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ON THE CRITICAL POINTS OF A POLYNOMIAL

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Let p be a complex polynomial, of the form $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$, where $|z_k| \geqslant 1$ when $1 \leqslant k \leqslant n-1$. Then $p'(z) \neq 0$ if $|z| < 1/n$.

Let $B(z,r)$ denote the open ball in $\mathbb{C}$ with centre z and radius r, and $\overline{B}(z,r)$ denote its closure. The Gauss–Lucas theorem states that every critical point of a complex polynomial p of degree at least 2 lies in the convex hull of its zeros. This theorem has been further investigated and developed. B. Sendov conjectured that, if all the zeros of p lie in $\overline{B}(0,1)$, then, for any zero ζ of p, the disc $\overline{B}(\zeta,1)$ contains at least one zero of p'; see [**3**, Problem 4.1]. This conjecture has attracted much attention–see, for example, [**1**], and the papers cited there. In connection with this conjecture, Brown [**2**] posed the following problem.

Let $\mathcal{Q}_n$ denote the set of all complex polynomials of the form $p(z) = z\prod_{k=1}^{n-1}(z - z_k)$, where $|z_k| \geqslant 1$ when $1 \leqslant k \leqslant n-1$. Find the best constant C_n such that p' does not vanish in $B(0, C_n)$, for all p in $\mathcal{Q}_n$.

Brown observed that, if $p(z) = z(z-1)^{n-1}$, then $p'(1/n) = 0$, and conjectured that $C_n = 1/n$. We show this here.

THEOREM *For all p in $\mathcal{Q}_n$, $p'(z) \neq 0$ if $z \in B(0, 1/n)$.*

PROOF: Clearly $p'(0) = \prod_{k=1}^{n-1}(-z_k) \neq 0$. If $0 < |z| < 1/n$, then $|z - z_k| > 1 - 1/n$, and so

$$\left|\frac{p'(z)}{p(z)}\right| = \left|\frac{1}{z} + \sum_{k=1}^{n-1}\frac{1}{z - z_k}\right| \geqslant \frac{1}{|z|} - \sum_{k=1}^{n-1}\frac{1}{|z - z_k|} > n - \sum_{k=1}^{n-1}\frac{n}{n-1} = 0.$$

It follows that p' does not vanish in $B(0, 1/n)$. ∎

Similarly, if $p(z) = z^m\prod_{k=1}^{n-m}(z - z_k)$, where $|z_k| \geqslant 1$ when $1 \leqslant k \leqslant n-m$, then $p'(z) \neq 0$ if $0 < |z| < m/n$.

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