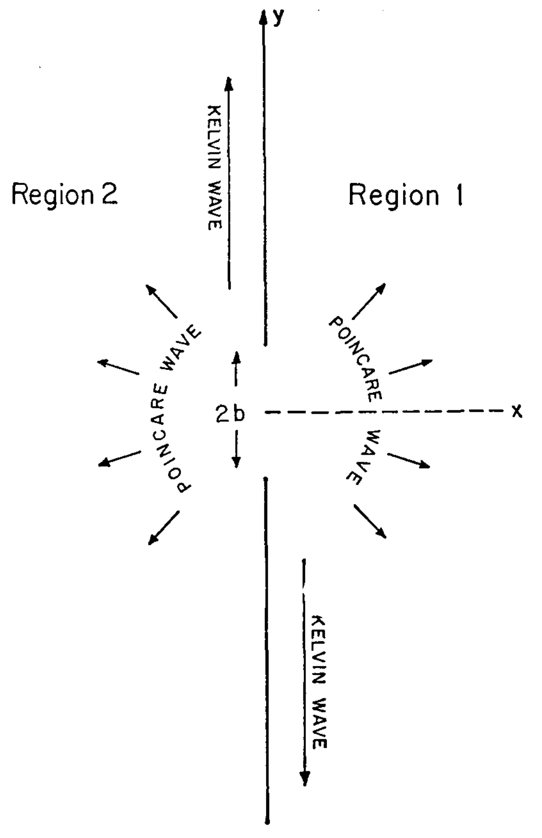

Figure 1. The diffracted Kelvin and Poincaré waves.

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